Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2006: Test 4

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts) A is an $n \times n$ matrix with entries 0 or 1. Let k be a positive integer. Show that if $n \geq R(2k, 2k)$ then there exists a set of rows I and columns J such that (i) $|I| = |J| = k$ and (ii) $i, i' \in I$ and $j, j' \in J$ implies $A(i, j) = A(i', j')$.
Solution This is HW9, Q2. Given A we construct a coloring τ of the edges of K_n as follows. If $i < j$ then we give the edge (i, j) of K_n the color Red if $A_{i,j} = 0$ and Blue if $A_{i,j} = 1$.
Since $n \geq R(2k, 2k)$ we see that K_n contains a mono-colored copy of K_{2k}. If the set of vertices of this copy is S, divide S into two parts S_1, S_2 of size k where $\max S_1 < \min S_2$. It follows that the sub-matrix given by $I = S_1, J = S_2$ satisfies our requirements.

Q2: (33pts) $A_1, A_2, \ldots, A_{mn+1}$ are non-empty subsets of $[n]$. Show that there exists $I \subseteq [mn+1]$ such that (i) $|I| = m+1$ and (ii) if $i, j \in I$ then $A_i \not\subseteq A_j$ and $A_j \not\subseteq A_i$.

Solution Consider the poset on $\{A_1, A_2, \ldots, A_{mn+1}\}$ with $\leq$ equal to $\subseteq$. The maximum length of a chain $X_1 \subset X_2 \subset \cdots \subset X_k$ in this poset is at most n, since $|X_k| \geq k$. Applying Dilworth's theorem, we see that there is an anti-chain $\{A_i : i \in I\}$ of size $\lceil (mn+1)/n \rceil = m+1$.

Q3: (34pts) Consider the following game: There is a pile of n chips. A move consists of removing 3^k chips for some $k \geq 0$.

(a) Compute the Sparague-Grundy numbers $g(n)$ for $n = 0, 1, 2, \ldots, 10$.

(b) Make a guess at the general form of $g(n)$.

(c) Give an inductive proof of your conjecture in (b).

Solution

(a)

n	0	1	2	3	4	5	6	7	8	9	10
$g(n)$	0	1	0	1	0	1	0	1	0	1	0

(b) $g(n) = n \mod 2$.

(c) This is true for $0 \leq n \leq 10$. Because 3^i is odd for $i \geq 0$ we see that for $k > 5$ we have

$$g(2k-1) = mex\{g(2k-1-3^i) : \; i \geq 0, \, 3^i \leq 2k-1\} = mex\{0, 0, \ldots, 0\} = 1.$$
$$g(2k) = mex\{g(2k-3^i) : \; i \geq 0, \, 3^i \leq 2k\} = mex\{1, 1, \ldots, 1\} = 0.$$