Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2006: Test 3

Name:________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
1 Total	100	

Q1: (33pts)
(a) A connected graph G has two paths P_1, P_2 of lengths k_1, k_2 respectively. Show that if P_1 and P_2 are vertex disjoint then G contains a path of length at least $\lceil k_1/2 \rceil + \lceil k_2/2 \rceil + 1$. (The length of path P is the number of edges in it.)
(b) Using (a), show that if G is a connected graph and P_1, P_2 are paths of *maximum* length then they share at least one common vertex.

Solution:
(a) G is connected and so there is a path joining a vertex in P_1 to a vertex in P_2. Take a shortest such path Q. Suppose it joins $a \in P_1$ to $b \in P_2$. Since Q is as short as possible, the internal vertices of Q are disjoint from P_1, P_2. Now let P_1' be the longer of the two sub-paths of P_1 which have end-point a and an end-point of P_1. Then P_1' has length $\geq \lceil k_1/2 \rceil$. Define P_2' similarly. Let P_1'' be the reversal of P_1'. Then the path P_1'', Q, P_2' has length at least $\lceil k_1/2 \rceil + \lceil k_2/2 \rceil + 1$.
(b) Suppose that P_1, P_2 are of maximum length k and are vertex disjoint. Part (a) implies that G has a path of length at least $2\lceil k/2 \rceil + 1 > k$, contradiction.

Q2: (33pts)

(a) A tree has n_i vertices of degree i for $i = 1, 2, 3$ and no vertex of degree 4 or more. Show that $n_1 = n_3 + 2$.

(b) What is the tree corresponding to the Prüfer code $1, 1, 1, 1, 1, \ldots, 1$ ($n-2$ 1's)?

(c) T is a tree on vertex set V. Vertices $x, y \in V$ and $e = (x, y)$ is not an edge of T. Show that $T + e$ contains a cycle.

(d) C is a cycle of $T + e$ in (c). f is an edge of C. Show that $T + e - f$ is a tree.

Solution:
(a) We have

$$\begin{aligned} n_1 + n_2 + n_3 &= n \\ n_1 + 2n_2 + 3n_3 &= 2n - 2. \end{aligned}$$

Subtracting the second equation from twice the first gives the answer.
(b) Vertex 1 has degree $n - 1$ and all other vertices have degree 1. So the tree consists of the $n - 1$ edges $(1, i), 2 \leq i \leq n$.
(c) $T + e$ is connected and has n edges. It has too many edges for a tree and so it must have a cycle.
(d) $T + e - f$ has $n - 1$ edges and it is connected and so it is a tree. (If $f = (x, y)$ and $P = P', x, y, P''$ is a walk in $T + e$ using (x, y) then we can replace it by the walk $P', C - f, P''$ i.e. $T + e - f$ is indeed connected).

Q3: (34pts)
Let m, n be positive integers and let S be a subset of $\{1, 2, \ldots, m+n\}$ such that $|S| > (m+n)/2$. Show that there exist $x, y \in S$ such that $|x - y| \in \{m, n\}$.
Hint: Consider the array

$$\begin{array}{llllllllll} 1 & 2 & 3 & \ldots & m & m+1 & m+2 & m+3 & \ldots & m+n \\ n+1 & n+2 & n+3 & \ldots & m+n & 1 & 2 & 3 & \ldots & n \end{array}$$

Solution:
Each $x \in [m+n]$ appears twice in the array. So the elements of S appear at least $m+n+1$ times. But then by the PHP some column of the array must have two entries from S, implying that the required x, y exist.