Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2006: Test 2

Name:______________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
1 Total	100	

Q1: (33pts) A box has m drawers. Drawer i contains g_i gold coins and s_i silver coins where $g_i + s_i \geq 2$, for $i = 1, 2, \ldots, m$. A person chooses a random drawer and then chooses two randoms coin from that draw. If both coins are silver then they are put back and the person tries again. What is the expected number of selections needed before at least one gold coin is selected.

Solution Let p be the probability that a gold coin is selected. Then the number of trials is distributed as a geometric random variable with probability p. Hence the expected number of drawings is $1/p$.
But,

$$p = 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{\binom{s_i}{2}}{\binom{g_i+s_i}{2}}.$$

Q2: (33pts)
(a) Let A, B, C be randomly chosen subsets of $[n]$. Show that

$$\mathbf{P}(A \cap B \subseteq C) = \left(\frac{7}{8}\right)^n.$$

(b) A family $\mathcal{A}$ of subsets of $[n]$ is said to *containment free* if $A \cap B \not\subseteq C$ for distinct $A, B, C \in \mathcal{A}$. Using the answer to (a) show that exists a containment free family $\mathcal{A}$ such that

$$|\mathcal{A}| \geq \left(\frac{8}{7}\right)^{n/3}.$$

Solution
(a)

$$A \cap B \subseteq C \ if \ x \notin (A \cap B) \setminus C \ \forall x \in [n].$$

By independence of choice, we see that

$$\mathbf{P}(A \cap B \subseteq C) = (1 - \mathbf{P}(1 \in (A \cap B) \setminus C))^n = (1 - 1/8)^n.$$

(b) Let $\mathcal{A}$ be a family of randomly chosen subsets $X_1, X_2, \ldots, X_p$ of $[n]$. Let Z be the number of triples i, j, k such that $X_i \cap X_j \subseteq X_k$. Then

$$\mathbf{P}(Z \geq 1) \leq \mathbf{E}(Z) \leq p(p-1)(p-2)\left(\frac{7}{8}\right)^n. \tag{1}$$

So, if $p \leq \left(\frac{8}{7}\right)^{n/3}$, then the RHS of (1) is < 1 and a union free family of size p will exist.

Q3: (34pts)
A particle sits in the middle 0 of a line $-L, 1-L, \ldots, -1, 0, 1, 2, \ldots, L$ where $L > 0$. When at $i \in [1-L, L-1]$ it makes a move to $i-1$ with probability $1/3$ and a move to $i+1$ with probability $2/3$. When at L it moves to $L-1$. When at $-L$ it stops.
Let E_k denote the expected number of visits to 0 before stopping if we started the walk at k.

1. Find a set of equations satisfied by the E_k.
2. Use backwards induction to show that $E_i = E_{i+1}$ for $0 \leq i < L$.
3. Use induction to show that $E_{i-L} = \left(\dfrac{2^{i+1}-2}{2^{i+1}-1}\right) E_{i+1-L}$ for $0 < i \leq L-1$.
4. Use your equation $E_0 = \cdots$ to deduce that $E_0 = 3(2^L - 1)$.

Solution

1. The equations are

$$\begin{aligned}
E_L &= E_{L-1} \\
E_{-L} &= 0 \\
E_0 &= \frac{1}{3}E_{-1} + \frac{2}{3}E_1 + 1 \\
E_i &= \frac{1}{3}E_{i-1} + \frac{2}{3}E_{i+1} \qquad for\ i \neq 0, \pm L
\end{aligned}$$

2. The basis of the induction is $i = L-1$. Assume then that $E_i = E_{i+1}$ for some $i > 0$. Then

$$E_i = \frac{1}{3}E_{i-1} + \frac{2}{3}E_{i+1} = \frac{1}{3}E_{i-1} + \frac{2}{3}E_i$$

and this implies $E_{i-1} = E_i$.

3. We have

$$E_{1-L} = \frac{1}{3}E_0 + \frac{2}{3}E_{2-L} = \frac{2}{3}E_{2-L}.$$

This is the basis of the induction. Assume then that

$$E_{i-L} = \left(\frac{2^{i+1}-2}{2^{i+1}-1}\right) E_{i+1-L}$$

for some $i > 0$. Then we have

$$\begin{aligned} E_{i+1-L} &= \frac{1}{3} \cdot \frac{2^{i+1}-2}{2^{i+1}-1} E_{i+1-L} + \frac{2}{3} E_{i+2-L} \\ (3(2^{i+1}-1) - (2^{i+1}-2))E_{i+1-L} &= 2(2^{i+1}-1)E_{i+2-L}. \end{aligned}$$

4.

$$E_0 = \frac{1}{3} \cdot \frac{2^L - 2}{2^L - 1} E_0 + \frac{2}{3} E_0 + 1.$$