Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2006: Test 1

Name:__________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
(a): Given integers $m, n > 0$ and an integer $a \geq 0$, show that the number of functions f from $[n]$ to $[m]$ which satisfy $f(i+1) \geq f(i) + a$ for $1 \leq i \leq n-1$ is

$$\binom{m+n-a(n-1)-1}{n}.$$

Hint: Let $x_i = f(i) - f(i-1)$ for $i = 2, 3, \ldots, n$. Define x_1 and x_{n+1} suitably and count the number of choices for $x_1, x_2, \ldots, x_{n+1}$.
(b): Assuming that $a \geq 1$, use the answer to (a) to find the number of subsets S of $[m]$ which have n elements and satisfy $|x - y| \geq a$ for $x, y \in S$, $x \neq y$.

Solution
(a): Define $x_1 = f(1)$ and $x_{n+1} = m - f(n)$. Then we have

$$x_1 + x_2 + \cdots + x_{n+1} = m$$

and $x_1 \geq 1, x_2, \ldots, x_n \geq a, x_{n+1} \geq 0$.
Furthermore, there is a bijection between the x's and our set of functions. The number of x's is given by the binomial coefficient.
b: Each set arises from a unique function and so the number of sets is

$$\binom{m+n-a(n-1)-1}{n}.$$

Q2: (33pts)
(a): We have n boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$. Show that there are $\frac{(2n)!}{2^n}$ ways to place the balls into the boxes so that each box gets two balls.
(b): An allocation of balls to boxes is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i}. Use the Inclusion-Exclusion formula to determine the number of scrambled allocations.
Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Solution
(a): Each permutation π of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box B_i, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly 2^n distinct permutations, giving the result.
(b): Let A be the set of all allocations of balls to boxes and let A_i be those allocations in which balls b_{2i-1}, b_{2i} are placed into box B_i, $i = 1, 2, \ldots, n$. Then for $S \subseteq [n]$, we have

$$|A_S| = \frac{(2(n - |S|))!}{2^{n-|S|}}.$$

Thus, the number of scrambled allocations is

$$\sum_{S \subseteq [N]} (-1)^{|S|} \frac{(2(n - |S|))!}{2^{n-|S|}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.$$

Q3: (34pts) The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1$ and

$$a_n - 3a_{n-1} = 1$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.

Solution
Multiply each equation by x^n and sum. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} x^n. \\
(a(x) - 1) - 3xa(x) &= \frac{1}{1-x} - 1. \\
a(x) &= \frac{1}{(1-x)(1-3x)} \\
&= \frac{-1/2}{1-x} + \frac{3/2}{1-3x} \\
&= -\frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n.
\end{aligned}$$

So,

$$a_n = \frac{3^{n+1} - 1}{2}.$$