Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2005: Test 4

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
4	34	
Total	100	

Q1: (33pts)
Consider the following general game involving one pile of chips . There is a finite set of positive integers S and each move involves choosing $s \in S$ and removing this number of chips from the pile. The game ends when there are no moves possible. Show that the SPRAGUE-GRUNDY function g satisfies $g(n) \leq |S|$ for all $n \geq 0$.
Solution First observe that if A is a set of non-negative integers, then $mex(A) \leq |A|$. This is because either $A = \{0, 1, 2 \ldots, |A| - 1\}$, in which case $mex(A) = |A|$ or there exists $i < |A|$ such that $i \notin A$, in which case $mex(A) \leq i$.
Now $g(n) = mex(A)$ where $A = \{g(n - s);\ s \in S\}$ and $|A| \leq |S|$.

Q2: (33pts)
Consider the following game involving two piles of chips. A move consists of removing one pile and splitting the remaining pile into two non-empty piles. There is a unique terminal position in which both piles have one chip.
Suppose that the two piles have m, n chips respectively. Here are the N, P positions for $1 \leq m, n \leq 6$.

	1	2	3	4	5	6
1	P	N	P	N	P	N
2	N	N	N	N	N	N
3	P	N	P	N	P	N
4	N	N	N	N	N	N
5	P	N	P	N	P	N
6	N	N	N	N	N	N

When in general is m, n a P position. Prove your claim.
Solution m, n is a P-position iff m, n are both odd. We can prove this by induction on $m + n$. The base case is $m + n = 2$ i.e. $m = n = 1$.
If m, n are both odd then any move will involve splitting one of the numbers into the sum of an odd and an even position. This is an N-position, by induction.
If m is even then we can remove the n and split m into the sum of two odd numbers, a P-position, by induction.
Thus our partition into N, P has the requisite properties.

Q3: (34pts)
Consider the following game involving one pile of chips. A move consists of removing 2^k chips where $k = 0, 1, 2, 3, \ldots$. The first few values of the SPRAGUE-GRUNDY function g are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	19	11	12	13	14	15	16
$g(n)$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

What is $g(n)$ in general? Prove your claim by induction.
Solution $g(n) = n \mod 3$. It is clearly true for $n \leq 16$.
Observe that $2^k \mod 3 \neq 0$ and so $n \neq n - 2^k \mod n$ for any $k \geq 0$. Since $n \mod 3 \neq n-1 \mod 3 \neq n-2 \mod 3$ we see by the induction hypothesis that

$$g(n) = mex(g(n-1), g(n-2), g(n-4), \ldots) = mex(g(n-1), g(n-2)).$$

$n = 3k$: $g(n) = mex(3k-1 \mod 3, 3k-2 \mod 3) = mex(2, 1) = 0$
$n = 3k+1$: $g(n) = mex(3k \mod 3, 3k-1 \mod 3) = mex(0, 2) = 1$
$n = 3k+2$: $g(n) = mex(3k+1 \mod 3, 3k \mod 3) = mex(1, 0) = 2$

Q4: (33pts)
An $m \times n$ 0,1 matrix A and a $p \times q$ 0,1 matrix B are *compatible* if $A(i,j) = B(i,j)$ for $1 \leq i \leq \min\{m,p\}$ and $1 \leq j \leq \min\{n,q\}$.
Suppose that A_i is an $m_i \times n_i$ $0,1$ matrix for $i = 1,2,\ldots,N$ and that there do **not** exist i,j such that A_i is compatible with A_j. Prove that

$$\sum_{i=1}^{N} \frac{1}{2^{m_i n_i}} \leq 1.$$

Solution Let $P = \max m_i$ and $Q = \max n_i$. Let X be a random $P \times Q$ 0,1 matrix and let $\mathcal{E}_i$ be the event that $X(p,q) = A_i(p,q)$ for $1 \leq p \leq m_i$, $1 \leq q \leq n_i$. Then the $\mathcal{E}_i$ are disjoint events, since if $\mathcal{E}_i$ and $\mathcal{E}_j$ both occur, then A_i and A_j must be a compatible pair.
Thus

$$1 \geq \sum_{i=1}^{N} \mathbf{Pr}(\mathcal{E}_i) = \sum_{i=1}^{N} \frac{1}{2^{m_i n_i}}.$$