**Department of Mathematics**
**Carnegie Mellon University**

# 21-301 Combinatorics, Fall 2005: Test 3

Name:____________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 33 | |
| 2 | 33 | |
| 3 | 34 | |
| Total | 100 | |

**Q1: (33pts)**
Let $n, p$ be positive integers and let $N = n^2p+1$. Suppose that $x_1, x_2, \ldots, x_N$ are real numbers. Show that either (i) there is a *strictly* increasing subsequence of length $n+1$ or (ii) a *strictly* decreasing subsequence of length $n+1$ or (iii) a *constant* subsequence of length $p+1$.
**Solution** Assume that there is no constant sequence of length $p+1$ i.e. no real value appears more than $p$ times in the sequence. Then the number of distinct values appearing is at least $\lceil \frac{N}{p} \rceil = n^2 + 1$.
Choose a subsequence consisting of these $\geq n^2 + 1$ distinct values. By the Erdős-Szekerés theorem this subsequence contains a monotone subsequence of length $\geq n+1$. Since the values are distinct, such a monotone subsequence is strict.

**Q2: (33pts)**
A tree $T$ has $n = 2m$ vertices. All vertices of $T$ have degree one or three. There are $n_1$ vertices of degree one and $n_3$ vertices of degree three. Determine the values of $n_1, n_3$ in terms of $m$. Justify your claim.
How many such trees are there on vertex set $\{1, 2, \ldots, n\}$?
**Solution** We have $n_1 + n_3 = n$ and $n_1 + 3n_3 = 2n - 2$ (sum of degrees = twice number of edges). Solving these equations gives

$$n_1 = m + 1 \text{ and } n_3 = m - 1.$$

The number of such trees is

$$\binom{2m}{m+1}\binom{2m-2}{2, 2, \ldots, 2, 0, 0, \ldots, 0} = \binom{2m}{m+1}\frac{(2m-2)!}{2^{m-1}}$$

where in the multi-nomial there are $m - 1$ 2's and $m + 1$ 0's.
The factor $\binom{2m}{m+1}$ counts the number of ways of choosing the vertices of degree 1 and the multi-nomial coefficent is the number of trees with a fixed degree sequence, consisting of $m - 1$ 3's and $m + 1$ 1's.

**Q3: (34pts)**
Let $p, q$ be positive integers and $n = p + q - 1$. Let $T$ be a fixed tree with $q$ vertices. Show that if we color the edges of $K_n$ Red or Blue then either (i) there is a vertex with Red degree $p$ or (ii) there is a Blue copy of $T$.
**Solution** If there is no vertex of Red degree $p$ then every vertex has Blue degree at least $n - 1 - (p - 1) = q - 1$. Thus the Blue sub-graph contains a copy of every tree with $q - 1 + 1 = q$ vertices. (See HW6,Q3).