Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2005: Test 2

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts) A box has four drawers; one contains three gold coins, one contains two gold coins and a silver coin, one contains a gold coin and two silver coins and one contains three silver coins. Assume that one drawer is selected randomly and that a randomly selected coin from that drawer turns out to be gold. What is the probability that the chosen drawer is the one with three gold coins?

Solution: Let the four drawers be A, B, C, D. Let G, S stand for the chosen coin being Gold/Silver respectively. Then what we want is

$$\Pr(A \mid G) = \frac{\Pr(A \wedge G)}{\Pr(G)}.$$

Now

$$\Pr(A \wedge G) = \Pr(A) = \frac{1}{4}.$$

$$\begin{aligned}
&\Pr(G) \\
&= \Pr(G \mid A)\Pr(A) + \Pr(G \mid B)\Pr(B) + \Pr(G \mid C)\Pr(C) + \Pr(G \mid D)\Pr(D) \\
&= 1 \times \frac{1}{4} + \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} + 0 \times \frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

So

$$\Pr(A \mid G) = \frac{1/4}{1/2} = \frac{1}{2}.$$

Q2: (33pts) Let $A_1, A_2, \ldots, A_n$ be subsets of A with $|A_i| = k$ for $1 \leq i \leq n$. Show that if $n(2^{1-k} + k2^{1-k}) < 1$ then it is possible to partition the set A into two sets R, B (i.e. color A red and blue) so that

$$|A_i \cap R| \geq 2 \text{ and } |A_i \cap B| \geq 2 \text{ for } i = 1, 2, \ldots, n.$$

Solution Let $\mathcal{E}_{i,X}$ be the event that color X is not used twice on A_i and let $\mathcal{E}_i = \mathcal{E}_{i,R} \cup \mathcal{E}_{i,B}$. Then

$$\Pr(\mathcal{E}_i) \leq \Pr(\mathcal{E}_{i,R}) + \Pr(\mathcal{E}_{i,B}) = 2\left(2^{-k} + \binom{k}{1}2^{-k}\right) = 2^{1-k} + k2^{1-k}.$$

Thus,

$$\Pr\left(\bigcup_{i=1}^{n} \Pr(\mathcal{E}_i)\right) \leq n(2^{1-k} + k2^{1-k}) < 1$$

and so there is a coloring for which none of the $\mathcal{E}_i$ occur.

Q3: (34pts)
A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L-1]$ it makes a move to $i-1$ with probability $1/3$ and a move to $i+1$ with probability $2/3$. When at L it stops.
Let E_k denote the expected number of visits to 0 if we started the walk at k.

1. Find a set of equations satisfied by the E_k.
2. Given that $E_k = \frac{A}{2^k} + B$ is a solution to your equations for some A, B, determine A, B and hence find E_0.

Solution: The equations are

$$\begin{aligned} E_L &= 0 \\ E_0 &= 1 + E_1 \\ E_k &= \frac{1}{3}E_{k-1} + \frac{2}{3}E_{k+1} \end{aligned}$$

for $0 < k < L$.
$E_L = 0$ implies then that $\frac{A}{2^L} + B = 0$ and so $B = -\frac{A}{2^L}$.
$E_0 = 1 + E_1$ implies then that $A + B = 1 + \frac{1}{2}A + B$ which implies that $A = 2$.
Thus $E_0 = 2 - 2^{1-L}$.