Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2005: Test 1

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts) Let a, m, n, p be positive integers. How many integer solutions are there to

$$x_1 + x_2 + \cdots + x_m = n$$

which satisfy $x_i \geq a$ for $i = 1, 2, \ldots, p$ and $x_i \geq 0$ for $i = p + 1, p + 2, \ldots, m$.

Solution Let $y_i = x_i - a$ for $i = 1, 2, \ldots, p$ and $y_i = x_i$ for $i > p$. Then $y_1, y_2. \ldots, y_m$ satisfy

$$y_1 + y_2 + \cdots + y_m = n - ap \text{ and } y_1, y_2, \ldots, y_m \geq 0$$

and we have described a bijection between the x's of the question and the y's that satisfy this.
Thus the number of x's is $\binom{m+n-ap-1}{m-1}$.

Q2: (33pts) Use the inclusion-exclusion formula

$$\left|\bigcap_{i=1}^{N} \overline{A}_i\right| = \sum_{S\subseteq[N]} (-1)^{|S|}|A_S|$$

to show that the number of permutations $\pi(1), \pi(2), \ldots, \pi(n)$ of $[n]$ which satisfy $\pi(i+1) \neq \pi(i)+1$ for $i = 1, 2, \ldots, n-1$ is

$$\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k}(n-k)!.$$

Solution Let

$$A_i = \{\pi :\ \pi(i+1) = \pi(i)+1\}.$$

In words, when we look at the sequence obtained by putting j into the $\pi(j)$th position for $j = 1, 2, \ldots, n$, we see that $i+1$ immediately follows i. Thus we can "merge" $i, i+1$ into one symbol and find that $|A_i| = (n-1)!$ for all i. Similarly, $|A_S| = (n-|S|)!$, since we get A_S by $|S|$ merges. (If $i, i+1$ are part of S then the merge contains $i, i+1, i+2$.).
The result follows directly from the inclusion-exclusion formula.

Q3: (34pts) The sequence $a_0, a_1, \ldots, a_n, \ldots$ satifies the following:
$a_0 = 1$, $a_1 = 4$ and

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

for $n \geq 2$.
Determine the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and hence find a_n.

Solution

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty}(a_n - 4a_{n-1} + 4a_{n-2})x^n \\ &= (a(x) - 1 - 4x) - 4x(a(x) - 1) + 4x^2 a(x) \\ &= a(x)(1 - 4x + 4x^2) - 1. \end{aligned}$$

So

$$\begin{aligned} a(x) &= \frac{1}{1 - 4x + 4x^2} \\ &= \frac{1}{(1 - 2x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)2^n x^n. \end{aligned}$$