Ordered Graph Removal Lemma

Omri Ben-Eliezer

The well-known triangle removal lemma, proved by Ruzsa and Szemerédi in 1976, states that if a graph G contains many pairwise-disjoint triangles, then a random (large enough) constant-size induced subgraph of G contains a triangle with good probability. In a series of works, culminating in a result of Alon and Shapira, it was shown that any hereditary graph property $\mathcal{P}$ satisfies a removal lemma of this type: Namely, if one must add or remove many edges in a graph G to make it satisfy the hereditary property $\mathcal{P}$, then with good probability, a random (large enough) constant-size induced subgraph of G doesn't satisfy $\mathcal{P}$. Here, a graph property is a collection of functions $f\colon \binom{[n]}{2} \to \{0,1\}$ that satisfies the following symmetry requirement: It must be closed under relabeling of the vertices.

In this work, we show that the symmetry is not needed, establishing a removal lemma for any hereditary property $\mathcal{P}$ of functions $f\colon \binom{[n]}{2} \to \{0,1\}$ (i.e., any hereditary vertex-ordered graph property). The proof also carries over to two-dimensional matrices over a finite alphabet. The result has direct implications in property testing, showing that any such hereditary property $\mathcal{P}$ is testable using a constant number of queries.

Joint work with Noga Alon and Eldar Fischer.