# Comparable pairs in families of sets

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**Abstract**

Given a family $\mathcal{F}$ of subsets of $[n]$, we say two sets $A, B \in \mathcal{F}$ are *comparable* if $A \subset B$ or $B \subset A$. Sperner's celebrated theorem gives the size of the largest family without any comparable pairs. This result was later generalised by Kleitman, who gave the minimum number of comparable pairs appearing in families of a given size.

In this talk we study a complementary problem posed by Erdős and Daykin and Frankl in the early '80s. They asked for the maximum number of comparable pairs that can appear in a family of $m$ subsets of $[n]$, a quantity we denote by $c(n, m)$. We first resolve an old conjecture of Alon and Frankl, showing that $c(n, m) = o(m^2)$ when $m = n^{\omega(1)}2^{n/2}$. We also show more accurate bounds for $c(n, m)$ for sparse and dense families, characterise the extremal constructions for certain values of $m$, and sharpen some other known results.

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