On the chromatic number of random regular graphs*

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Determining the chromatic number of random graphs is one of the longest-standing challenges in probabilistic combinatorics. For the Erdős-Rényi model, the single most intensely studied model in the random graphs literature, the question dates back to the seminal 1960 paper that started the theory of random graphs [4].

Apart from $G_{\mathrm{ER}}(n, m)$, the model that has received the most attention certainly is the random regular graph $G(n, d)$. We provide an almost complete solution to the chromatic number problem on $G(n, d)$, at least in the case that d remains fixed as $n \to \infty$. The strongest previous result on the chromatic number of $G(n, d)$ is due to Kemkes, Pérez-Giménez and Wormald [5]. They proved that w.h.p. for $k \geq 3$ if $d \in ((2k-3)\ln(k-1), (2k-2)\ln(k-1))$ then $\chi(G(n, d)) = k$ and if $d \in [(2k-2)\ln(k-1), (2k-1)\ln k]$ then $\chi(G(n, d)) \in \{k, k+1\}$. These bounds imply that $G(n, d)$ is k-colorable w.h.p. if $d < (2k-2)\ln(k-1)$, while $G(n, d)$ fails to be k-colorable w.h.p. if $d > (2k-1)\ln k$. Our main result is

Theorem 1 *There is a sequence $(\varepsilon_k)_{k\geq 3}$ with $\lim_{k\to\infty} \varepsilon_k = 0$ such that the following is true.*

1. *If $d \leq (2k-1)\ln k - 2\ln 2 - \varepsilon_k$, then $G(n, d)$ is k-colorable w.h.p.*
2. *If $d \geq (2k-1)\ln k - 1 + \varepsilon_k$, then $G(n, d)$ fails to be k-colorable w.h.p.*

This implies that for every integer k exceeding a certain constant k_0 we identify a number $d_{k-\mathrm{col}}$ such that $G(n, d)$ is k-colorable w.h.p. if $d < d_{k-\mathrm{col}}$ and non-k-colorable w.h.p. if $d > d_{k-\mathrm{col}}$.

The best current results on coloring $G_{\mathrm{ER}}(n, m)$ as well as the best prior result on $\chi(G(n, d))$ are obtained via the *second moment method* [1, 3, 5]. So are the present results. Recently, Coja-Oghlan and Vilenchik [3] improved the result from [1] on the chromatic number of $G_{\mathrm{ER}}(n, m)$. This improvement is obtained by considering a different random variable, namely the number $Z_{k,\mathrm{good}}$ of "good" k-colorings instead of $Z_{k-\mathrm{col}}$ the number of all k-colorings. The definition of this random variable draws on intuition from non-rigorous statistical mechanics work on random graph coloring [6, 8]. Crucially, the concept of good colorings facilitates the computation of the second moment. Theorem 1 provides a result matching [3] for $G(n, d)$. Following [5], we combine the second moment bound from [3] with *small subgraph conditioning*.

The previous *lower* bound on the chromatic number of $G(n, d)$ is based on a simple first moment argument over the number of k-colorings. The bound that can be obtained in this way, attributed to Molloy and Reed [7], is that $G(n, d)$ is non-k-colorable w.h.p. if $d > (2k-1)\ln k$. By contrast, the second assertion in Theorem 1 marks a strict improvement. The proof is via an adaptation of techniques developed in [2] for the random k-NAESAT problem. Extending this argument to the chromatic number problem on $G(n, d)$ requires substantial technical work.

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