# HOW DOES THE CORE SIT INSIDE THE MANTLE?

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ABSTRACT. The"giant component" has remained a guiding theme in the theory of random graphs ever since the seminal paper of Erdős and Rényi [Magayar Tud. Akad. Mat. Kutato Int. Kozl. **5** (1960) 17–61]. Because for any $k \geq 3$ the $k$-core, defined as the (unique) maximal subgraph of minimum degree $k$, is identical to the largest $k$-connected subgraph of the random graph w.h.p., the $k$-core is perhaps the most natural generalisation of the "giant component". Pittel, Wormald and Spencer were the first to determine the precise threshold $d_k$ beyond which the $k$-core $\mathscr{C}_k(\boldsymbol{G})$ of $\boldsymbol{G} = \boldsymbol{G}(n, d/n)$ with $d > 0$ fixed is non-empty w.h.p. [Journal of Combinatorial Theory, Series B **67** (1996) 111–151]. Specifically, for any $k \geq 3$ there is a function $\psi_k : (0, \infty) \to [0,1]$ such that for any $d \in (0,\infty) \setminus \{d_k\}$ the sequence $(n^{-1}|\mathscr{C}_k(\boldsymbol{G})|)_n$ converges to $\psi_k(d)$ in probability.

The aim of the present paper is to enhance the branching process perspective of the $k$-core problem pointed out in their paper. More specifically, we are concerned with the following question. Fix $k \geq 3$, $d > d_k$ and let $s > 0$ be an integer. Generate a random graph $\boldsymbol{G}$ and mark each vertex according to $\sigma_{k,\boldsymbol{G}} : V(\boldsymbol{G}) \to \{0,1\}$, $v \mapsto \mathbf{1}\{v \in \mathscr{C}_k(\boldsymbol{G})\}$. For a vertex $v$ let $\boldsymbol{G}_v$ denote its component. Now, pick a vertex $\boldsymbol{v}$ uniformly at random and let $\partial^s[\boldsymbol{G}_{\boldsymbol{v}}, \boldsymbol{v}, \sigma_{k,\boldsymbol{G}_{\boldsymbol{v}}}]$ denote the isomorphism class of the finite rooted $\{0,1\}$-marked graph obtained by deleting all vertices at distance greater than $s$ from $\boldsymbol{v}$ from $\boldsymbol{G}_{\boldsymbol{v}}$. Our aim is to determine the distribution of $\partial^s[\boldsymbol{G}_{\boldsymbol{v}}, \boldsymbol{v}, \sigma_{k,\boldsymbol{G}_{\boldsymbol{v}}}]$.

To accomodate the non-trivial correlations between the $k$-core and the "mantle" (i.e., the vertices outside the core) we introduce a Galton-Watson process $\hat{\boldsymbol{T}}(d,k,p)$ that posseses five vertex types, denoted by 000, 001, 010, 110, 111. Setting $q = q(d,k,p) = \mathbb{P}\left[\mathrm{Po}(dp) = k-1 | \mathrm{Po}(dp) \geq k-1\right]$, we let $p_{000} = 1-p$, $p_{010} = pq$, $p_{110} = p(1-q)$. The process starts with a single vertex, whose type is chosen from $\{000, 010, 111\}$ according to the distribution $(p_{000}, p_{010}, p_{111})$. Subsequently, each vertex of type $z_1z_2z_3$ spawns a random number of vertices of each type. The offspring distributions are defined by the generating functions $g_{z_1z_2z_2}(\boldsymbol{x})$ detailed in Figure 1, where $\boldsymbol{x} = (x_{000}, x_{001}, x_{010}, x_{110}, x_{111})$ and $\bar{q} = \bar{q}(d,k,p) = \mathbb{P}\left[\mathrm{Po}(dp) = k-2 | \mathrm{Po}(dp) \leq k-2\right]$. Let $\boldsymbol{T}(d,k,p)$ signify the random rooted $\{0,1\}$-marked tree obtained by giving mark 0 to all vertices of type 000, 001 or 010, and mark 1 to all others.

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\begin{aligned}
g_{000}(\boldsymbol{x}) &= \exp(d(1-p)x_{000})\frac{\sum_{h=0}^{k-2}(dp)^h(qx_{010}+(1-q)x_{110})^h/h!}{\sum_{h=0}^{k-2}(dp)^h/h!},\\
g_{001}(\boldsymbol{x}) &= \bar{q}\left(\exp(d(1-p)x_{001})\left(qx_{010}+(1-q)x_{110}\right)^{k-2}\right)\\
&\quad + (1-\bar{q})\left(\exp(d(1-p)x_{000})\frac{\sum_{h=0}^{k-3}(dp)^h(qx_{010}+(1-q)x_{110})^h/h!}{\sum_{h=0}^{k-3}(dp)^h/h!}\right),\\
g_{010}(\boldsymbol{x}) &= \exp(d(1-p)x_{001})\left(qx_{010}+(1-q)x_{110}\right)^{k-1},\\
g_{110}(\boldsymbol{x}) &= \exp(d(1-p)x_{001})\frac{\sum_{h\geq k}(dpx_{111})^h/h!}{\sum_{h\geq k}(dp)^h/h!},\\
g_{111}(\boldsymbol{x}) &= \exp(d(1-p)x_{001})\frac{\sum_{h\geq k-1}(dpx_{111})^h/h!}{\sum_{h\geq k-1}(dp)^h/h!}.
\end{aligned}
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FIGURE 1. The generating functions $g_{z_1z_2z_3}(\boldsymbol{x})$.

**Theorem.** *Assume that $k \geq 3$ and $d > d_k$. Let $s \geq 0$ be an integer and let $\tau$ be a rooted $\{0,1\}$-marked tree. Moreover, let $p^*$ be the largest fixed point of $\phi_{d,k} : [0,1] \to [0,1]$, $p \mapsto \mathbb{P}\left[\mathrm{Po}(dp) \geq k-1\right]$. Then*

$$\frac{1}{n}\sum_{v \in V(\boldsymbol{G})} \mathbf{1}\left\{\partial^s[\boldsymbol{G}, v, \sigma_{k,\boldsymbol{G}_v}] = \partial^s[\tau]\right\} \text{ converges to } \mathbb{P}\left[\partial^s[\boldsymbol{T}(d,k,p^*)] = \partial^s[\tau]\right] \text{ in probability.}$$

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