Asymptotic distribution of the numbers of vertices and arcs of the giant strong component in sparse random digraphs.

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(Joint work with Daniel Poole)

Abstract. Two models of a random digraph on n vertices, $D(n, \text{number of arcs} = m)$ and $D(n, \text{Prob(arc)} = p)$ are studied. In 1990 R. Karp for $D(n, p = c/n)$ and independently T. Łuczak for $D(n, m = cn)$ proved that for $c > 1$, with probability tending to 1, there is an unique strong component of size of order n. Karp showed, in fact, that the giant component has likely size asymptotic to $n\theta^2$, where $\theta = \theta(c)$ is the unique positive root of $1 - \theta = e^{-c\theta}$. We prove that, for both random digraphs, the joint distribution of the number of vertices and number of arcs in the giant strong component is asymptotically Gaussian with the same mean vector $n\boldsymbol{\mu}(c)$, $\boldsymbol{\mu}(c) := (\theta^2, c\,\theta^2)$, and two distinct 2×2 covariance matrices, $n\mathbf{B}(c)$ and $n\big[\mathbf{B}(c) + c\boldsymbol{\mu}'(c)^T\boldsymbol{\mu}'(c)\big]$. To this end, we introduce and analyze a randomized deletion process which determines the directed $(1,1)$-core, the maximal sub-digraph with minimum in-degree and out-degree at least 1. This $(1,1)$-core contains all non-trivial strong components. However, we show that the likely numbers of peripheral vertices and arcs in the $(1,1)$-core, those outside the largest strong component, are of log-polynomial order, thus dwarfed by anticipated fluctuations, on the scale of $n^{1/2}$, of the giant component parameters. By approximating the likely realization of the deletion algorithm with a deterministic trajectory, we obtain our main result via exponential supermartingales and Fourier-based techniques.