Discrepancy of random graphs and hypergraphs

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(Joint work with Benny Sudakov and Jie Ma)

Abstract

The *discrepancy* of an n-vertex k-uniform hypergraph H with edge density $\rho_H = e(H)/\binom{n}{k}$ is $\mathrm{disc}(H) = \max_{S \subseteq V(H)} \left| e(S) - \rho_H \binom{|S|}{k} \right|$, where $e(S) = e(H[S])$ is the number of edges in the sub-hypergraph induced by S. This important concept appears naturally in various branches of combinatorics and was studied by many researchers in recent years. The discrepancy can be viewed as a measure of how uniformly the edges of H are distributed among the vertices, and it is closely related to the theory of quasi-random graphs, as the property $\mathrm{disc}(G) = o(|V(G)|^2)$ implies the quasi-randomness of the graph G. A natural generalization is the relative discrepancy of two hypergraphs. Let G and H be two k-uniform hypergraphs over the same vertex set V, with $|V| = n$. The *discrepancy of G with respect to H* is $\mathrm{disc}(G, H) = \max_\pi \left| e(G_\pi \cap H) - \rho_G \rho_H \binom{n}{k} \right|$, over all bijections $\pi : V \to V$. Thus $\mathrm{disc}(G, H)$ measures by how much the overlap $e(G_\pi \cap H)$ can deviate from its average. Bollobás and Scott introduced and studied this notion, and they asked the following question. For two random n-vertex graphs G, H with constant edge probability p, what is the expected value of $\mathrm{disc}(G, H)$? Answering in a strong form this question, we determine the discrepancy between two random k-uniform hypergraphs, up to a constant factor depending only on k. We will also discuss other related results and problems.

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