Unit distance graphs with no large cliques or short cycles and high chromatic number

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Unit distance graph is a graph whose set of vertices is a subset of $\mathbb{R}^d$ and two vertices are connected by an edge only if they are at the unit distance apart. It is known that for any unit distance graph G in $\mathbb{R}^d$ we have $\chi(G) \leq (3+o(1))^d$. On the other hand, there exists a sequence G_d of unit distance graphs in $\mathbb{R}^d$ such that $\chi(G_d) \geq (1.239+o(1))^d$. We are interested in the chromatic number of distance graphs that satisfy certain conditions.

In 1959 Erdős proved that for any $k, l \in \mathbb{N}$ there exists a graph G such that $\chi(G) \geq l$, $g(G) \geq k$. Here $g(G)$ is the girth of G, that is, the length of the shortest cycle in G. Later, in 1976 Erdős asked whether for any $k \in \mathbb{N}$ there exists a unit distance graph on the plane such that $\chi(G) = 4$ and $g(G) \geq k$. This question was answered positively by O'Donnell.

We study analogous questions in higher dimensions. The main theorem we prove is the following: for any $k \in \mathbb{N}$ there exists a constant $c > 1$ and a sequence G_d of unit distance graphs in $\mathbb{R}^d$ such that $\chi(G_d) \geq (c+o(1))^d$ and $g(G_d) > k$. We also investigate analogous questions for unit distance graphs without cliques of size $k \geq 3$. We use both probabilistic arguments and explicit constructions to obtain better bounds on the constant c in terms of k.

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