Average subtree density of series-reduced trees

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Define $\mu(T)$ to be the average order of a subtree of a fixed tree T, chosen uniformly at random, and the average subtree density $D(T) = \mu(T)/n(T)$, where $n(T)$ is the number of vertices of T. These invariants were introduced by Jamison [1], who showed that $D(T) > \frac{1}{3}$ for any tree T, but that there exist trees with $D(T)$ arbitrarily close to 1. Among trees of a fixed order, the path minimises $\mu(T)$.

We consider particularly the *series-reduced* trees, which are those with no vertex of degree 2. Jamison conjectured that $D(T) > \frac{1}{2}$ for any such tree T; this, along with the upper bound $D(T) < \frac{3}{4}$ (if T has more than one vertex), was proved by Vince and Wang [2]. Both constants are best possible.

Vince and Wang posed the problem of finding necessary and sufficient conditions for a sequence of series-reduced trees to approach either bound. After introducing some natural parameters we give stronger upper and lower bounds on $\mu(T)$, and obtain simple necessary and sufficient conditions for a sequence of series-reduced trees to have average subtree density tending to $\frac{1}{2}$ or $\frac{3}{4}$.

References

[1] R. E. Jamison, On the average number of nodes in a subtree of a tree, *J. Combin. Theory Ser. B* **35** (1983), 207–223.

[2] A. Vince and H. Wang, The average order of a subtree of a tree, *J. Combin. Theory Ser. B* **100** (2010), 161–170.