On vertices of given degree in Polya trees

Bernhard Gittenberger and Veronika Kraus*

April 8, 2011

Let $\mathcal{T}_n$ be the set of rooted unlabelled non-plane trees (Polya trees) on n vertices. It has been shown that the random variable $X_n^{(d)}$, counting the number of vertices of degree d in a tree $T \in \mathcal{T}_n$ drawn uniformly at random, has expected value $\mathbb{E}(X_n^{(d)}) = \mu_d n + \mathcal{O}(1)$, with $\mu_d = \frac{2C}{b^2\sqrt{\rho}}\rho^d$, where $\rho \approx 0.3383219$ is the singularity of the generating function $T(x)$ of Polya trees and $C \approx 7.7581604$ and $b \approx 2.6811266$ are known constants.

Let $L_n^{(d)}(k)$ be the number of nodes of degree d at distance k from the root in a random Polya tree $T \in \mathcal{T}_n$ of size n, and $L_n^{(d)}(t)$ be the stochastic process obtained by linear interpolation. We prove the following refinement.

Theorem 1. *Let $l_n^{(d)}(t) = \frac{1}{\sqrt{n}} L_n^{(d)}(t\sqrt{n})$, and $l(t)$ denote the local time of a standard Brownian excursion. Then $l_n^{(d)}(t)$ converges weakly to the local time of a Brownian excursion, i.e., we have*

$$(l_n^{(d)}(t))_{t\geq 0} \xrightarrow{w} \mu_d \frac{b\sqrt{b}}{2\sqrt{2}} \cdot l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}}t\right)_{t\geq 0}.$$

We further compute the correlation of two different degrees d_1 and d_2 on a given level k and prove that the correlation coefficient is 1, asymptotically as n tends to infinity.