On the Two-Phase Stokes Flow Problem with Surface Tension

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A Two-Phase Stokes Flow Problem with Surface Tension

Consider two 2-D immiscible fluids of the same viscosity $\mu = 1$, separated by a simple closed curve Γ. Their dynamics are governed by

$$\mu \Delta \boldsymbol{u} - \nabla p = \mathbf{0} \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{2}$$

$$[\boldsymbol{u}] = \mathbf{0}, \tag{3}$$

$$[\Sigma(\boldsymbol{u}, p)\boldsymbol{n}] = -\gamma\kappa\boldsymbol{n} \tag{4}$$

where

$\boldsymbol{u}$: fluid velocity; p: pressure; $\Sigma(\boldsymbol{u}, p)$: Newtonian stress tensor;

$[\cdot]$: interior value minus exterior value; μ: Newtonian viscosity;

γ: surface tension coefficient; $\boldsymbol{n}$: outward unit normal;

κ: signed curvature of the interface

The Model Diagram

Related Literature

- The Navier-Stokes Problem with Surface Tension
  1. The One-Phase Problem
     1.1 Solonnikov [23, 26, 22, 27, 28, 29, 2, 25], Mogilevskii and Solonnikov [14], Solonnikov [26], Shibata and Shimizu [18, 19, 20], Allain [1], Beale [4], Beale and Nishida [5], Tani [31], Tani and Tanaka [32]
  2. The Two-Phase Problem
     2.1 Denisova [6, 8], Denisova and Solonnikov [9, 7], Tanaka [30], Shimizu [21], Prüss and Simonett [15, 3, 17]
- The Stokes Problem with Surface Tension
  1. The One-Phase Problem
     1.1 Günther and Prokert [13], Prokert [16], Solonnikov [24], Escher and Prokert [10], Günther and Prokert [13], Friedman and Reitich [11]

Interface Parametrization

The interface $\Gamma = z(\alpha, t)$ is parametrized such that the tangent vector's magnitude has no dependence on α, i.e.,

$$z_\alpha(\alpha, t) = \frac{L(t)}{2\pi} e^{i(\alpha+\theta(\alpha,t))}$$

where $L(t)$ is the interface length at time t.
We can then derive evolution equations for $L(t)$ and $\theta(\alpha, t)$:

$$L_t(t) = -\int_{-\pi}^{\pi} (1 + \theta_\alpha(\alpha))U(\alpha)d\alpha \quad (5)$$

$$\theta_t(\alpha, t) = \frac{2\pi}{L(t)} U_\alpha(\alpha) + \frac{2\pi}{L(t)} T(\alpha)(1 + \theta_\alpha(\alpha)). \quad (6)$$

Reformulation of the Evolution Equation for $L(t)$

Using the interior fluid's incompressibility, we obtain

$$\left(\frac{L(t)}{2\pi}\right)^2 = R^2\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right)^{-1}. \tag{7}$$

This analytical expression for $L(t)$ can be shown to be equivalent to equation (5).

Steady State Solutions

For any constants $c \in \mathbb{R}$ and $R > 0$,

$$(\theta(\alpha, t), L(t)) = (c, 2\pi R)$$

is a steady state solution to (5) and (6), which corresponds to a stationary circle of radius R.

Solution Space

For a periodic function f defined on $[-\pi, \pi)$, its Fourier transform is defined as

$$\mathcal{F}(f)(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) e^{-ik\alpha} d\alpha. \tag{8}$$

The corresponding Fourier series is given as

$$f(\alpha) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik\alpha}. \tag{9}$$

Solution Space

We let $\mathcal{F}_\nu^{0,1}$ and $\dot{\mathcal{F}}_\nu^{s,1}$, $s \geq 0$, be spaces of periodic functions on $[-\pi, \pi)$ whose norms

$$\|f\|_{\mathcal{F}_\nu^{0,1}} = \sum_{k \in \mathbb{Z}} e^{\nu(t)|k|} \left| \hat{f}(k) \right|, \tag{10}$$

$$\|f\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k \neq 0} e^{\nu(t)|k|} |k|^s \left| \hat{f}(k) \right|, \tag{11}$$

where

$$\nu(t) = \frac{t}{1+t}\nu_0, \tag{12}$$

are finite.

Solution Space

We also use a family of Banach spaces $\mathcal{F}^{0,1}$ and $\dot{\mathcal{F}}^{s,1}$, $s \geq 0$, equipped respectively with norms

$$\|f\|_{\mathcal{F}^{0,1}} = \sum_{k\in\mathbb{Z}} \left|\hat{f}(k)\right|, \tag{13}$$

$$\|f\|_{\dot{\mathcal{F}}^{s,1}} = \sum_{k\neq 0} |k|^s \left|\hat{f}(k)\right|. \tag{14}$$

The space $\mathcal{F}^{0,1}$ equipped with the norm (13) is the classical Wiener algebra, i.e., the space of absolutely convergent Fourier series.

Main Result

Theorem (C.)

Fix $\gamma > 0$. If the initial datum $\theta^0 \in \dot{\mathcal{F}}^{1,1}$ such that $\left|\mathcal{F}(\theta^0)(0)\right|$ and $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$ are sufficiently small, then for any $T \in (0, \infty)$ there exists a unique solution

$$\theta(\alpha, t) \in C([0, T]; \dot{\mathcal{F}}_\nu^{1,1}) \cap L^1([0, T]; \dot{\mathcal{F}}_\nu^{2,1}) \tag{15}$$

to the equations (5) and (6), where ν is given in (12) and $\nu_0 > 0$ is dependent on θ^0. The solution becomes instantaneously analytic. In particular, for any $t \in [0, T]$

$$\|\theta(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\theta(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau \le \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}, \tag{16}$$

where $\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 > 0$. Moreover, $\|\theta(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}}$ decays exponentially in time.

Boundary Integral Formulation

The fluid velocity, which appears in the evolution equation, is represented by the single-layer potential form, i.e.,

$$u_j(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Gamma (-\gamma\kappa(s)\boldsymbol{n}(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds, \quad \boldsymbol{x}\in\mathbb{R}^2, \tag{17}$$

where $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ and $G = (G_{ij})$ given by

$$G_{ij}(\boldsymbol{w}) = -\delta_{ij}\log|\boldsymbol{w}| + \frac{w_i w_j}{|\boldsymbol{w}|^2} \tag{18}$$

is the Green's function for two-dimensional infinite unbounded incompressible Stokes flow.

Key Proof Strategy

We linearize the evolution equation for θ around a steady state solution:

$$\partial_t \phi + (-\Delta)^{1/2} \phi = \mathfrak{R}. \tag{19}$$

The $\mathfrak{R}$ part can be shown to be "small" in the norm of the solution space $\dot{\mathcal{F}}^{1,1}_{\nu}$.

In the Fourier space, this equation becomes

$$\partial_t \hat{\phi}(k) = -|k|\,\hat{\phi}(k) + \hat{\mathfrak{R}}(k), \tag{20}$$

which clearly reveals that the principal linear part is diagonalized.

Derivation of the A Priori Estimate

Take the time derivative of

$$\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s \left|\hat{\phi}(k)\right| = 2\sum_{k\geq 1} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right| \tag{21}$$

to obtain

$$\begin{aligned} &\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| \\ &+2\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)}+\overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{2\left|\hat{\phi}(k)\right|}. \end{aligned} \tag{22}$$

Derivation of the A Priori Estimate

Using careful estimates, we obtain

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \leq \nu'(t)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s+1,1}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| \tag{23}$$

$$+ \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \tag{24}$$

$$+ 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|. \tag{25}$$

Handling the Dissipation Term

Note that

$$\int_{-\pi}^{\pi} z_\alpha(\alpha, t) d\alpha = 0. \tag{26}$$

In HLS parametrization, this identity yields

$$0 = \int_{-\pi}^{\pi} e^{i(\alpha + \hat{\phi}(1)e^{i\alpha} + \hat{\phi}(-1)e^{-i\alpha} + \sum_{|k|>1} \hat{\phi}(k)e^{ik\alpha})} d\alpha. \tag{27}$$

Handling the Dissipation Term

Proposition (Gancedo, García-Juárez, Patel, and Strain)

Let $r \in (0, \frac{1}{2}\log\frac{5}{4})$. *Consider* $\|\phi\|_{\mathcal{F}^{0,1}} < r$. *Then*

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \leq C_I(r) r \sum_{|k|\geq 2} \left|\hat{\phi}(k)\right|,$$

where

$$C_I(r) = \frac{1}{r} \cdot \frac{2e^r(e^r - 1)}{1 - 4(e^{2r} - 1)}.$$

Here, $C_I(r) > 0$ *is a strictly increasing function of* r *where*

$$\lim_{r\to 0^+} C_I(r) = 2,$$
$$\lim_{r\to \log\frac{5}{4}^-} C_I(r) = \infty.$$

Derivation of the A Priori Estimate

$$
\begin{aligned}
&\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
&\le \|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} \Big(R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad + R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \\
&\quad + R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
&\quad + 3\left(H_3 \|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4 \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \\
&\quad + 3\left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \left(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \\
&\quad + \left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \right) \left(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \\
&\quad + 6\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \left(H_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4 \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \\
&\quad + 2\left(H_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4 \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \right) \Big).
\end{aligned}
$$

Derivation of the A Priori Estimate

Using the estimate for $\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}$ and the implicit function theorem for $\left|\hat{\phi}(\pm 1)\right|$, we obtain

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \leq -\left(\Lambda(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(t)\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} \tag{28}$$

for some function Λ, which is a monotone decreasing function of $\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}$, and

$$\nu'(t) = \frac{\nu_0}{(1+\tau)^2}. \tag{29}$$

Regularization Argument

Theorem (Picard-Lindelöf)

Let $O \subseteq B$ be an open subset of a Banach space B with norm $\|\cdot\|_B$ and let $F : O \to B$ be a nonlinear operator satisfying the following conditions:

1. *F maps O into B.*
2. *F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subseteq O$ of X such that*

$$\left\| F(\tilde{X}) - F(\hat{X}) \right\|_B \leq L \left\| \tilde{X} - \hat{X} \right\|_B$$

   for all $\tilde{X}, \hat{X} \in U_X$.

Then for any $X_0 \in O$, there exists a time T such that the ordinary differential equation

$$\frac{dX}{dt} = F(X)$$
$$X(0) = X_0 \in O$$

has a unique local solution $X \in C^1((-T, T); O)$. If F does not depend explicitly on time, then solutions to the above ODE can be continued until they leave the set O.

Regularization Argument

Cast our original evolution equation

$$\theta_t(\alpha) = \frac{2\pi}{L(t)}(U_\alpha(\theta)(\alpha) + T(\theta)(\alpha)(1 + \theta_\alpha(\alpha))),$$

$$L(t) = 2\pi R\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta(\alpha) - \theta(\eta))^n d\eta d\alpha\right)^{-\frac{1}{2}}$$

into an ODE on an infinite-dimensional Banach space:

$$\frac{d\theta_N}{dt} = (\mathcal{J}_N^1 \circ G_N)(\theta_N). \quad (30)$$

Regularization Argument

where

$$
\begin{aligned}
& G_N(\theta_N) \\
=& R^{-1}\left(1+\frac{1}{2\pi}\operatorname{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N(\alpha)-\theta_N(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \\
& \cdot\left((U_\alpha)_N(\theta_N)+T_N(\theta_N)\left(1+(\theta_N)_\alpha\right)\right).
\end{aligned}
$$

Regularization Argument

Apply the Picard-Lindelöf Theorem by setting $B = H_N^m$, $O = O^M$, and $F = \mathcal{J}_N^1 \circ G_N$, where

$$H_N^m = \left\{ f \in H^m([-\pi, \pi]) : \operatorname{supp}(\hat{f}) \subseteq [-N, N],\ \hat{f}(\pm 1) = 0,\ \operatorname{Im}(f) = 0 \right\}$$

and

$$O^M = \{ f \in H_N^m : \|f\|_{H^m} < M \}.$$

Regularization Argument

Lemma (Aubin-Lions)

Let X_0, X, and X_1 be Banach spaces such that

$$X_0 \subseteq X \subseteq X_1,$$

with compact embedding $X_0 \hookrightarrow X$, and let $p \in (1, \infty]$. Let G be a set of functions mapping $[0, T]$ into X_1 such that G is bounded in $L^p([0, T]; X) \cap L^1_{loc}([0, T]; X_0)$ and $\partial_t G$ is bounded in $L^1_{loc}([0, T]; X_1)$. Then G is relatively compact in $L^q([0, T]; X)$, where $q \in [1, p)$.

Regularization Argument

- Aubin-Lions' Lemma allows us to extract a subsequence of these solutions that is convergent in $L^2([0, T]; \dot{\mathcal{F}}^{1,1}_\nu)$ for any $T > 0$.
- To apply Aubin-Lions' Lemma, we set $X_0 = \dot{\mathcal{F}}^{2,1}_\nu$, $X = \dot{\mathcal{F}}^{1,1}_\nu$, $X_1 = \dot{\mathcal{F}}^{0,1}_\nu$, $p = \infty$, and let

$$G = \{\theta_N : N \in \mathbb{N}\}.$$

The limit of the extracted subsequence is a weak solution to the original equation.

Inheritance of the A Priori Estimate

For all $N \in \mathbb{N}$ and for all $t \in [0, T]$,

$$\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \leq \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{31}$$

Inheritance of the A Priori Estimate

By Fatou's lemma, for any $t \in [0, T]$,

$$\int_0^t \liminf_{N\to\infty} \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\, d\tau \le \liminf_{N\to\infty} \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\, d\tau.$$

Then we obtain for all $t \in [0, T]$

$$\begin{aligned}
&\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\, d\tau \\
&\le \liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \liminf_{N\to\infty} \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\, d\tau \\
&\le \liminf_{N\to\infty} \left(\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\, d\tau \right) \\
&\le \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}$$

Therefore,

$$\theta \in L^\infty([0, T]; \dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0, T]; \dot{\mathcal{F}}^{2,1}_\nu). \tag{32}$$

Uniqueness of Solutions

Taking the time derivative of

$$\|\theta_1 - \theta_2\|_{\dot{\mathcal{F}}^{1,1}} = 2\sum_{k>0} |k| \left|\mathcal{F}(\theta_1 - \theta_2)(k)\right|, \tag{33}$$

we obtain

$$\begin{aligned} &\frac{d}{dt}\|\theta_1 - \theta_2\|_{\dot{\mathcal{F}}^{1,1}} \\ =&\sum_{k>0} \frac{|k|}{|\mathcal{F}(\theta_1 - \theta_2)(k)|} \\ &\cdot \left(\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k) \cdot \overline{\mathcal{F}(\theta_1 - \theta_2)(k)} \right. \\ &\left. + \mathcal{F}(\theta_1 - \theta_2)(k) \cdot \overline{\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k)}\right). \end{aligned} \tag{34}$$

Uniqueness of Solutions

After careful estimates, we obtain that for sufficiently small $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$,

$$\frac{d}{dt}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{1,1}} \leq \mathcal{E}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{1,1}}, \tag{35}$$

where $\mathcal{E}$ is a coefficient that may depend on $\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}$, $\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}$, $\|\phi_1\|_{\dot{\mathcal{F}}^{2,1}}$, and $\|\phi_2\|_{\dot{\mathcal{F}}^{2,1}}$, and is integrable in time.

An Associated Problem I: The Muskat Problem

- The Muskat problem describes the dynamics of incompressible fluids of different nature (e.g., oil and water) permeating porous media (e.g., tar sands) under gravity.
- Gancedo, García-Juárez, Patel, and Strain [12] established global-in-time existence, uniqueness, and instantaneous analyticity of solutions for small initial data of low regularity for a 2-D Muskat bubble immersed in Muskat flow.

An Associated Problem II: The Peskin Problem

- The Peskin problem is a fluid-structure interaction (FSI) problem that describes the dynamics of a 1-D closed elastic string separating 2-D Stokes fluids.
- The only mathematical difference between the Peskin model and mine is the nature of the driving force.
- The Peskin model is driven by the elasticity of the string, which obeys the following general law of elasticity:

$$\partial_\theta \left(T(|\partial_\theta \boldsymbol{X}|) \cdot \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right) \cdot |\partial_\theta \boldsymbol{X}|^{-1} . \tag{36}$$

- The most general setting in which well-posedness has been established for the Peskin problem is where $T(\alpha) > 0$ and $T'(\alpha) > 0$.

Computational Verification

To verify the analytical results, we numerically solve the following dynamics equation for the fluid interface

$$\partial_t \boldsymbol{X}(\theta, t) = \frac{1}{4\pi} \int_\Gamma (-\gamma\kappa(s)\boldsymbol{n}(s)) \boldsymbol{G}(\boldsymbol{X}(\theta, t) - \boldsymbol{X}(s, t)) ds, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

We discretize the interface with N points for some fixed even integer N. For a fixed time step size $dt > 0$, let

$$\boldsymbol{X}^n = (\boldsymbol{X}_0^n, \boldsymbol{X}_1^n, \ldots, \boldsymbol{X}_{N-1}^n)$$

be the position of the interface at time $n \cdot dt$.

Computational Verification

Given the initial position $\boldsymbol{X}^0$ of the interface, the boundary integral can be written as

$$
\begin{aligned}
&\partial_t \boldsymbol{X}(\theta, t) \\
&= -\frac{1}{4\,|\partial_\theta \boldsymbol{X}^0|}\mathcal{H}(\partial_\theta \boldsymbol{X}) - \frac{1}{4}\mathcal{H}\left(\left(\frac{1}{|\partial_\theta \boldsymbol{X}|} - \frac{1}{|\partial_\theta \boldsymbol{X}^0|}\right)\partial_\theta \boldsymbol{X}\right)(\theta) \\
&\quad - \frac{1}{4\pi}\int_{S^1} \partial_{\theta'}\left(-\log\left(\frac{|\Delta \boldsymbol{X}|}{2\left|\sin(\frac{\theta-\theta'}{2})\right|}\right) I + \frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right) \cdot \frac{\partial_{\theta'} \boldsymbol{X}}{|\partial_{\theta'} \boldsymbol{X}|} d\theta',
\end{aligned}
$$

where $\Delta \boldsymbol{X} = \boldsymbol{X}(\theta, t) - \boldsymbol{X}(s, t)$.

Computational Verification

Given $\boldsymbol{X}^n$, ensure that any adjacent pair of the N points in the interface have the same chordal length. Then $\boldsymbol{X}^{n+1}$ is obtained by solving

$$\begin{aligned}
\frac{\boldsymbol{X}^{n+1/2} - \boldsymbol{X}^n}{\Delta t/2} &= -\frac{1}{4\left|\mathcal{D}_N \boldsymbol{X}^n\right|}\mathcal{H}_N\left(\mathcal{D}_N \boldsymbol{X}^{n+1/2}\right) + R_2(\boldsymbol{X}^n) \\
\frac{\boldsymbol{X}^{n+1} - \boldsymbol{X}^n}{\Delta t} &= -\frac{1}{4\left|\mathcal{D}_N \boldsymbol{X}^n\right|}\mathcal{H}_N\left(\frac{\mathcal{D}_N \boldsymbol{X}^n + \mathcal{D}_N \boldsymbol{X}^{n+1}}{2}\right) \\
&\quad + R_1(\boldsymbol{X}^{n+1/2}, \boldsymbol{X}^n) + R_2(\boldsymbol{X}^{n+1/2}),
\end{aligned}$$

Computational Verification

where

$$
\begin{aligned}
&R_1(\boldsymbol{X}^{n+1/2}, \boldsymbol{X}^n) \\
=&\frac{1}{4}\mathcal{H}_N\left(\frac{\mathcal{D}_N(\boldsymbol{X}^{n+1/2}-\boldsymbol{X}^n)\cdot\mathcal{D}_N(\boldsymbol{X}^{n+1/2}+\boldsymbol{X}^n)}{\left|\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right|\cdot\left|\mathcal{D}_N\boldsymbol{X}^n\right|\cdot\left(\left|\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right|+\left|\mathcal{D}_N\boldsymbol{X}^n\right|\right)}\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right)
\end{aligned}
$$

and $R_2(\boldsymbol{X})$ is a numerical computation of the integral

$$
-\frac{1}{4\pi}\int_{S^1}\partial_{\theta'}\left(-\log\left(\frac{|\Delta\boldsymbol{X}|}{2\left|\sin(\frac{\theta-\theta'}{2})\right|}\right)I+\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\cdot\frac{\partial_{\theta'}\boldsymbol{X}}{|\partial_{\theta'}\boldsymbol{X}|}d\theta'.
$$

Computational Verification

To compute the perturbation, we need to devise a way to "project away" circles from the interface. To that end, we parametrize a circle of radius $A^2 + B^2 > 0$ centered at (C_1, C_2) by

$$\boldsymbol{X}(\theta) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + A \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + B \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Since $|\partial_\theta \boldsymbol{X}| = \sqrt{A^2 + B^2}$ is independent of θ, the points $\boldsymbol{X}(k \cdot \frac{2\pi}{N})$ for $k = 0, 1, \ldots, N-1$ that make up the discretized circle will be uniformly spaced, as in the case of the points forming the interface from our numerical scheme. For discrete periodic functions $\boldsymbol{V}$ and $\boldsymbol{W}$, the discrete inner product is defined by

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle_N = \sum_{k=0}^{N-1} (\boldsymbol{V}_k \cdot \boldsymbol{W}_k) \cdot \frac{2\pi}{N}.$$

Computational Verification

Let $\mathbf{e_1^N}$, $\mathbf{e_2^N}$, $\mathbf{e_3^N}$, and $\mathbf{e_4^N}$ be

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{e_4} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

evaluated at $\theta_k = k \cdot \frac{2\pi}{N}$ for $k = 0, 1, \ldots, N-1$, respectively. We define the discrete perturbation operator by

$$\Pi_N \boldsymbol{V} = \boldsymbol{V} - \mathcal{P}_N \boldsymbol{V},$$

where

$$\begin{aligned} &\mathcal{P}_N \boldsymbol{V} \\ =&\frac{1}{2\pi}\Bigg(\quad \left\langle \boldsymbol{V}, \mathbf{e_1^N} \right\rangle_N \mathbf{e_1} + \left\langle \mathbf{V}, \mathbf{e_2^N} \right\rangle_{\mathbf{N}} \mathbf{e_2} + \left\langle \mathbf{V}, \mathbf{e_3^N} \right\rangle_{\mathbf{N}} \mathbf{e_3} + \left\langle \mathbf{V}, \mathbf{e_4^N} \right\rangle_{\mathbf{N}} \mathbf{e_4} \Bigg). \end{aligned}$$

We measure the perturbation using the discrete L^∞ norm

$$\left\| \boldsymbol{V} \right\|_\infty = \sup_k \left| \boldsymbol{V}_k \right|.$$

Computational Verification

We plot $\log \|\Pi_{100}(\boldsymbol{X}^n)\|_\infty$ against n for $dt = 0.1$, 0.05, and 0.01 up to $t = 50$ for the initial condition on the interface

$$\boldsymbol{X}^0 = \begin{pmatrix} \left(1 + \frac{e^{\cos(3\theta)}}{4}\right) \cos\theta \\ \left(1 + \frac{e^{\cos(4\theta)}}{4}\right) \sin\theta \end{pmatrix}.$$

Computational Verification

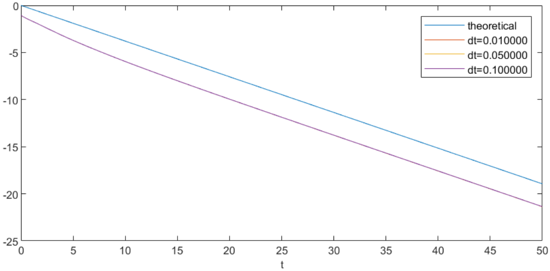
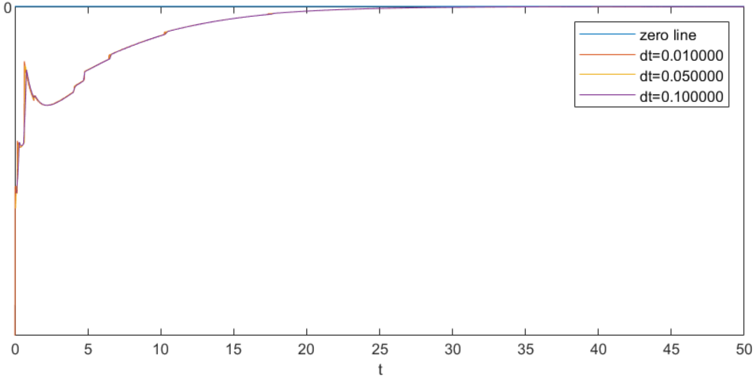


Figure: The plot of $\log \|\Pi_{100}(\boldsymbol{X}^n)\|_\infty$ against n for $dt = 0.1$, 0.05, and 0.01, up to $t = 50$.

The blue "theoretical" line has a slope of $-\frac{\sqrt{\pi}}{2\sqrt{A}}$, where A is the area enclosed by the initial interface. This plot suggests that the perturbation decays at an exponential rate of $-\frac{\sqrt{\pi}}{2\sqrt{A}}$.

Computational Verification



The Order of the Numerical Scheme

Let $\boldsymbol{X}_{dt}^{N,T}$ be the discretized interface at time T computed by our numerical scheme with time step size $dt > 0$. Suppose that for sufficiently large $n \in \mathbb{N}$,

$$E_n^{N,T} = \left\| \boldsymbol{X}_{2^{-(n-1)}}^{N,T} - \boldsymbol{X}_{2^{-n}}^{N,T} \right\|_\infty \leq C \cdot 2^{-nk}$$

for some constants $C > 0$ and $k > 0$.

The Order of the Numerical Scheme

If $\boldsymbol{X}^T$ is the unique analytical solution at time T evaluated at an equal arclength grid, and our numerical scheme converged to it, then

$$\begin{aligned}
\left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^T\right\|_\infty &\leq \left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^{N,T}_{2^{-n}}\right\|_\infty + \left\|\boldsymbol{X}^{N,T}_{2^{-n}} - \boldsymbol{X}^T\right\|_\infty \\
&\leq \left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^{N,T}_{2^{-n}}\right\|_\infty + \left\|\boldsymbol{X}^{N,T}_{2^{-n}} - \boldsymbol{X}^{N,T}_{2^{-(n+1)}}\right\|_\infty \\
&\quad + \cdots \\
&\leq C\left(2^{-nk} + 2^{-(n+1)k} + \cdots\right) \\
&= \frac{C}{1-2^{-k}} \cdot 2^{-nk}.
\end{aligned}$$

The Order of the Numerical Scheme

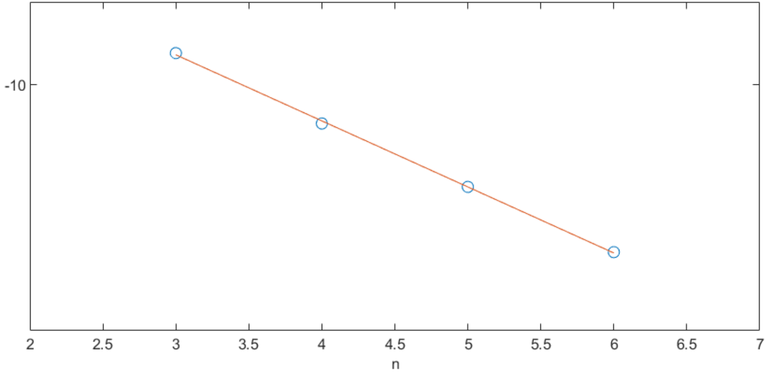


Figure: The plot of $\log_2 E_n^{100,40}$ against n for $n = 3, 4, 5, 6$.

Room for Exploration

- Global well-posedness in a scaling critical space
- The case of distinct viscosities
- Well-posedness of closely related non-Stokes fluids
- Convergence analysis of a numerical scheme utilizing the "equal arclength" parametrization

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