# Rectifiability and existence of principal values in rough Riemannian settings

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# Rectifiability: "geometric structure of $\mu$"

A measure $\mu$ is *m-rectifiable* if $\mu \ll \mathcal{H}^m$ and there exist Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right) = 0$$

# Analytic condition: existence of principal values

Example:

$$\lim_{\varepsilon \downarrow 0} \int_{(-c,c)\setminus B(a,\varepsilon)} \frac{1}{x-a}\,dx$$

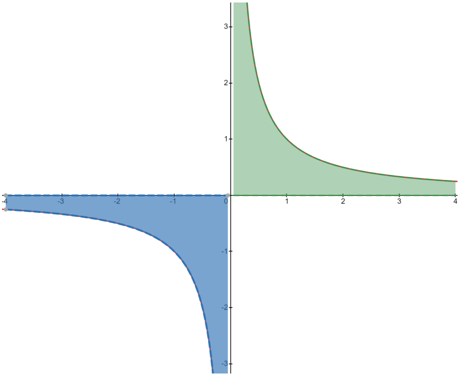


Figure: $\lim_{\varepsilon \to 0} \int_{(-c,c)\setminus B(0,\varepsilon)} \frac{1}{x}\,dx < \infty$

# Analytic condition: existence of principal values

Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ be a *Calderón-Zygmund kernel*. Example: $\frac{y-x}{|y-x|^{m+1}}$

# Analytic condition: existence of principal values

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For a finite Borel measure, $\mu$, on $\mathbb{R}^n$ we say the *principal value exists with respect to* $\mu$ at $x \in \mathbb{R}^n$ if

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(y-x)\,d\mu(y) \in \mathbb{R}^n.$$

Examples:

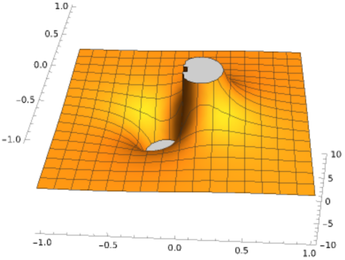
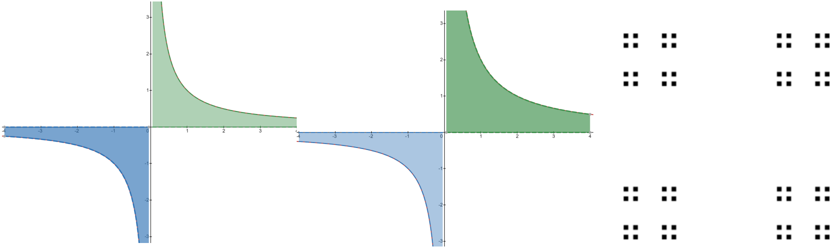



Figure: The $e_1$ component of $K(y-x) = \frac{y-x}{|y-x|^{m+1}}$

# Examples

When do principal values exist? Let $\mu$ be a Radon measure on $\mathbb{R}$.



(a) $\mu = \mathcal{L}^1$

(b) $\mu = 2\mathcal{H}^1 \llcorner \mathbb{R}_+ + \mathcal{H}^1 \llcorner \mathbb{R}_-$

(c) $\mu = \mathcal{H}^1 \llcorner \mathcal{C}$*

Figure: $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(a,\varepsilon)} \frac{y-a}{|y-a|^2} \, d\mu(y)$

*(Cufí, Ponce, Verdera, 2022)

# Density

$\mu$ is a Radon measure on $\mathbb{R}^n$. For $m < n$ the *m-dimensional lower density* is defined

$$\theta_*^m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m},$$

and the *m-dimensional upper density* is defined

$$\theta^{m,*}(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m}.$$

If both exist, the *m-dimensional density* is

$$\theta^m(\mu, x) = \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m}.$$

# David-Semmes Conjecture: Quantitative question

- (1991) David & Semmes: Suppose $\mu$ is an $m$-Ahlfors regular measure. $\mu$ is uniformly rectifiable $\iff$ all $m$-dimensional Calderón-Zygmund operators are bounded in $L^2(\mu)$.

$$\int_{\mathbb{R}^n} \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(y-x)f(y)d\mu(y) \right|^2 d\mu(x) \leq C \int_{\mathbb{R}^n} |f|^2 d\mu,$$

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If

$$\int_{\mathbb{R}^n} \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{y-x}{|y-x|^{m+1}} f(y) d\mu(y) \right|^2 d\mu(x) \leq C \int_{\mathbb{R}^n} |f|^2 d\mu,$$

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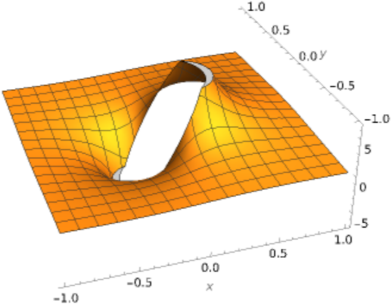
- (1996) Mattila, Melnikov, Verdera (m=1,n=2)
- (2014) Nazarov, Tolsa, Volberg ($m = n-1, n \geq 2$)

# Qualitative analog: rectifiability and principal values

- (1995) Mattila, Mattila & Preiss: Suppose $\mu$ is a Radon measure on $\mathbb{R}^n$. If for $\mu$-a.e. $x \in \mathbb{R}^n$ $0 < \theta_*^m(\mu, x) < \infty$ and

$$\lim_{\varepsilon \downarrow 0} \int_{|x-y| \geq \varepsilon} \frac{y-x}{|y-x|^{m+1}} d\mu(y) \in \mathbb{R}^n,$$

then $\mu$ is $m$-rectifiable. Converse known.



# Rectifiability and principal values

## Theorem (C.-Goering-Toro-Wilson)

Suppose $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ is a measurable function and $\mu$ is a finite Borel measure. Then $\mu$ is $m$-rectifiable if and only if $0 < \theta_*^m(\mu, x) < \infty$ and

$$\lim_{\varepsilon \downarrow 0} \int_{|\Lambda(x)^{-1}(y-x)| \geq \varepsilon} \frac{\Lambda(x)^{-1}(y-x)}{|\Lambda(x)^{-1}(y-x)|^{m+1}} d\mu(y) \in \mathbb{R}^n,$$

for $\mu$-a.e. $x \in \mathbb{R}^n$.

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- $\frac{y-x}{|y-x|^n} \approx_n \nabla_1 \Gamma_{I_n}(x, y)$, since $\Delta = -div(\nabla \cdot)$
- Let $A \in \mathbb{R}^{n \times n}$ be *nice* and let $L_A = -\mathrm{div}(A\nabla \cdot)$. Then,

$$\nabla_1 \Gamma_A(x, y) = \frac{\Lambda^{-2}(y-x)}{|\Lambda^{-1}(y-x)|^n},$$

where $\Lambda$ unique pd matrix s.t. $\Lambda^2 = A$.

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- Let $A \in \mathbb{R}^{n \times n}$ be *nice* and let $L_A = -\mathrm{div}(A\nabla \cdot)$, and $\Lambda$ linear transformation:

$$\lim_{\varepsilon \downarrow 0} \int_{|\Lambda^{-1}(y-x)| \geq \varepsilon} \frac{\Lambda^{-2}(y-x)}{|\Lambda^{-1}(y-x)|^n} d\mu(y)$$
$$= \Lambda^{-1} \lim_{\varepsilon \downarrow 0} \int_{|\Lambda^{-1}(y-x)| \geq \varepsilon} \frac{\Lambda^{-1}(y-x)}{|\Lambda^{-1}(y-x)|^n} d\mu(y)$$

# Rectifiability and principal values

**Theorem (Molero, Mourgoglou, Puliatti, Tolsa, 2023)**

If $A \in \mathbb{R}^{n\times n}$ is *nice* and $\mu$ is an $(n-1)$-rectifiable Radon measure on $\mathbb{R}^n$, then

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} \nabla_1 \Gamma_A(x,y) d\mu(y) \in \mathbb{R}^n \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

where $\Gamma_A(x,y)$ is the fundamental solution for $L_A = -div(A\nabla \cdot)$ with pole at $x$.

# Rectifiability and principal values

## Corollary (C.-Goering-Toro-Wilson)

If $A \in \mathbb{R}^{n \times n}$ is *nice*, $\mu$ satisfies some mild density assumptions, and

$$\lim_{\varepsilon \downarrow 0} \int_{|\Lambda(x)^{-1}(y-x)|>\varepsilon} \nabla_1 \Gamma_A(x,y) d\mu(y) \in \mathbb{R}^n \quad \text{for } \mu\text{-a.e. } x,$$

where $\Lambda^2 = A$, then $\mu$ is *m*-rectifiable.

# Rectifiability and principal values

**Theorem (C.-Goering-Toro-Wilson)**

Suppose $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ is a measurable function and $\mu$ is a finite Borel measure. Then $\mu$ is $m$-rectifiable if and only if $0 < \theta^m_*(\mu, x) < \infty$ and

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for $\mu$-a.e. $x \in \mathbb{R}^n$.

# Geometry implies analysis

Let $\mu$ be a finite $m$-rectifiable Borel measure on $\mathbb{R}^n$. Then

$$\lim_{\varepsilon\downarrow 0}\int_{|\Lambda(x)^{-1}(y-x)|\geq\varepsilon}\frac{\Lambda(x)^{-1}(y-x)}{|\Lambda(x)^{-1}(y-x)|^{m+1}}d\mu(y)\in\mathbb{R}^n,\quad \text{for } \mu\text{-a.e. } x\in\mathbb{R}^n.$$

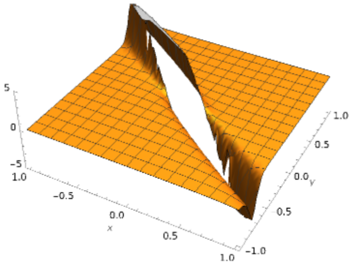


Figure: first component of $\frac{(\Lambda(0)^{-1}y)}{|\Lambda(0)^{-1}y|^3}$ on $|\Lambda(x)^{-1}(y-x)|\geq\varepsilon$

# Geometry implies analysis

Let $\mu$ be a finite $m$-rectifiable Borel measure on $\mathbb{R}^n$. Suppose for each $x$, $K_x(z)$ is a smooth CZ kernel. Then for any norm

$$\lim_{\varepsilon \downarrow 0} \int_{\|y-x\|>\varepsilon} K_x(y-x)\,d\mu(y) \in \mathbb{R}^n$$

for $\mu$-a.e. $x \in \mathbb{R}^n$.

# $\mathring{\alpha}$-numbers

The *centered $\alpha$-numbers* are defined by

$$\mathring{\alpha}_{\mu}^{m}(x,r) := \inf_{\sigma \in \mathcal{F}_{m,n}(x)} \sup_{f \in \mathrm{Lip}_1(B(x,r))} \frac{1}{r^{m+1}} \left| \int_{B(x,r)} f(y)\, d(\mu - \sigma) \right|,$$

where $\mathcal{F}_{m,n}(x) = \{\sigma = c\mathcal{H}^m \llcorner (V + x) : V \in G(m,n)\}$.

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- how close the support of the measure is to being a plane (locally)
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For $\mu$ such that $0 < \theta_*^m(\mu, x) \leq \theta^{m,*}(\mu, x) < \infty$,

$$\int_0^1 \mathring{\alpha}_\mu^m(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \iff \mu \text{ is m-rectifiable.}$$

(Azzam, Tolsa, Toro, Kolasinski, Dąbrowski)

# Motivating the proof

1. Existence of round principal values
   $\mu$ $m$-rectifiable $\implies$

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)d\mu(y) < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

   (Similar ideas from Puliatti & Mattila)

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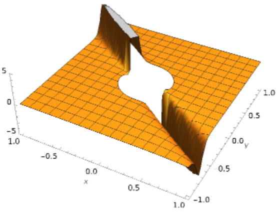
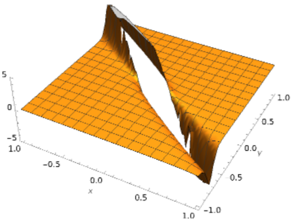
2. Round principal values $\iff$ Normed principal values

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)d\mu(y) \in \mathbb{R}^n \iff \lim_{\varepsilon \downarrow 0} \int_{\|y-x\|>\varepsilon} K_x(y-x)d\mu(y) \in \mathbb{R}^n.$$



# Motivating the proof

2. Round principal values $\iff$ Normed principal values

$$\left| \int_{\mathbb{R}^n} \left( \chi_{|y-x|>\varepsilon} - \chi_{\|y-x\|>\varepsilon} \right) K_x(y-x) d\mu(y) \right| \xrightarrow[\varepsilon \to 0]{} 0.$$

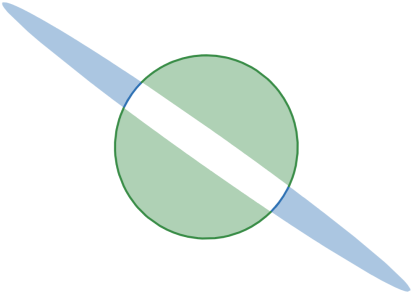


Figure: $\operatorname{spt} \chi_{|y-x|>\varepsilon} - \chi_{\|y-x\|>\varepsilon}$

# Proof sketch Part I: Round world

1. Existence of round principal values
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2. (Orponen-Villa, 2023): Round rough to smooth cutoffs

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)\,d\mu(y) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \phi_\varepsilon(|y-x|) K_x(y-x)\,d\mu(y),$$

   where $\phi$ is any reasonable smooth approximation of $\chi_{(1,\infty)}$.

# Proof sketch Part II: Normed world

3. Transition from round to normed world
   For any norm $\|\cdot\|$,

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\phi_\varepsilon(|y-x|)K_x(y-x)d\mu(y)=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\phi_\varepsilon(\|y-x\|)K_x(y-x)d\mu(y).$$

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4. Normed smooth to rough cutoffs

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\phi_\varepsilon(\|y-x\|)K_x(y-x)d\mu(y)=\lim_{\varepsilon\downarrow 0}\int_{\|y-x\|>\varepsilon}K_x(y-x)d\mu(y)$$

□

# 3. Transition from round to normed world

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We show:
$$\left|\int_{\mathbb{R}^n}\overbrace{\left(\phi_\varepsilon(|y-x|)-\phi_\varepsilon(\|y-x\|)\right)}^{\psi_\varepsilon(y-x)}K_x(y-x)d\mu(y)\right|\xrightarrow{\varepsilon\to 0}0$$

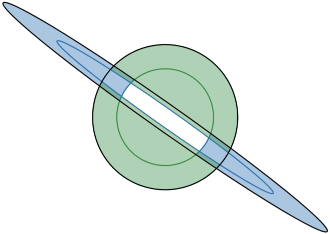

Figure: $\operatorname{spt}\psi_\varepsilon(y-x)$

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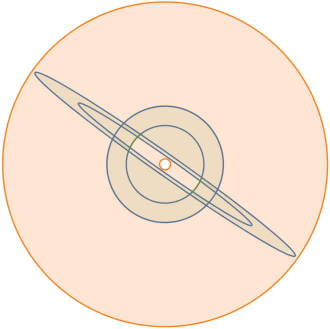

Figure: $\operatorname{spt} \psi_\varepsilon(y-x) \subseteq B(x, 2^M \varepsilon) \setminus B(x, 2^{-M}\varepsilon)$

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for *any* flat measure, $\sigma$, through $x$.

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  for *any* flat measure, $\sigma$, through $x$.

Thus,

$$(2^M\varepsilon)^{-(m+1)} \left| \int_{\mathbb{R}^n} \frac{1}{(2^M\varepsilon)^{-(m+1)}} \psi_\varepsilon(y-x)K_x(y-x)d\mu(y) \right| \lesssim \mathring{\alpha}_\mu^m(x, 2^M\varepsilon)$$

$$\xrightarrow[\varepsilon\to 0]{} 0$$

# 1. Existence of round principal values

1. $\mu$ $m$-rectifiable $\implies$

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)\, d\mu(y) \in \mathbb{R}^n \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

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- Known for $\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K(y-x)d\mu(y)$;

# 1. Existence of round principal values

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$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)\,d\mu(y) \in \mathbb{R}^n \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

(Similar ideas from Puliatti & Mattila)

- Known for $\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K(y-x)\,d\mu(y)$;
  Choose a CZ kernel for each $x \in \mathbb{R}^n$,

$$\lim_{\varepsilon \downarrow 0} \int_{|y-z|>\varepsilon} K_x(y-z)\,d\mu(y)$$

exists for $\mu$-a.e. $z$, but not necessarily at $z = x$.

# 1. Existence of round principal values

1. If $\mu$ is $m$-rectifiable, then $\lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} K_x(y-x)d\mu(y) \in \mathbb{R}^n$ for $\mu$-a.e. $x$.
   - Write $K_x(y-x)$ as a linear combination of "nice" kernels

$$K_x(y-x) = \sum_j a_j(x) K_j(y-x) d\mu(y),$$

   where for each $j$

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_j(y-x)d\mu(y) \in \mathbb{R}^n \quad \text{for } \mu\text{-a.e. } x.$$

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   - Then,

     $$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)d\mu(y) = \sum_j a_j(x) \lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_j(y-x)d\mu(y)$$

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$$\sum_j a_j(x) \lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_j(y-x)\,d\mu(y)$$
$$= \sum_j a_j(x) \lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_j(y-x) f(y)\, d(\mathcal{H}^m \llcorner \Gamma_i)(y)$$
$$+ \sum_j a_j(x) \lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_j(y-x)\,d\sigma_i(y)$$

(Pulliati 2022 & see Mattila "Geometry of Sets and Measures")

4. Normed smooth to rough cutoffs

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \phi_\varepsilon(\|y-x\|) K_x(y-x) d\mu(y) = \lim_{\varepsilon \downarrow 0} \int_{\|y-x\|>\varepsilon} K_x(y-x) d\mu(y)$$

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Show:

$$\left| \int_{\mathbb{R}^n} (\phi_\varepsilon(\|y-x\|) - \chi_{\|x-y\|>\varepsilon}) K_x(y-x)\, d\mu(y) \right| \xrightarrow[\varepsilon \to 0]{} 0$$

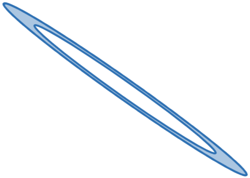

Figure: $\operatorname{spt}(\phi_\varepsilon(\|y-x\|) - \chi_{\|x-y\|>\varepsilon}) \subseteq B_{\|\cdot\|}(x, \varepsilon(1+\delta)) \setminus B(x,\varepsilon)$

# 4. Normed smooth to rough cutoffs

$$\left| \int_{\mathbb{R}^n} (\phi_\varepsilon(\|y-x\|) - \chi_{\|x-y\|>\varepsilon}) K_x(y-x) d\mu(y) \right|$$

$$\leq \varepsilon^{-m} \mu \left( B_{\|\cdot\|}(x, \varepsilon(1+\delta)) \setminus B_{\|\cdot\|}(x, \varepsilon) \right)$$

$$\lesssim \frac{\mathring{\alpha}_\mu(B(x, 2C\varepsilon))}{\delta} + \theta_\mu^{m,*}(x, C\varepsilon)\delta$$

$$\xrightarrow[\varepsilon \to 0]{} 0,$$

for $\delta$ correctly chosen.

(Jaye & Merchán, 2020)

# Summary

1. Existence of round principal values
   $\mu$ $m$-rectifiable $\implies$
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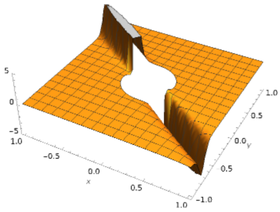
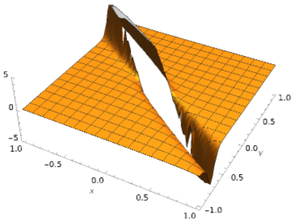
2. Round principal values $\iff$ Normed principal values

$$\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} K_x(y-x)d\mu(y) \in \mathbb{R}^n \iff \lim_{\varepsilon \downarrow 0} \int_{\|y-x\|>\varepsilon} K_x(y-x)d\mu(y) \in \mathbb{R}^n.$$



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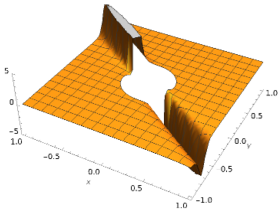
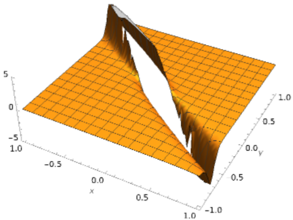
# Thank you!




(a) $|y - x| > \varepsilon$




(b) $\|y - x\| > \varepsilon$