A model for phase transitions with competing terms

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Abstract. In this paper we study, via Γ-convergence techniques, the asymptotic behaviour of a family of coupled singular perturbations of a non-convex functional of the type

$$\int_\Omega f(u(x), \nabla u(x), \rho(x))\, dx$$

as a variational model to address two-phase transition problems under the volume constraints $\int_\Omega u(x)\, dx = V_f$, $\displaystyle\int_\Omega \rho(x)\, dx = V_s$, and where the additional unknown ρ interplays with ∇u in the formation of interfaces.

Keywords: two-phase transition problems, functions of bounded variation, Γ-convergence, surfactants

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Contents

1. Introduction

Our aim in this paper is to study the asymptotic behaviour of a family of coupled singular perturbations of a non-convex functional of the type

$$\int_\Omega f(u(x), \nabla u(x), \rho(x))\, dx,$$

where $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ represents the vector-valued fluid density of d fluids present in a container Ω and is subject to the volume constraint $\displaystyle\int_\Omega u(x)\, dx = V_f$. The energy density $f(\cdot, 0, 0)$ is assumed to vanish if and only if $u \in \{\alpha, \beta\}$, for some fixed $\{\alpha, \beta\} \subset \mathbb{R}^d_+$. Our model includes an additional unknown ρ, that is taken to be a non-negative L^1 function whose volume is also fixed, and which interplays with the gradient of u in the formation of interfaces. One possible application of this model is within the theory of phase transitions in the presence of surfactants, in which case ρ represents the density of the surfactant.

A *surfactant* (a contraction of the term *surface acting agent*) is usually an organic compound that when present in a system has the property of altering its interfacial energy, in general reducing it (see [18] for a comprehensive study of the properties and applications of surfactants).

In [14], Fonseca, Morini and Slastikov studied a two-phase field model to explain the role of surfactants in the formation of bubbles in foams. Their energy, which is based on a modification of the van der Waals-Cahn-Hilliard model for fluid-fluid phase transitions (c.f [8], [19]) suggested by Perkins, Sekerka, Warren and Langer, includes an additional term that describes the influence of the surfactant in preventing the coalescence of bubbles and in encouraging the formation of interfaces. Precisely, in [14], the authors considered a penalized energy of the form

$$\int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 + \varepsilon(\rho - |\nabla u|)^2\, dx$$

where W is a double-well potential and ε is the scaling that is used to drive systems towards phase separation. The limiting energy functional, obtained by Γ-convergence, reveals that, on the one hand, the surfactant is essentially located on the interfaces separating the foam bubbles and that, on the other hand, interfaces are created where the surfactant is present.

The Γ-convergence of a more general coupled class of energies of the form

$$\frac{1}{\varepsilon} \int_\Omega f\big(u(x), \varepsilon \nabla u(x), \varepsilon \rho(x)\big)\, dx \tag{1.1}$$

was addressed by Acerbi and Bouchitté [1] under some convexity hypothesis on f and still in the case of scalar fluid density.

The objective of this work is to generalise the results of [14] to the coupled case, and of [1] to the case of vector-valued fluid densities and under non-convexity hypotheses on f.

We point out that our analysis also holds in the case where a mixture of surfactants is considered, i.e. when $\rho : \Omega \to \mathbb{R}^m$, $m \geq 1$, however, for simplicity of notation, we consider ρ taking nonnegative real values.

Precisely, we consider a family of energy functionals as in (1.1) where $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ represents the vector-valued fluid density of d fluids, $d \in \mathbb{N}$, $\rho : \Omega \to [0, +\infty)$ is the density of a surfactant and Ω is an open bounded subset of $\mathbb{R}^N$ which represents the container where the d fluids and the surfactant are mixed. We assume further that each scalar component of u, u_i, $i \in \{1, \dots, d\}$, which identifies the density of the ith-ingredient of the mixture, is nonnegative, that is, $u \in \mathbb{R}^d_+$ (we refer to Section 2 for all the notations used throughout this work). In addition, the bulk energy density $f \colon \mathbb{R}^d_+ \times \mathbb{R}^{d\times N} \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to satisfy the following hypotheses:

(H1) f is nonnegative and continuous on $\mathbb{R}^d_+ \times \mathbb{R}^{d\times N} \times \mathbb{R}_+$;
(H2) $f(u,0,0)=0$ if and only if $u \in \{\alpha,\beta\}$, for some fixed $\{\alpha,\beta\} \subset \mathbb{R}^d_+$.

Also, there exists $p \in [1,+\infty)$ such that,

(H3)
$$\frac{1}{C}\Big(g(u)+|\xi|^p\Big) \le f(u,\xi,\rho) \le C\Big(h(u,\rho)+|\xi|^p\Big)$$
for all $(u,\xi,\rho) \in \mathbb{R}^d_+ \times \mathbb{R}^{d\times N} \times [0,+\infty)$ and for some $C \ge 1$, where $g \in L^\infty_{\rm loc}(\mathbb{R}^d_+;[0,+\infty))$, $h \in L^\infty_{\rm loc}(\mathbb{R}^d_+ \times [0,+\infty);[0,+\infty))$ and g is such that $g(u)=0 \Leftrightarrow u \in \{\alpha,\beta\}$ and $\inf\limits_{|u|\ge L} g(u) > 0$;

(H4) for every $M>0$ there exists a constant $C_M>0$ such that for every $(u,\xi) \in \mathbb{R}^d_+ \times \mathbb{R}^{d\times N}$ and every $\rho_1,\rho_2 \in [0,+\infty)$,
$$\big|f(u,\xi,\rho_1)-f(u,\xi,\rho_2)\big| \le C_M\big|\rho_1-\rho_2\big|\big(1+|\xi|^p\big),$$
whenever $\rho_1,\rho_2 \le M$ and $|u| \le M$;

(H5) $\lim\limits_{\rho\to 0^+} \dfrac{1}{\rho} f(\alpha,0,\rho)=0$, $\lim\limits_{\rho\to 0^+} \dfrac{1}{\rho} f(\beta,0,\rho)=0$;

(H6) for every $M>0$ there exists a constant $C_M>0$ such that if $|u-\alpha|<\delta$ then
$$|f(u,\xi,\rho)-f(\alpha,\xi,\rho)| \le C_M|u-\alpha|(1+|\xi|^p),$$
respectively, if $|u-\beta|<\delta$ then
$$|f(u,\xi,\rho)-f(\beta,\xi,\rho)| \le C_M|u-\beta|(1+|\xi|^p),$$
whenever $\rho \le M$.

The counterpart of (H3) for $p=+\infty$ leads to sequences of gradients and of surfactants that are bounded in L^∞ which is not physically interesting. A prototype for f, satisfying the above hypothesis in the case $p=2$, is
$$f(u,\xi,\rho) := |u-\alpha|^2|u-\beta|^2+|\xi|^2+(\rho-|\xi|)^2.$$
However, we point out that our hypothesis are weaker than those considered in [14] and include functions which do not satisfy polynomial type growth conditions, for example
$$f(u,\xi,\rho) := |u-\alpha|^p|u-\beta|^p+|\xi|^p+\rho^2 e^{|u|\rho}.$$

In our setting, the volume of the surfactant is given a priori and fixed, and the total amount of bulk material is preserved, i.e., $(u,\rho) \in \mathcal{V}$ where
$$\mathcal{V} = \left\{(u,\rho) \in W^{1,p}(\Omega;\mathbb{R}^d_+) \times L^1\big(\Omega;[0,+\infty)\big) : \int_\Omega \rho(x)\,dx = V_s,\ \int_\Omega u(x)\,dx = V_f\right\} \tag{1.2}$$
for some $V_s>0$ and some $V_f=(V_f^1,\dots,V_f^d) \in \mathbb{R}^d_+$ satisfying
$$|\Omega|\,\min(\alpha^i,\beta^i) \le V_f^i \le |\Omega|\,\max(\alpha^i,\beta^i) \qquad \text{for every } i=1,\dots,d, \tag{1.3}$$
where α^i and β^i are the i-th components of α and β, respectively.

Our aim in this article is to characterise the asymptotic behaviour, as $\varepsilon \to 0^+$, of the family of functionals
$$E_\varepsilon(u,\rho) := \frac{1}{\varepsilon}\int_\Omega f(u(x),\varepsilon\nabla u(x),\varepsilon\rho(x))\,dx, \qquad \text{for } (u,\rho) \in W^{1,p}(\Omega;\mathbb{R}^d_+) \times L^1\big(\Omega;[0,+\infty)\big), \tag{1.4}$$
subject to the constraints defined in (1.2). We will use a Γ-convergence argument. For this purpose we consider the space
$$X(\Omega) := L^1(\Omega;\mathbb{R}^d_+) \times \mathcal{M}^+(\Omega)$$

endowed with the product topology $\tau_1 \times \tau_2$, where τ_1 denotes the strong convergence in $L^1(\Omega;\mathbb{R}^d_+)$ and τ_2 stands for the weak*-convergence in the space $\mathcal{M}^+(\Omega)$ of nonnegative finite Radon measures supported on Ω, and we define

$$\mathcal{W} = \left\{ (u,\mu) \in X(\Omega) : \; \mu(\Omega) = V_s, \; \int_\Omega u(x)\,dx = V_f \right\}. \tag{1.5}$$

Note that if $(u,\rho) \in W^{1,p}(\Omega;\mathbb{R}^d_+) \times L^1\big(\Omega;[0,+\infty)\big)$ are such that $(u,\rho) \in \mathcal{V}$ and $\mu = \rho\mathcal{L}^N \llcorner \Omega$, then $(u,\mu) \in \mathcal{W}$.

We now extend the functional E_ε to the whole space $X(\Omega)$ by setting, for every $(u,\mu) \in X(\Omega)$,

$$F_\varepsilon(u,\mu) := \begin{cases} E_\varepsilon(u,\rho) & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d_+), \; \mu = \rho\mathcal{L}^N \llcorner \Omega\,, \\ & \rho \in L^1\big(\Omega;[0,+\infty)\big), \; (u,\rho) \in \mathcal{V} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

Given $\nu \in \mathbb{S}^{N-1}$ and $\theta \in [0,+\infty)$ the set

$$\begin{aligned} \mathcal{A}(\nu,\theta) := \Bigg\{ & (u,\rho) \in W^{1,p}_{\mathrm{loc}}(S_\nu;\mathbb{R}^d_+) \times L^1_{\mathrm{loc}}\big(\mathbb{R}^N;[0,+\infty)\big) : \; u(y) = \alpha \text{ if } y\cdot\nu = -\frac{1}{2}, \\ & u(y) = \beta \text{ if } y\cdot\nu = \frac{1}{2}, \int_{Q_\nu} \rho(y)\,dy \le \theta, \\ & u \text{ and } \rho \text{ are periodic with period one in the directions of } \nu_1,\dots,\nu_{N-1} \Bigg\}, \end{aligned} \tag{1.7}$$

represents the class of admissible pairs of density functions and surfactants for (ν,θ), where the boundary values of u are understood in the sense of traces, $\{\nu_1,\dots,\nu_{N-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^N$ and S_ν is the strip

$$S_\nu := \left\{ x \in \mathbb{R}^N : \; |x\cdot\nu| < \frac{1}{2} \right\}.$$

Finally, we introduce the anisotropic surface energy density $\sigma : \mathbb{S}^{N-1} \times [0,+\infty) \to [0,+\infty)$ defined by

$$\sigma(\nu,\theta) := \inf\left\{ \int_{Q_\nu} \frac{1}{t} f\big(u(y), t\nabla u(y), t\rho(y)\big)\,dy : \; t>0, \; (u,\rho) \in \mathcal{A}(\nu,\theta) \right\}, \tag{1.8}$$

and the limit energy functional $F : L^1(\Omega;\mathbb{R}^d_+) \times \mathcal{M}^+(\Omega) \to \mathbb{R}$ that is given by

$$F(u,\mu) := \begin{cases} \displaystyle\int_{S_u} \sigma\left(\nu_u(x), \frac{d\mu}{d\big(\mathcal{H}^{N-1}\llcorner S_u\big)}(x)\right) d\mathcal{H}^{N-1}(x) & \\ \qquad\qquad \text{if } (u,\mu) \in [BV\big(\Omega;\{\alpha,\beta\}\big) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W} & \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{1.9}$$

Our first result establishes that the topology in $X(\Omega)$ is compact for sequences with bounded energy. In the case $p=1$, this is a direct consequence of Poincaré's inequality and the fact that $Cf(u,\xi,\rho) \ge |\xi|$. For other values of p, to prove this result we need to make use of the coercivity condition in (H3), $Cf(u,\xi,\rho) \ge g(u)$, where $\inf\limits_{|u|\ge L} g(u) > 0$ (cf. [17]). Precisely, our theorem reads as follows.

Theorem 1.1. *Assume that hypotheses (H1)-(H3) hold. Let* $\varepsilon_n \to 0$ *and let* $(u_n, \rho_n) \in [W^{1,p}(\Omega;\mathbb{R}^d_+) \times L^1(\Omega;[0,+\infty)] \cap \mathcal{V}$ *be such that*

$$\sup_n F_{\varepsilon_n}(u_n, \rho_n) < +\infty.$$

Then there exist a subsequence $\{(u_{n_k}, \rho_{n_k})\}_k \subset \{(u_n, \rho_n)\}_n$ *and* $(u, \mu) \in [BV(\Omega;\{\alpha,\beta\}) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ *such that*

$$(u_{n_k}, \rho_{n_k}) \to (u, \mu) \quad \text{in } X(\Omega).$$

The main result of this paper is the following.

Theorem 1.2. *Under hypotheses (H1)–(H6), the family of functionals* F_ε *in* (1.6) $\Gamma[X(\Omega)]$*-converge to the functional* F *in* (1.9)*.*

The proof of Theorem 1.2 is based on the blow-up method, introduced by Fonseca & Müller (see e.g [15] and [16]), which allows us to consider the case where Ω is a small cube and the target function has planar interface. We also rely on a slicing argument (cf. Lemma 3.1), enabling us to modify a sequence near the boundary of the cube without increasing the total energy, as well as on periodicity arguments based on the Riemann-Lebesgue Lemma.

The following property is an immediate consequence of Theorem 1.2 and Theorem 1.1 (see also Theorem 2.11).

Corollary 1.3. *Assume that hypotheses (H1)–(H6) hold and let* $\{(u_\varepsilon, \mu_\varepsilon)\}_\varepsilon$ *be a sequence such that* $(u_\varepsilon, \mu_\varepsilon)$ *is a minimum point of* F_ε*. Then the sequence* $\{(u_\varepsilon, \mu_\varepsilon)\}$ *is relatively compact with respect to the* $(\tau_1 \times \tau_2)$*-convergence of* $X(\Omega)$*, and any cluster point* $(\bar{u}, \bar{\mu})$ *of* $\{(u_\varepsilon, \mu_\varepsilon)\}_\varepsilon$ *belongs to* $BV(\Omega;\{\alpha,\beta\}) \times \mathcal{M}^+(\Omega)$ *and is a solution of the minimisation problem*

$$\min\{F(u,\mu) : \ (u,\mu) \in X(\Omega) \cap \mathcal{W}\}.$$

This article is organized as follows: in Section 2 we set up the notation and state some preliminary results on measure theory, functions of bounded variation and Γ-convergence which will be used throughout the paper. In Section 3 we prove some auxiliary results which will be needed in the sequel. Section 4 is devoted to the proof of a compactness result for sequences with bounded energy, whereas the statements and proofs of our main results can be found in Section 5.

2. Preliminaries

In this section we set up the notation used throughout this work and recall some well-known facts about measure theory, functions of bounded variation and Γ-convergence. Standard references on these topics include [3, 10, 11, 12, 13], on which most of the presentation is based.

2.1. **Notation.**

Throughout the text, unless otherwise specified, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, will denote an open bounded set with Lipschitz boundary and we will use the following notations.

- $|\Omega|$ denotes the Lebesgue measure of Ω.
- $\mathbb{R}^d_+ := [0,+\infty)^d$.
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ stand, respectively, for the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.
- $|x|$ denotes the Euclidean norm of a vector x.
- Given $x \in \mathbb{R}^N$ we write $x = (x', x_N)$, where x' stands for its first $N-1$ coordinates and x_N for the N-th one.
- Q is the open unit cube centered at the origin with faces normal to the coordinates axes.
- $B(x,r)$ denotes the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.
- $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : \ |x| = 1\}$.

- Given $\nu \in \mathbb{S}^{N-1}$, the set S_ν represents the strip
$$S_\nu := \left\{ x \in \mathbb{R}^N : |x \cdot \nu| < \frac{1}{2} \right\}$$
and Q_ν denotes an open unit cube centered at the origin with two of its faces normal to ν, i.e., if $\{\nu_1, \dots, \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$ then
$$Q_\nu := \left\{ x \in \mathbb{R}^N : |x \cdot \nu| < \frac{1}{2}, \, |x \cdot \nu_i| < \frac{1}{2}, \, i = 1, \dots, N-1 \right\}. \tag{2.1}$$
- $Q_\nu(x_0, r) := x_0 + rQ_\nu$ for $x_0 \in \mathbb{R}^N$, $r > 0$ and $\nu \in \mathbb{S}^{N-1}$. If $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$ then $Q_{e_N}(x_0, r) = x_0 + rQ =: Q(x_0, r)$.
- $SO(N)$ denotes the set of rotations in $\mathbb{R}^N$.
- $a \otimes b$ is the $N \times d$-matrix given by $(a \otimes b)_{ij} = a_i b_j$, $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^d$.
- C denotes a generic positive constant whose value might change from line to line.
- $\displaystyle\lim_{n,m \to +\infty} := \lim_{n \to +\infty} \lim_{m \to +\infty}$.

2.2. **Periodic functions and the Riemann-Lebesgue Lemma.** We state here the Riemann-Lebesgue Lemma that, due to the periodicity in the first $N-1$ variables of the admissible functions for the limit energy functional, will be useful in the proof of Theorem 1.2 (more precisely Lemma 5.4). We recall that a function v defined in $\mathbb{R}^N$ is periodic with period one in the direction of a vector ν if $v(y) = v(y + k\nu)$, for all $y \in \mathbb{R}^N$, and is said to be Q-periodic if it is periodic with period one in all the directions of the canonical basis of $\mathbb{R}^N$.

Lemma 2.1 (Riemann-Lebesgue Lemma; cf. Lemma 2.85 in [13]). *Let $f \in L^p_{loc}(\mathbb{R}^N)$, $1 \leq p \leq +\infty$, be a Q-periodic function. Define $f_n(x) = f\left(\frac{x}{\varepsilon_n}\right)$, where $\{\varepsilon_n\}_n$ is a given fixed sequence of positive real numbers converging to zero. Then the sequence f_n converges weakly in $L^p_{loc}(\mathbb{R}^N)$, $1 \leq p < +\infty$ (weakly-* in L^∞) to the function $\displaystyle ⨍_Q f(x)\,dx$.*

The following corollary can be found in [9].

Corollary 2.2. *Let $f_n \in L^p_{loc}(\mathbb{R}^N)$, $1 < p < +\infty$, be a sequence of Q-periodic functions such that $\|f_n\|_{L^p(Q)} \leq C$ and $\displaystyle\lim_{n \to +\infty} ⨍_Q f_n(x)\,dx = \overline{f}$. Define $g_n(x) = f_n\left(\frac{x}{\varepsilon_n}\right)$, where $\{\varepsilon_n\}_n$ is a given fixed sequence of positive real numbers converging to zero. Then the sequence g_n converges weakly in $L^p_{loc}(\mathbb{R}^N)$ to the function $\overline{f}$.*

2.3. **Remarks on measure theory.** Let X be a locally compact separable metric space and let $\mathcal{B}(X)$ denote its Borel σ-algebra. We represent by $\mathcal{M}(X;\mathbb{R}^N)$ the space of finite $\mathbb{R}^N$-valued Radon measures, that is, the set of all $\mu : \mathcal{B}(X) \to \mathbb{R}^N$, $\mu = (\mu_1, ..., \mu_N)$, such that
$$< \mu, \varphi >:= \int_X \varphi \, d\mu \equiv \sum_{i=1}^N \int_X \varphi_i \, d\mu_i$$
for all $\varphi = (\varphi_1, ..., \varphi_N) \in C_0(X;\mathbb{R}^N)$, and we endow this space with the weak*-topology. In particular, a sequence $\{\mu_n\} \subset \mathcal{M}(X;\mathbb{R}^N)$ is said to weak*-converge to $\mu \in \mathcal{M}(X;\mathbb{R}^N)$ (indicated by $\mu_n \overset{\star}{\rightharpoonup} \mu$) if for all $\varphi \in C_0(X;\mathbb{R}^N)$
$$\lim_{n \to +\infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.$$
If $N = 1$ we write by simplicity $\mathcal{M}(X)$ and we denote by $\mathcal{M}^+(X)$ its subset of positive measures.

The following is a well known result on measure theory which is important to understand the structure of the class of BV-functions.

Theorem 2.3 (Lebesgue-Radon-Nikodým Theorem). *Let $\mu \in \mathcal{M}^+(X)$ and $\nu \in \mathcal{M}(X;\mathbb{R}^N)$. Then*

(i) *there exist two $\mathbb{R}^N$-valued measures ν_a and ν_s such that*

$$\nu = \nu_a + \nu_s \tag{2.2}$$

with $\nu_a << \mu$ and $\nu_s \perp \mu$. Moreover, the decomposition (2.2) is unique, that is, if $\nu = \bar\nu_a + \bar\nu_s$ for some measures $\bar\nu_a, \bar\nu_s$, with $\bar\nu_a << \mu$ and $\bar\nu_s \perp \mu$, then $\nu_a = \bar\nu_a$ and $\nu_s = \bar\nu_s$;

(ii) *there is a μ-measurable function $u \in L^1(\Omega;\mathbb{R}^N)$ such that*

$$\nu_a(E) = \int_E u\, d\mu$$

for every $E \in \mathcal{B}(\Omega)$. The function u is unique up to a set of μ measure zero.

The decomposition $\nu = \nu_a + \nu_s$ is called the *Lebesgue decomposition* of ν with respect to μ (see [13, Theorem 1.115]), ν_a and ν_s are called, respectively, the absolutely continuous part and the singular part of ν with respect to μ and the function u is called the *Radon-Nikodým derivative* of ν with respect to μ, denoted by $u = d\nu/d\mu$ (see [13, Theorem 1.101]).

In the sequel, we will often identify a function $f \in L^1\big(\Omega;[0,+\infty)\big)$ with the measure $f\mathcal{L}^N \llcorner \Omega$. Given $\mu \in \mathcal{M}(X;\mathbb{R}^N)$ its *total variation* will be indicated by $|\mu|$ and its *support* by supp μ. In addition, given $E \in \mathcal{B}(X)$ we will denote by $\mu \llcorner E$ the measure given by $\mu \llcorner E(A) := \mu(E \cap A)$ for every $A \in \mathcal{B}(X)$.

The next result is a strong version of the Besicovitch Derivation Theorem due to Ambrosio and Dal Maso [2] (see also [3, Theorem 2.22 and Theorem 5.52] or [13, Theorem 1.155]) and it is crucial for the proof of Theorem 1.2 (see Lemma 5.1).

Theorem 2.4. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $\nu \in \mathcal{M}(\Omega;\mathbb{R}^N)$. Then there exists a Borel set $E \subset \Omega$ with $\mu(E) = 0$ such that for every $x \in$ (supp $\mu)\backslash E$*

$$\frac{d\nu}{d\mu}(x) = \frac{d\nu_a}{d\mu}(x) = \lim_{\varepsilon\to 0^+} \frac{\nu\big((x+\varepsilon D)\cap\Omega\big)}{\mu\big((x+\varepsilon D)\cap\Omega\big)} \in \mathbb{R}$$

and

$$\frac{d\nu_s}{d\mu}(x) = \lim_{\varepsilon\to 0^+} \frac{\nu_s\big((x+\varepsilon D)\cap\Omega\big)}{\mu\big((x+\varepsilon D)\cap\Omega\big)} = 0,$$

where D is any bounded, convex, open set containing the origin and the exceptional set E is independent of the choice of D.

2.4. **Functions of bounded variation.** We recall that a function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$ (or $BV(\Omega)$ for $d=1$), if all its first order distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$. It is well known that $BV(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega)$$

where Du is the matrix-valued measure whose entries are $D_j u_i$.

Clearly, we have that any $u \in W^{1,1}(\Omega;\mathbb{R}^d)$ is a BV-function with $Du \in L^1(\Omega;\mathbb{R}^d)$ and the measures Du^i_j are absolutely continuous with respect to the Lebesgue measure.

Given $u \in BV(\Omega;\mathbb{R}^d)$, let Ω_u be the set of points $x \in \Omega$ where the approximate limit of u exists, i.e. such that there exists $z \in \mathbb{R}^d$ with

$$\lim_{\varepsilon\to 0^+} \fint_{B(x,\varepsilon)} |u(y)-z|\,dy = 0.$$

If $x \in \Omega_u$ and $z = u(x)$ we say that u is *approximately continuous* at x (or that x is a Lebesgue point of u). The function u is approximately continuous $\mathcal{L}^N$-a.e. $x \in \Omega_u$ and

$$\mathcal{L}^N(S_u) = 0$$

where we denote by S_u the set of points where u is not approximately continuous, ie., $S_u = \Omega \setminus \Omega_u$. We say that $x \in S_u$ is an approximate jump point of u if there exists $\nu_u(x) \in \mathbb{S}^{N-1}$ and $u^\pm(x) \in \mathbb{R}^d$ such that

$$\lim_{r\to 0^+} \frac{1}{r^N}\left(\int_{B^+(x,r)} \big|u(y)-u^+(x)\big|\,dy + \int_{B^-(x,r)} \big|u(y)-u^-(x)\big|\,dy\right) = 0,$$

with $B^\pm(x,r) := \{y \in B(x,r) : \pm(y-x)\cdot\nu_u(x) > 0\}$. The triple $(\nu_u(x), u^+(x), u^-(x))$ is unique up to a change of sign of $\nu_u(x)$ and a permutation of $u^+(x)$ and $u^-(x)$. The set of approximate jump points is denoted by J_u.

By the Lebesgue-Radon-Nikodým Theorem 2.3, if $u \in BV(\Omega;\mathbb{R}^d)$ then

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u,$$

where ∇u is the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$.

We recall that an $\mathcal{H}^{N-1}$-measurable set $E \subset \mathbb{R}^N$ is said to be a *countably $\mathcal{H}^{N-1}$-rectifiable set* if it can be covered $\mathcal{H}^{N-1}$-almost everywhere by a countable family of $(N-1)$-dimensional surfaces of class C^1. The proof of the well known Structure Theorem for BV-functions that we present below can be found in [3, Theorem 3.78 (Federer-Vol'pert) and Proposition 3.92].

Theorem 2.5 (Structure Theorem for BV-functions)**.** *If $\Omega \subset \mathbb{R}^N$ is open and $u \in BV(\Omega;\mathbb{R}^d)$, then J_u is a countably $\mathcal{H}^{N-1}$-rectifiable set oriented by ν_u, $\mathcal{H}^{N-1}(S_u \setminus J_u) = |Du|(S_u \setminus J_u) = 0$ and $D^s u$ can be decomposed as $D^c u + D^j u$, where $|D^c u|(E) = 0$ for every Borel set E with $\mathcal{H}^{N-1}(E) < +\infty$, and*

$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

$D^c u$ and $D^j u$ are called the *Cantor part* and the *jump part* of the measure Du, respectively.

We also recall that a $\mathcal{L}^N$-measurable subset $E \subset \mathbb{R}^N$ is a *set of finite perimeter* in Ω if the characteristic function χ_E of E is a function of bounded variation. In this case, the perimeter of E in Ω is given by the total variation of χ_E in Ω, i.e., $\mathrm{Per}_\Omega(E) := |D\chi_E|(\Omega)$.

Definition 2.6 (Reduced boundary)**.** Let E be a $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$ and Ω be the largest open set such that E is locally of finite perimeter in Ω, i.e., such that $\chi_E \in BV_{\mathrm{loc}}(\Omega)$. The *reduced boundary* of E, $\partial^* E$, is the collection of all points $x_0 \in \Omega$ such that

(i) $\big|D\chi_E\big|\big(B(x_0,r)\big) > 0$ for all $r > 0$, that is, $x_0 \in \mathrm{supp}|D\chi_E|$;

(ii) the limit $\nu_E(x_0) := \displaystyle\lim_{r\to 0^+} \frac{D\chi_E\big(B(x_0,r)\big)}{|D\chi_E|\big(B(x_0,r)\big)}$ exists in $\mathbb{R}^N$;

(iii) $|\nu_E(x_0)| = 1$.

The function $\nu_E : \partial^* E \to \mathbb{S}^{N-1}$ is called the *generalized unit inner normal* to E.

It can be easily checked that $\partial^* E$ is a Borel set and that ν_E is a Borel map. By the Besicovitch Derivation Theorem 2.4 the measure $|D\chi_E|$ is concentrated on $\partial^* E$ and $D\chi_E = \nu_E |D\chi_E|$. In addition, by De Giorgi's Rectifiability Theorem, see [3, Theorem 3.59], $|D\chi_E|$ coincides with $\mathcal{H}^{N-1}\llcorner\partial^*E$, and for every $x \in \partial^*E$ the following properties hold

$$\lim_{r\to 0^+} \frac{1}{r^{N-1}}\mathcal{H}^{N-1}\Big(\partial^*E \cap Q_{\nu_E(x)}(x,r)\Big) = 1 \tag{2.3}$$

$$\lim_{r\to 0^+} \frac{1}{r^{N}}\,\mathcal{L}^{N}\big(\big\{y \in Q_{\nu_E(x)}(x,r)\setminus E:\,(y-x)\cdot\nu_E(x) \geq 0\big\}\big) = 0 \tag{2.4}$$

$$\lim_{r\to 0^+} \frac{1}{r^{N}}\,\mathcal{L}^{N}\big(\big\{y \in Q_{\nu_E(x)}(x,r)\cap E:\,(y-x)\cdot\nu_E(x) \leq 0\big\}\big) = 0 \tag{2.5}$$

(see also Evans & Gariepy [11, § 5.7.2, Corollary 1] and [16] for the proof of (2.3) when cubes are considered instead of balls).

Remark 2.7. (The set $BV(\Omega;\{\alpha,\beta\})$) Given $\alpha,\beta \in \mathbb{R}^d$, $\alpha \neq \beta$, we denote by $BV\big(\Omega;\{\alpha,\beta\}\big)$ the set of all vector-valued functions u of bounded variation in Ω such that $u(x) \in \{\alpha,\beta\}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. If $u \in BV\big(\Omega;\{\alpha,\beta\}\big)$, that is, $u = \beta\chi_E + \alpha\chi_{\Omega\setminus E}$ for some $\mathcal{L}^N$-measurable set E of finite perimeter, then S_u, the reduced boundary ∂^*E and the jump set J_u of u have the same $\mathcal{H}^{N-1}$-measure in Ω. By (2.4) and (2.5), we also have $\nu_u(x) = \nu_E(x)$, $u^+(x) = \beta$ and $u^-(x) = \alpha$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^*E$.

The following theorem is a variant of a well-known approximation result for sets of finite perimeter and it will be used in the proof of Theorem 1.2 for the construction of the recovery sequence for the limit energy functional since it will allow us to reduce our study to the case where the limit target is suitably regular.

Theorem 2.8. *Let Ω be an open, bounded set with Lipschitz boundary and let E be a subset of Ω with* $\mathrm{Per}_\Omega(E) < +\infty$. *There exists a sequence $\{E_n\}$ of polyhedral sets (i.e., for each n, E_n is a bounded Lipschitz domain with $\partial E_n = H_{1,n} \cup H_{2,n} \cup \ldots H_{L_n,n}$, where each $H_{j,n}$ is a closed subset of a hyperplane $\{x \in \mathbb{R}^N : x\cdot\nu_j = c_j\}$, for some $c_j \in \mathbb{R}$ and $\nu_j \in \mathbb{S}^{N-1}$, $j = 1,\ldots,L_n$, $L_n \in \mathbb{N}$) satisfying the following properties:*

(i) $\chi_{E_n} \to \chi_E$ *in* $L^1(\Omega)$, *as* $n \to +\infty$,

(ii) $\lim\limits_{n\to+\infty} \mathrm{Per}_\Omega(E_n) = \mathrm{Per}_\Omega(E)$,

(iii) $\mathcal{H}^{N-1}(\partial^*E_n \cap \partial\Omega) = 0$,

(iv) $\mathcal{L}^N(E_n) = \mathcal{L}^N(E)$.

For the construction of the sets E_n in Theorem 2.8 we refer to Lemma 3.1 in [4].

2.5. **Γ-convergence and its main properties.** Let X denote a metric space.

Definition 2.9. (Γ-convergence of a sequence of functionals) *Let $F_n, F : X \to \mathbb{R}\cup\{+\infty\}$. The functional F is said to be the* Γ-$\liminf$ *(resp.* Γ-$\limsup$) *of $\{F_n\}_n$ with respect to the metric of X if for every $u \in X$*

$$F(u) = \inf_{\{u_n\}}\left\{\liminf_{n\to+\infty} F_n(u_n):\; u_n \in X,\, u_n \to u \text{ in } X\right\} \;(\text{resp.}\;\; \limsup_{n\to+\infty}).$$

In this case we write

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_n \qquad \left(\text{resp.}\quad F = \Gamma\text{-}\limsup_{n\to+\infty} F_n\right).$$

Moreover, F is said to be the Γ-$\lim$ *of $\{F_n\}_n$ if*

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_n = \Gamma\text{-}\limsup_{n\to+\infty} F_n,$$

and in this case we write

$$F = \Gamma\text{-}\lim_{n\to+\infty} F_n.$$

For every $\varepsilon > 0$ let F_ε be a functional defined in X with values in $\mathbb{R} \cup \{+\infty\}$, $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$.

Definition 2.10. (Γ-convergence of a family of functionals) *A functional $F : X \to \mathbb{R} \cup \{+\infty\}$ is said to be the* $\Gamma\text{-}\liminf$ *(resp.* $\Gamma\text{-}\limsup$ *or* $\Gamma\text{-}\lim$*) of $\{F_\varepsilon\}_\varepsilon$ with respect to the metric of X, as $\varepsilon \to 0^+$, if for every sequence $\varepsilon_n \to 0^+$,*

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_{\varepsilon_n} \quad \left(\text{resp.} \quad F = \Gamma\text{-}\limsup_{n\to+\infty} F_{\varepsilon_n} \ \text{ or } \ F = \Gamma\text{-}\lim_{n\to+\infty} F_{\varepsilon_n}\right),$$

and we write

$$F = \Gamma\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon \quad \left(\text{resp.} \quad F = \Gamma\text{-}\limsup_{\varepsilon\to0^+} F_\varepsilon \ \text{ or } \ F = \Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon\right).$$

One of the most important properties of Γ-convergence is that under appropriate compactness assumptions it implies the convergence of minimisers of a family of functionals to the minimum of the limiting functional, as a consequence of the following result (see Corollary 7.20 in [10]).

Theorem 2.11. (Fundamental Theorem of Γ-convergence) *Let $\{F_\varepsilon\}_\varepsilon$ be a family of functionals defined in X and let*

$$F = \Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon.$$

If u_ε is a minimiser of F_ε in X and $u_\varepsilon \to u$ in X then u is a minimiser of F in X and

$$F(u) = \lim_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon).$$

3. Auxiliary results

In this section we present some auxiliary results for the proof of Theorem 1.2. Our first lemma is crucial to apply a blow-up argument in the lower bound estimate for the limit energy (see Proposition 5.2). It relies on a slicing argument applied in the cube Q_ν, for $\nu \in \mathcal{S}^{N-1}$, and to a target function of the type

$$u_0(x) := \begin{cases} \beta, & \text{if } x\cdot\nu > 0, \\ \alpha, & \text{if } x\cdot\nu < 0, \end{cases} \tag{3.1}$$

allowing us, given a fixed $\theta \geq 0$, to replace a sequence $\{(u_k, \rho_k)\}$, converging to (u_0, θ) by a sequence $\{(w_k, \gamma_k)\}$ of admissible pairs in $\mathcal{A}(\nu, \theta)$, still converging to (u_0, θ) in $X(\Omega)$ and without increasing the total energy.

Given $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\varepsilon > 0$, we denote by u_ε the standard mollification of u. We recall that,

i) if u is bounded, then for every $1 \leq p < +\infty$,

$$u_\varepsilon \to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N), \quad \|u_\varepsilon\|_\infty \leq \|u\|_\infty, \quad \|\nabla u_\varepsilon\|_\infty \leq C\frac{\|u\|_\infty}{\varepsilon}. \tag{3.2}$$

ii) if $u = \beta\chi_E + \alpha\chi_{\mathbb{R}^N\setminus E}$ for some set $E \subset \mathbb{R}^N$ with Lipschitz boundary, then

$$\begin{aligned}\|u_\varepsilon - u\|_{L^1(\Omega)} &= \int_{\{x\in\Omega:\, \mathrm{dist}(x,\Omega\cap\partial E)\leq\varepsilon\}} |u_\varepsilon(x) - u(x)|\,dx \\ &\leq C(\alpha,\beta,N)\,\mathcal{L}^N(\{x\in\Omega:\ \mathrm{dist}(x,\Omega\cap\partial E)\leq\varepsilon\}) = O(\varepsilon).\end{aligned}$$

Finally we note that the convolution with the target function defined in (3.1), which we denote in the sequel by $\tilde{u}_\varepsilon$, has the following properties

$$\widetilde{u}_\varepsilon(x) = \beta \quad \text{if } x \cdot \nu > \varepsilon, \qquad \widetilde{u}_\varepsilon(x) = \alpha \quad \text{if } x \cdot \nu < -\varepsilon, \tag{3.3}$$

$$\nabla \widetilde{u}_\varepsilon(x) = 0 \qquad \text{if } |x \cdot \nu| > \varepsilon. \tag{3.4}$$

Lemma 3.1. *Assume that (H1)–(H4) hold. Let $\varepsilon_k \to 0^+$ as $k \to +\infty$. If $\{v_k\} \subset W^{1,p}(Q_\nu; \mathbb{R}^d_+)$ converges in $L^1(Q_\nu; \mathbb{R}^d_+)$ to u_0, and if $\{\lambda_k\} \subset L^1\big(Q_\nu; [0, +\infty)\big)$ is such that*

$$\lim_{k \to +\infty} \int_{Q_\nu} \lambda_k(x)\, dx \le \theta, \tag{3.5}$$

then there exist a subsequence $\{k'\}$ of $\{k\}$ and $\{(w_{k'}, \gamma_{k'})\} \subseteq W^{1,p}(Q_\nu; \mathbb{R}^d_+) \times L^1\big(Q_\nu; [0, +\infty)\big)$ such that

(i) *$w_{k'} \to u_0$ in $L^1(Q_\nu; \mathbb{R}^d_+)$ and $w_{k'} = u_0 * \Psi_{\varepsilon_{k'}}$ near the boundary ∂Q_ν;*

(ii) *$\displaystyle\int_{Q_\nu} \gamma_{k'}(x)\, dx \le \theta$ for every k';*

(iii) $$\limsup_{k' \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_{k'}} f\big(w_{k'}(x), \varepsilon_{k'} \nabla w_{k'}(x), \varepsilon_{k'} \gamma_{k'}(x)\big)\, dx \le \liminf_{k \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)\, dx.$$

Proof. The outline of the proof is as follows. We begin by proving the lemma in the particular case where v_k and λ_k are uniformly bounded in L^∞, i.e., we assume that there exists $M > 0$ such that, for all k, $||v_k||_\infty < M$ and $||\lambda_k||_\infty < M$. In a second step, we prove that given $\delta > 0$, for every k, there exist $M(k, \delta)$ and $\overline{v}_{k,\delta} \in W^{1,p}(Q_\nu; \mathbb{R}^d_+) \cap L^\infty(Q_\nu; \mathbb{R}^d_+)$, $\overline{\lambda}_{k,\delta} \in L^1\big(Q_\nu; [0, +\infty)\big) \cap L^\infty\big(Q_\nu; [0, +\infty)\big)$ such that $||\overline{v}_{k,\delta}||_\infty < M(k, \delta)$, $||\overline{\lambda}_{k,\delta}||_\infty < M(k, \delta)$ and

$$\int_{Q_\nu} \frac{1}{\varepsilon_k} f\big(\overline{v}_{k,\delta}(x), \varepsilon_k \nabla \overline{v}_{k,\delta}(x), \varepsilon_k \overline{\lambda}_{k,\delta}(x)\big)\, dx \le \int_{Q_\nu} \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)\, dx + \delta.$$

The result then follows by a diagonalisation argument.

Without loss of generality, we assume that $\nu = e_N$ and we denote Q_ν by Q. Extracting a subsequence if necessary, we may also assume that $v_k(x) \to u_0(x)$ for $\mathcal{L}^N$-a.e. $x \in Q$, and that

$$\liminf_{k \to +\infty} \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big) dx = \lim_{k \to +\infty} \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big) dx < +\infty. \tag{3.6}$$

Step 1: case of L^∞ uniformly bounded sequences.

Step 1a: construction of w_k. Assume that there exists $M > 0$ such that, for all k, $||v_k||_\infty < M$ and $||\lambda_k||_\infty < M$. We first notice that, if $p > 1$, then

$$\lim_{k \to +\infty} \int_Q |v_k(x) - u_0(x)|^p\, dx = 0. \tag{3.7}$$

(if $p = 1$ this holds by hypothesis). In fact, since $||v_k||_\infty < M$ and $u_0 \in L^\infty$, the claim follows immediately since

$$\int_Q |v_k(x) - u_0(x)|^p\, dx \le ||v_k - u_0||_1 ||v_k - u_0||_\infty^{p-1}.$$

Notice also that, by (H3) and (3.6), we have

$$\limsup_{k \to +\infty} \int_Q \varepsilon_k^{p-1} |\nabla v_k(x)|^p\, dx < +\infty. \tag{3.8}$$

For $0 < s < 1/2$, define

$$Q_s := \Big\{ x \in Q : \operatorname{dist}(x, \partial Q) > s \Big\}.$$

Choose a sequence $s_m \downarrow 0^+$, and for each $m \in \mathbb{N}$ and $h \in \mathbb{N}$ define

$$L_{m,h} := \Big\{ x \in Q : s_m \leq \operatorname{dist}(x, \partial Q) \leq s_m + \frac{1}{h} \Big\}.$$

If v_k equals the mollified target function $\widetilde{u}_k := \widetilde{u}_{\varepsilon_k}$ for infinitely many k, we choose $w_k = \widetilde{u}_k$ to achieve the conclusions of the lemma. Otherwise, without loss of generality, we may assume that for every $k \in \mathbb{N}$, $\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)} > 0$. Partition $L_{m,h}$ into $T_{m,h,k}$ pairwise disjoint layers

$$L^{(i)}_{m,h,k} := \big\{ x \in L_{m,h} : \delta_{i-1} < \operatorname{dist}\big(x, \partial Q_{s_m}\big) \leq \delta_i \big\}, \qquad i = 1, \ldots, T_{m,h,k},$$

of constant width $\delta_i - \delta_{i-1} = \varepsilon_k \|v_k - \widetilde{u}_k\|^{1/p}_{L^p(Q;\mathbb{R}^d_+)}$, with $\delta_0 = 0$ and $\delta_{T_{m,h,k}} = O(1/h)$, so that

$$T_{m,h,k} \varepsilon_k \big\|v_k - \widetilde{u}_k\big\|^{1/p}_{L^p(Q;\mathbb{R}^d_+)} = O\bigg(\frac{1}{h}\bigg), \tag{3.9}$$

and

$$\begin{aligned}\sum_{i=1}^{T_{m,h,k}} \int_{L^{(i)}_{m,h,k}} & \bigg[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p + \frac{|v_k(x) - \widetilde{u}_k(x)|^p}{\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)}} \bigg] dx \\ &= \int_{L_{m,h}} \bigg[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p + \frac{|v_k(x) - \widetilde{u}_k(x)|^p}{\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)}} \bigg] dx.\end{aligned}$$

Thus, there exists $i_* = i_*(m,h,k)$ such that

$$\begin{aligned}\int_{L^{(i_*)}_{m,h,k}} & \bigg[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p + \frac{|v_k(x) - \widetilde{u}_k(x)|^p}{\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)}} \bigg] dx \\ &\leq \frac{1}{T_{m,h,k}} \int_{L_{m,h}} \bigg[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p + \frac{|v_k(x) - \widetilde{u}_k(x)|^p}{\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)}} \bigg] dx.\end{aligned} \tag{3.10}$$

Consider cut-off functions $\varphi_{m,h,k} \in C^\infty_c(Q_{s_m}; [0,1])$ such that

$$\varphi_{m,h,k} = 0 \qquad \text{on } \big(Q \setminus Q_{s_m}\big) \cup \bigcup_{i=1}^{i_*-1} L^{(i)}_{m,h,k} =: A_{m,h,k}$$

$$\varphi_{m,h,k} = 1 \qquad \text{on } \big(Q_{s_m} \setminus L_{m,h}\big) \cup \Bigg(\bigcup_{i=i_*+1}^{T_{m,h,k}} L^{(i)}_{m,h,k} \Bigg) =: B_{m,h,k},$$

and

$$\|\nabla \varphi_{m,h,k}\|_\infty = O\Big(\varepsilon_k^{-1} \|v_k - \widetilde{u}_k\|^{-1/p}_{L^p(Q;\mathbb{R}^d_+)} \Big). \tag{3.11}$$

Define

$$w_{m,h,k}(x) := \varphi_{m,h,k}(x) v_k(x) + (1 - \varphi_{m,h,k}(x)) \widetilde{u}_k(x), \qquad x \in Q, \tag{3.12}$$

and

$$\rho_{m,h,k}(x) := \begin{cases} \lambda_k(x) & \text{if } x \in Q \setminus A_{m,h,k}, \\ 0 & \text{if } x \in A_{m,h,k}. \end{cases}$$

Then $w_{m,h,k} \in W^{1,p}(Q;\mathbb{R}^d_+)$, and by (3.5)

$$\limsup_{m,h,k \to +\infty} \int_Q \rho_{m,h,k}(x)\, dx \leq \limsup_{m,h,k \to +\infty} \int_{Q \setminus A_{m,h,k}} \lambda_k(x)\, dx \leq \theta. \tag{3.13}$$

From (3.7) and $(3.2)_1$, we have that for each $m \in \mathbb{N}$

$$\lim_{m,h,k\to+\infty} \|w_{m,h,k} - u_0\|_{L^p(Q;\mathbb{R}^d_+)} = 0.$$

Furthermore,

$$\begin{aligned}
&\int_Q \frac{1}{\varepsilon_k} f\big(w_{m,h,k}(x), \varepsilon_k \nabla w_{m,h,k}(x), \varepsilon_k \rho_{m,h,k}(x)\big)dx \\
&\quad\le \int_{A_{m,h,k}} \frac{1}{\varepsilon_k} f\big(\widetilde{u}_k(x), \varepsilon_k \nabla \widetilde{u}_k(x), 0\big)dx \\
&\qquad + \int_{L^{(i_*)}_{m,h,k}} \frac{1}{\varepsilon_k} f\big(w_{m,h,k}(x), \varepsilon_k \nabla w_{m,h,k}(x), \varepsilon_k \lambda_k(x)\big)dx \\
&\qquad + \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)dx.
\end{aligned} \tag{3.14}$$

By (H2), (H3), (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
&\limsup_{m,h,k\to+\infty} \int_{A_{m,h,k}} \frac{1}{\varepsilon_k} f\big(\widetilde{u}_k(x), \varepsilon_k \nabla \widetilde{u}_k(x), 0\big)dx \\
&\quad\le C \limsup_{m,h,k\to+\infty} \int_{\{x\in A_{m,h,k} :\, |x_N|<\varepsilon_k\}} \frac{1}{\varepsilon_k}\big[h(\widetilde{u}_k(x),0) + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p\big]\,dx \\
&\quad\le C \limsup_{m,h\to+\infty} \left\{ \limsup_{k\to+\infty} \frac{1}{\varepsilon_k} \mathcal{L}^N\big(\{x \in (Q\setminus Q_{s_m}) \cup L_{m,h} :\, |x_N| < \varepsilon_k\}\big)\right\} \\
&\quad= C \limsup_{m\to+\infty} \mathcal{L}^N\big(\partial Q_{s_m}\big) = 0.
\end{aligned}$$

Given that $h \in L^\infty_{\mathrm{loc}}$ and $w_{m,h,k}$ and $\varepsilon_k\lambda_k$ are uniformly bounded in L^∞, in view of (H3), (3.11), (3.10) and (3.9), in this order, we have for each $m \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{h,k\to+\infty} \int_{L^{(i_*)}_{m,h,k}} \frac{1}{\varepsilon_k} f\big(w_{m,h,k}(x), \varepsilon_k \nabla w_{m,h,k}(x), \varepsilon_k \lambda_k(x)\big)dx \\
&\le \limsup_{h,k\to+\infty} \frac{C}{\varepsilon_k T_{m,h,k}} \int_{L_{m,h}} \left[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p + \frac{|v_k(x) - \widetilde{u}_k(x)|^p}{\|v_k - \widetilde{u}_k\|_{L^p(Q;\mathbb{R}^d_+)}}\right] dx \\
&\le \limsup_{h,k\to+\infty} Ch \big\|v_k - \widetilde{u}_k\big\|^{1/p}_{L^p(Q;\mathbb{R}^d_+)} \left\{ \int_{L_{m,h}} \left[1 + \varepsilon_k^p |\nabla v_k(x)|^p + \varepsilon_k^p |\nabla \widetilde{u}_k(x)|^p\right] dx \right. \\
&\qquad\qquad \left. + \|v_k - \widetilde{u}_k\|^{p-1}_{L^p(Q;\mathbb{R}^d_+)} \right\} = 0,
\end{aligned}$$

where in the last equality we used (3.2), (3.7), and (3.8). Thus, (3.14) becomes

$$\begin{aligned}
&\limsup_{m,h,k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(w_{m,h,k}(x), \varepsilon_k \nabla w_{m,h,k}(x), \varepsilon_k \rho_{m,h,k}(x)\big)dx \\
&\qquad\qquad \le \lim_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)dx.
\end{aligned}$$

Finally, using a diagonalisation process (see [7, Lemma 7.1]) we can extract subsequences $\{m(k)\}$ and $\{h(k)\}$ such that setting $w_k := w_{m(k),h(k),k}$ and $\rho_k := \rho_{m(k),h(k),k}$, we have that

$$\begin{aligned} &\lim_{k\to+\infty} \|w_k - u_0\|_{L^1(Q;\mathbb{R}^d_+)} = 0, \qquad \lim_{k\to+\infty} \int_Q \rho_k(x)\,dx \le \theta, \\ &\limsup_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \rho_k(x)\big) dx \le \liminf_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big) dx, \end{aligned} \tag{3.15}$$

where we used (3.6) and the fact that the limit along any subsequence equals the limit along the original sequence.

Step 1b: construction of γ_k. We now need to modify the sequence $\{\rho_k\}_k$ in order to obtain a new sequence $\{\gamma_k\}_k$ satisfying $\int_Q \gamma_k(x)\,dx \le \theta$ for all k (and not only in the limit as $k \to +\infty$). Set

$$\gamma_k := c_k \rho_k, \qquad \text{where} \quad c_k := \min\left\{1, \frac{\theta}{\int_Q \rho_k(x)\,dx}\right\}.$$

If $\int_Q \rho_k(x)\,dx = 0$ then we take $c_k = 1$. It is clear that γ_k satisfies (ii) for every $k \in \mathbb{N}$. We also claim that

$$\limsup_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \gamma_k(x)\big) dx \le \limsup_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \rho_k(x)\big) dx. \tag{3.16}$$

In order to prove (3.16), we begin by noting that, by (3.6) and (3.15),

$$\limsup_{k\to+\infty} \int_Q \frac{1}{\varepsilon_k} f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \rho_k(x)\big) dx < +\infty,$$

so (H3) yields

$$\limsup_{k\to+\infty} \int_Q \varepsilon_k^{p-1} |\nabla w_k(x)|^p\,dx < +\infty. \tag{3.17}$$

Since by construction $c_k \to 1$ as $k \to +\infty$, using hypothesis (H4), (3.17) and the uniform bounds $||v_k||_\infty < M$, $||\rho_k||_\infty \le ||\lambda_k||_\infty < M$, we have

$$\begin{aligned} &\limsup_{k\to+\infty} \frac{1}{\varepsilon_k} \int_Q \big| f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \gamma_k(x)\big) - f\big(w_k(x), \varepsilon_k \nabla w_k(x), \varepsilon_k \rho_k(x)\big)\big|\,dx \\ &\le \limsup_{k\to+\infty} \frac{C_M}{\varepsilon_k} \int_Q \big|\varepsilon_k \rho_k(x) - \varepsilon_k \gamma_k(x)\big| \big(1 + |\varepsilon_k \nabla w_k(x)|^p\big)\,dx \\ &\le \limsup_{k\to+\infty} C(1 - c_k) \int_Q \rho_k(x) \big(1 + |\varepsilon_k \nabla w_k(x)|^p\big)\,dx = 0. \end{aligned}$$

Thus the proof of (3.16) is complete.

Step 2: truncation. Now let $\varepsilon_k \to 0^+$ as $k \to +\infty$, $\{v_k\} \subset W^{1,p}(Q_\nu;\mathbb{R}^d_+)$ converge in $L^1(Q_\nu;\mathbb{R}^d_+)$ to u_0 and $\{\lambda_k\} \subset L^1\big(Q_\nu;[0,+\infty)\big)$ be such that

$$\lim_{k\to+\infty} \int_{Q_\nu} \lambda_k(x)\,dx \le \theta. \tag{3.18}$$

We will use a truncation argument to show that, for each $\delta > 0$ and for each fixed k, there exist $M = M(k,\delta)$ and functions $\overline{v}_k$ and $\overline{\lambda}_k$ such that $||\overline{v}_k||_\infty < M(k,\delta)$, $||\overline{\lambda}_k||_\infty < M(k,\delta)$, $\overline{v}_k \to u_0$ in $L^1(Q_\nu;\mathbb{R}^d_+)$, $\lim_{k\to+\infty} \int_Q \overline{\lambda}_k(x)\,dx \le \theta$ and

$$\int_Q \frac{1}{\varepsilon_k} f\left(\overline{v}_k(x), \varepsilon_k \nabla \overline{v}_k(x), \varepsilon_k \overline{\lambda}_k(x)\right)\,dx \le \int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)\,dx + \delta. \tag{3.19}$$

For each $k \in \mathbb{N}$, $M > \|u_0\|_\infty$, define

$$v_{k,M}(x) = \begin{cases} v_k(x) & \text{if } |v_k(x)| < M, \\ u_0(x) & \text{if } |v_k(x)| \geq M+1 \end{cases}$$

and such that, for all $x \in Q$, $|v_{k,M}(x)| \leq |v_k(x)|$, $|\nabla v_{k,M}(x)| \leq |\nabla v_k(x)|$ (cf. [5]), and

$$\lambda_{k,M}(x) = \begin{cases} \lambda_k(x) & \text{if } \lambda_k(x) < M, \\ 0 & \text{if } \lambda_k(x) \geq M. \end{cases}$$

Comparing the energies we have that

$$\begin{aligned}
&\int_Q \frac{1}{\varepsilon_k} f\left(v_{k,M}(x), \varepsilon_k \nabla v_{k,M}(x), \varepsilon_k \lambda_{k,M}(x)\right)\, dx \\
&\qquad = \int_{Q\cap\{|v_k|<M,\, \lambda_k<M\}} \frac{1}{\varepsilon_k} f\left(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\right)\, dx \\
&\qquad + \int_{Q\cap\{|v_k|<M,\, \lambda_k\geq M\}} \frac{1}{\varepsilon_k} f\left(v_k(x), \varepsilon_k \nabla v_k(x), 0\right)\, dx \\
&\qquad + \int_{Q\cap\{|v_k|\geq M+1,\, \lambda_k<M\}} \frac{1}{\varepsilon_k} f(u_0(x), 0, \varepsilon_k \lambda_k(x))\, dx \\
&\qquad + \int_{Q\cap\{|v_k|\geq M+1,\, \lambda_k\geq M\}} \frac{1}{\varepsilon_k} f(u_0(x), 0, 0)\, dx \\
&\qquad + \int_{Q\cap\{M\leq|v_k|<M+1,\, \lambda_k<M\}} \frac{1}{\varepsilon_k} f\left(v_{k,M}(x), \varepsilon_k \nabla v_{k,M}(x), \varepsilon_k \lambda_k(x)\right)\, dx \\
&\qquad + \int_{Q\cap\{M\leq|v_k|<M+1,\, \lambda_k\geq M\}} \frac{1}{\varepsilon_k} f\left(v_{k,M}(x), \varepsilon_k \nabla v_{k,M}(x), 0\right)\, dx,
\end{aligned}$$

where, by (H2), the fourth term is zero. Using hypothesis (H3) yields,

$$\begin{aligned}
\int_Q \frac{1}{\varepsilon_k} f\left(v_{k,M}(x), \varepsilon_k \nabla v_{k,M}(x), \varepsilon_k \lambda_{k,M}(x)\right)\, dx &\leq \int_Q \frac{1}{\varepsilon_k} f\left(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\right)\, dx \\
&+ C \int_{Q\cap\{\lambda_k\geq M\}} \frac{1}{\varepsilon_k} (h(v_k(x), 0) + |\varepsilon_k \nabla v_k(x)|^p)\, dx \\
&+ C \int_{Q\cap\{|v_k|\geq M+1\}} \frac{1}{\varepsilon_k} h(u_0(x), \varepsilon_k \lambda_k(x))\, dx \\
&+ C \int_{Q\cap\{|v_k|\geq M\}} \frac{1}{\varepsilon_k} \left(h(v_{k,M}(x), \varepsilon_k \lambda_k(x)) + (\varepsilon_k |\nabla v_k(x)|)^p\right)\, dx \\
&+ C \int_{Q\cap\{\lambda_k\geq M\}} \frac{1}{\varepsilon_k} \left(h(v_{k,M}(x), 0) + (\varepsilon_k |\nabla v_k(x)|)^p\right)\, dx, \\
&\leq \int_Q \frac{1}{\varepsilon_k} f\left(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\right)\, dx + \delta,
\end{aligned}$$

provided $M = M(k, \delta)$ large enough. Defining $\overline{v}_k = v_{k,M(k,\delta)}$ and $\overline{\lambda}_k = \lambda_{k,M(k,\delta)}$ it is clear that these functions satisfy the required properties.

Step 3: diagonalisation. Fix $M > \|u_0\|_\infty$. Since the sequences $\{v_{k,M}\}_k$ and $\{\lambda_{k,M}\}_k$ are uniformly bounded in L^∞, by Step 1, there exist sequences $w_{k,M}$ and $\gamma_{k,M}$, satisfying the conditions in the statement of the lemma and such that

$$\limsup_{k\to+\infty}\int_Q \frac{1}{\varepsilon_k} f\big(w_{k,M}(x), \varepsilon_k \nabla w_{k,M}(x), \varepsilon_k \gamma_{k,M}(x)\big)\, dx$$
$$\leq \liminf_{k\to+\infty}\int_Q \frac{1}{\varepsilon_k} f\big(v_{k,M}(x), \varepsilon_k \nabla v_{k,M}(x), \varepsilon_k \lambda_{k,M}(x)\big)\, dx.$$

Thus, for all j, there exists $k(j)$ such that

$$\lim_{j\to+\infty}\int_Q \frac{1}{\varepsilon_{k(j)}} f\big(w_{k(j),M}(x), \varepsilon_{k(j)} \nabla w_{k(j),M}(x), \varepsilon_{k(j)} \gamma_{k(j),M}(x)\big)\, dx$$
$$\leq \lim_{j\to+\infty}\int_Q \frac{1}{\varepsilon_{k(j)}} f\big(v_{k(j),M}(x), \varepsilon_{k(j)} \nabla v_{k(j),M}(x), \varepsilon_{k(j)} \lambda_{k(j),M}(x)\big)\, dx$$

and so, for every $\delta > 0$, there exists $k(j,\delta)$ such that

$$\int_Q \frac{1}{\varepsilon_{k(j,\delta)}} f\big(w_{k(j,\delta),M}(x), \varepsilon_{k(j,\delta)} \nabla w_{k(j,\delta),M}(x), \varepsilon_{k(j,\delta)} \gamma_{k(j,\delta),M}(x)\big)\, dx$$
$$\leq \int_Q \frac{1}{\varepsilon_{k(j,\delta)}} f\big(v_{k(j,\delta),M}(x), \varepsilon_{k(j,\delta)} \nabla v_{k(j,\delta),M}(x), \varepsilon_{k(j,\delta)} \lambda_{k(j,\delta),M}(x)\big)\, dx + \delta.$$

Thus, if in the previous inequality we set $\delta = \dfrac{1}{j}$ and $M_j = M(k(j,\frac{1}{j}),\frac{1}{j})$ from Step 2, and we define $\varepsilon_j = \varepsilon_{k(j,\frac{1}{j})}$, $w_j = w_{k(j,\frac{1}{j}),M_j}$ and $\gamma_j = \gamma_{k(j,\frac{1}{j}),M_j}$, we get, using the estimate obtained in Step 2,

$$\limsup_{j\to+\infty}\int_Q \frac{1}{\varepsilon_j} f\big(w_j(x), \varepsilon_j \nabla w_j(x), \varepsilon_j \gamma_j(x)\big)\, dx$$
$$\leq \liminf_{j\to+\infty}\left(\int_Q \frac{1}{\varepsilon_{k(j,\frac{1}{j})}} f\big(v_{k(j,\frac{1}{j}),M_j}(x), \varepsilon_{k(j,\frac{1}{j})} \nabla v_{k(j,\frac{1}{j}),M_j}(x), \varepsilon_{k(j,\frac{1}{j})} \lambda_{k(j,\frac{1}{j}),M_j}(x)\big)\, dx + \frac{1}{j}\right)$$
$$\leq \liminf_{j\to+\infty}\left(\int_Q \frac{1}{\varepsilon_{k(j,\frac{1}{j})}} f\big(v_{k(j,\frac{1}{j})}(x), \varepsilon_{k(j,\frac{1}{j})} \nabla v_{k(j,\frac{1}{j})}(x), \varepsilon_{k(j,\frac{1}{j})} \lambda_{k(j,\frac{1}{j})}(x)\big)\, dx + \frac{2}{j}\right)$$
$$\leq \liminf_{k\to+\infty}\int_Q \frac{1}{\varepsilon_k} f\big(v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k \lambda_k(x)\big)\, dx.$$

This completes the proof of the lemma. □

Remark 3.2. Notice that by (3.3) and Lemma 3.1 (i)-(ii), we have that $(w_k, \gamma_k) \in \mathcal{A}(\nu,\theta)$ for sufficiently large k.

We will now analyse some properties of the surface energy density σ given in (1.8). These properties will be useful for the construction of recovery sequences to obtain an upper bound for the limit energy (see Proposition 5.3) as they will allow us to reduce the target function to a suitably regular class of functions.

Proposition 3.3. *If (H1) and (H3) hold, then*

(i) $0 \leq \sigma(\nu,\theta) \leq C\big(1 + |\alpha|^p + |\beta|^p\big)$ *for all* $(\nu,\theta) \in \mathbb{S}^{N-1}\times[0,+\infty)$*;*

(ii) *for all* $(\nu,\theta) \in \mathbb{S}^{N-1}\times[0,+\infty)$, $\sigma(\nu,\theta) = \sigma_\infty(\nu,\theta)$ *where*

$$\sigma_\infty(\nu,\theta) := \inf\left\{\int_{Q_\nu} \frac{1}{t} f\big(u(y), t\nabla u(y), t\rho(y)\big)\, dy :\ t>0,\right.$$
$$\left.(u,\rho) \in \mathcal{A}(\nu,\theta)\cap L^\infty(S_\nu;\mathbb{R}^d_+)\times L^\infty(\mathbb{R}^N;[0,+\infty))\right\};$$

(iii) σ *is upper semicontinuous on* $\mathbb{S}^{N-1}\times[0,+\infty)$ *and non-increasing with respect to* θ*.*

Notice that the continuity of f is not used to prove (i).

Proof. (i) Fix $(\nu,\theta)\in\mathbb{S}^{N-1}\times[0,+\infty)$, and let

$$w(x):=(\beta-\alpha)(x\cdot\nu)+\frac{\alpha+\beta}{2}. \tag{3.20}$$

Since $(w,0)\in\mathcal{A}(\nu,\theta)$, $f\geq 0$ and (H3) holds, we have

$$\begin{aligned}0\leq\sigma(\nu,\theta)&\leq\int_{Q_\nu}f\big(w(x),\nabla w(x),0\big)\,dx\leq\int_{Q_\nu}C\Big(h(w(x),0)+|\nabla w(x)|^p\Big)\,dx\\&\leq\int_{Q_\nu}C\left(1+\left|(\beta-\alpha)\otimes\nu\right|^p\right)\,dx\leq C\big(1+|\alpha|^p+|\beta|^p\big).\end{aligned}$$

(ii) Clearly $\sigma(\nu,\theta)\leq\sigma_\infty(\nu,\theta)$. To show the reverse inequality fix $\varepsilon>0$ and let $(u_\varepsilon,\rho_\varepsilon)\in\mathcal{A}(\nu,\theta)$, $t_\varepsilon>0$ be such that

$$\int_Q\frac{1}{t_\varepsilon}f\big(u_\varepsilon(x),t_\varepsilon\nabla u_\varepsilon(x),t_\varepsilon\rho_\varepsilon(x)\big)\,dx<\sigma(\nu,\theta)+\frac{\varepsilon}{2}.$$

We follow the truncation argument given in the proof of Lemma 3.1 to obtain sequences $\{u_{\varepsilon,j}\}$ and $\{\rho_{\varepsilon,j}\}$, bounded in L^∞, and such that, for j large enough,

$$\int_Q\frac{1}{t_\varepsilon}f\left(u_{\varepsilon,j}(x),t_\varepsilon\nabla u_{\varepsilon,j}(x),t_\varepsilon\rho_{\varepsilon,j}(x)\right)\,dx\leq\int_Q\frac{1}{t_\varepsilon}f\left(u_\varepsilon(x),t_\varepsilon\nabla u_\varepsilon(x),t_\varepsilon\rho_\varepsilon(x)\right)\,dx+\frac{\varepsilon}{2}<\sigma(\nu,\theta)+\varepsilon.$$

Thus,

$$\sigma_\infty(\nu,\theta)\leq\int_Q\frac{1}{t_\varepsilon}f\left(u_{\varepsilon,j}(x),t_\varepsilon\nabla u_{\varepsilon,j}(x),t_\varepsilon\rho_{\varepsilon,j}(x)\right)\,dx<\sigma(\nu,\theta)+\varepsilon,$$

so to conclude the result it suffices to let $\varepsilon\to 0^+$.

(iii) The fact that σ is non-increasing with respect to its second variable is obvious from the definition. To prove upper semicontinuity, we observe that, by a change of variable argument it is clear that, for every $(\nu,\theta)\in\mathbb{S}^{N-1}\times[0,+\infty)$,

$$\sigma(\nu,\theta)=\inf\left\{\int_Q\frac{1}{t}f\big(w(x),t\nabla w(x)R^T,t\gamma(x)\big)\,dx:\,t>0,\,(w,\gamma)\in\mathcal{A}(e_N,\theta),\,Re_N=\nu,\,R\in SO(N)\right\}, \tag{3.21}$$

where Q stands for Q_{e_N}. Let $(\nu_n,\theta_n)\in\mathbb{S}^{N-1}\times[0,+\infty)$ be such that $(\nu_n,\theta_n)\to(\nu,\theta)$ and choose a rotation R such that $Re_N=\nu$. Given $\varepsilon\in(0,1)$, let $t_\varepsilon>0$ and $(w_\varepsilon,\gamma_\varepsilon)\in\mathcal{A}(e_N,\theta)$ be such that

$$\left|\sigma(\nu,\theta)-\int_Q\frac{1}{t_\varepsilon}f\big(w_\varepsilon(x),t_\varepsilon\nabla w_\varepsilon(x)R^T,t_\varepsilon\gamma_\varepsilon(x)\big)\,dx\right|<\varepsilon. \tag{3.22}$$

Since by (ii), $\sigma(\nu,\theta)=\sigma_\infty(\nu,\theta)$, we may also assume that $\|w_\varepsilon\|_\infty\leq C$ and $\|\gamma_\varepsilon\|_\infty\leq C$, for all $\varepsilon\in(0,1)$.

By (3.22) and (i),

$$\sup_{\varepsilon\in(0,1)}\int_Q\frac{1}{t_\varepsilon}f\big(w_\varepsilon(x),t_\varepsilon\nabla w_\varepsilon(x)R^T,t_\varepsilon\gamma_\varepsilon(x)\big)dx<+\infty. \tag{3.23}$$

Notice also that, if $\theta=0$, then $\int_Q\gamma_\varepsilon(x)\,dx\leq 0$, that is, $\gamma_\varepsilon(x)=0$ for a.e. $x\in Q$. For every $n\in\mathbb{N}$, choose $R_n\in SO(N)$ such that $R_ne_N=\nu_n$ and $R_n\to R$ as $n\to+\infty$, and define $\gamma_{n,\varepsilon}\in L^1\big(Q;[0,+\infty)\big)$ by setting $\gamma_{n,\varepsilon}=0$ if $\theta=0$ and $\gamma_{n,\varepsilon}:=\frac{\theta_n}{\theta}\,\gamma_\varepsilon$ if $\theta\neq 0$. Clearly,

$$\int_Q\gamma_{n,\varepsilon}(x)\,dx\leq\theta_n\qquad\text{and}\qquad\lim_{n\to+\infty}\|\gamma_{n,\varepsilon}-\gamma_\varepsilon\|_{L^1(Q)}=\lim_{n\to+\infty}\left|1-\frac{\theta_n}{\theta}\right|\,\|\gamma_\varepsilon\|_{L^1(Q)}=0.$$

Since $(w_\varepsilon, \gamma_{n,\varepsilon}) \in \mathcal{A}(e_N, \theta_n)$, in view of (3.21), we have

$$\sigma(\nu_n, \theta_n) \leq \int_Q \frac{1}{t_\varepsilon} f\big(w_\varepsilon(x), t_\varepsilon \nabla w_\varepsilon(x) R_n^T, t_\varepsilon \gamma_{n,\varepsilon}(x)\big)\, dx. \tag{3.24}$$

By (H1) and (H3), for a.e. $x \in Q$

$$\lim_{n\to+\infty} \frac{1}{t_\varepsilon} f\big(w_\varepsilon(x), t_\varepsilon \nabla w_\varepsilon(x) R_n^T, t_\varepsilon \gamma_{n,\varepsilon}(x)\big) = \frac{1}{t_\varepsilon} f\big(w_\varepsilon(x), t_\varepsilon \nabla w_\varepsilon(x) R^T, t_\varepsilon \gamma_\varepsilon(x)\big),$$

and

$$f\big(w_\varepsilon, t_\varepsilon \nabla w_\varepsilon R_n^T, t_\varepsilon \gamma_{n,\varepsilon}\big) \leq C\Big(h(w_\varepsilon, t_\varepsilon \gamma_{n,\varepsilon}) + t_\varepsilon^p |\nabla w_\varepsilon|^p\Big) \in L^1(Q),$$

since, by (3.23) and (H3), $t_\varepsilon^p |\nabla w_\varepsilon|^p \in L^1\big(Q; [0,+\infty)\big)$, and by the uniform L^∞ bounds on w_ε and $t_\varepsilon \gamma_\varepsilon$, $h(w_\varepsilon, t_\varepsilon\gamma_\varepsilon) \in L^1\big(Q;[0,+\infty)\big)$, and so also $h(w_\varepsilon, t_\varepsilon \gamma_{n,\varepsilon}) \in L^1\big(Q;[0,+\infty)\big)$. Thus, by the Lebesgue Dominated Convergence Theorem, we obtain that

$$\lim_{n\to+\infty} \int_Q \frac{1}{t_\varepsilon} f\big(w_\varepsilon(x), t_\varepsilon \nabla w_\varepsilon(x) R_n^T, t_\varepsilon \gamma_{n,\varepsilon}(x)\big)\, dx = \int_Q \frac{1}{t_\varepsilon} f\big(w_\varepsilon(x), t_\varepsilon \nabla w_\varepsilon(x) R^T, t_\varepsilon \gamma_\varepsilon(x)\big)\, dx$$

so, in view of (3.24) and (3.22), we conclude that

$$\limsup_{n\to+\infty} \sigma(\nu_n, \theta_n) \leq \sigma(\nu, \theta) + \varepsilon.$$

It suffices now to let $\varepsilon \to 0^+$. □

In view of the previous proposition, it is possible to extend σ to the whole $\mathbb{R}^N \times [0,+\infty)$ by setting

$$\sigma(z,\theta) := \begin{cases} |z|\, \sigma\left(\dfrac{z}{|z|}, \dfrac{\theta}{|z|}\right), & \text{for every } z \in \mathbb{R}^N \setminus \{0\} \text{ and every } \theta \in [0,+\infty), \\ 0, & \text{for } z = 0 \text{ and every } \theta \in [0,+\infty). \end{cases} \tag{3.25}$$

so that σ is upper semicontinuous, positively homogeneous of degree one, and non-increasing with respect to θ on $\mathbb{R}^N \times [0,+\infty)$. Moreover, $\sigma(z,\theta) \leq C|z|$ for every $(z,\theta) \in \mathbb{R}^N \times [0,+\infty)$.

4. Proof of the compactness Theorem 1.1

To prove Theorem 1.1 let $\varepsilon_n \to 0$ and let $(u_n, \rho_n) \in [W^{1,p}(\Omega; \mathbb{R}^d_+) \times L^1(\Omega;[0,+\infty))] \cap \mathcal{V}$ be such that

$$\sup_n F_{\varepsilon_n}(u_n, \rho_n) < +\infty.$$

We must see that there exist a subsequence $\{(u_{n_k}, \rho_{n_k})\}_k \subset \{(u_n,\rho_n)\}_n$ and $(u,\mu) \in [BV(\Omega;\{\alpha,\beta\}) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ such that

$$(u_{\varepsilon_k}, \rho_{\varepsilon_k}) \to (u,\mu) \quad \text{in } X(\Omega).$$

In the case $p = 1$, by (H3) any sequence with bounded energy satisfies

$$\sup_n \int_\Omega |\nabla u_n(x)|\, dx < +\infty.$$

Hence, by Poincaré's inequality, u_n is bounded in $W^{1,1}(\Omega;\mathbb{R}^d_+)$, and thus (up to a subsequence) $u_n \to u$ for some $u \in L^1(\Omega;\mathbb{R}^d)$.

For other values of p, we use the coercivity condition in (H3) given by

$$\frac{1}{C}\Big(g(u) + |\xi|^p\Big) \leq f(u,\xi,\rho)$$

where $g(u) = 0 \Leftrightarrow u \in \{\alpha, \beta\}$ and $\inf_{|u|\geq L} g(u) > 0$, for some $L > 0$. In this case, to achieve the L^1 convergence of u_n to some $u \in L^1(\Omega;\mathbb{R}^d)$ we argue as follows. By (H3), for each $n \in \mathbb{N}$, it follows by Young's inequality that

$$\begin{aligned} F_{\varepsilon_n}(u_n, \rho_n) &\geq \int_\Omega \frac{C}{\varepsilon_n}\big(g(u_n(x)) + \varepsilon_n^p |\nabla u_n(x)|^p\big)\,dx \\ &\geq C\int_\Omega (g(u_n(x)))^{\frac{1}{q}} |\nabla u_n(x)|\,dx \geq C\int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\,dx, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By the argument given in [17, Theorem 1.1], using the volume constraint $\int_\Omega u_n(x)\,dx = V_f$, we conclude that $\{u_n\}_n$ is bounded in $L^1(\Omega;\mathbb{R}^d_+)$ and equi-integrable. Thus (up to a subsequence) $u_n \to u$ for some $u \in L^1(\Omega;\mathbb{R}^d_+)$.

For any $p \geq 1$, the proof that $u \in BV(\Omega;\{\alpha,\beta\})$ and is such that $\int_\Omega u(x)\,dx = V_f$ relies on the fact that $g(u) = 0 \Leftrightarrow u \in \{\alpha,\beta\}$ and can be achieved following an argument analogous to the one used in Lemma 4.3 in [6].

As for the sequence $\{\rho_n\}$, since $\int_\Omega \rho_n(x) = V_s$ for every $n \in \mathbb{N}$ then it follows (up to a subsequence) that $\rho_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+(\Omega)$, for some $\mu \in \mathcal{M}^+(\Omega)$. Moreover,

$$\mu(\overline{\Omega}) \geq \limsup_{n\to+\infty} \int_{\overline{\Omega}} \rho_n(x)\,dx = V_s,$$

because $\overline{\Omega}$ is compact. On the other hand, setting ρ_n equal to zero outside of Ω, we obtain

$$\mu(\Omega) \leq \mu(\mathbb{R}^N) \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} \rho_n(x)\,dx = V_s,$$

because $\mathbb{R}^N$ is open. Hence $(u,\mu) \in [BV(\Omega;\{\alpha,\beta\}) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$.

5. Main result

The following lemma addresses the proof of Theorem 1.2 in the case where $(u,\mu) \in X(\Omega) \setminus \big[\big(BV(\Omega;\{\alpha,\beta\}) \times \mathcal{M}^+(\Omega)\big) \cap \mathcal{W}\big]$.

Lemma 5.1. *Let* $(u,\mu) \in X(\Omega) \setminus \big[\big(BV\big(\Omega;\{\alpha,\beta\}\big) \times \mathcal{M}^+(\Omega)\big) \cap \mathcal{W}\big]$. *Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} F_\varepsilon(u,\mu) = +\infty \tag{5.1}$$

Proof. Given $(u,\mu) \in X(\Omega) \setminus \big[\big(BV\big(\Omega;\{\alpha,\beta\}\big) \times \mathcal{M}^+(\Omega)\big) \cap \mathcal{W}\big]$ to show (5.1) it is enough to see that for every sequence $\varepsilon_n \to 0^+$ and for every sequence $(u_n,\mu_n) \subset X(\Omega)$ such that $(u_n,\mu_n) \to (u,\mu)$ in $X(\Omega)$ we have that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n,\mu_n) = +\infty.$$

Without loss of generality we can consider the case $u \in L^1(\Omega;\mathbb{R}^d_+) \setminus BV\big(\Omega;\{\alpha,\beta\}\big)$, $(u,\mu) \in \mathcal{W}$ and $(u_n,\mu_n) \to (u,\mu)$ in $X(\Omega)$ with $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d_+)$, $\mu_n = \rho_n \mathcal{L}^N \lfloor \Omega$, $\rho_n \in L^1(\Omega;[0,+\infty))$ and $(u_n,\rho_n) \in \mathcal{V}$ (otherwise there is nothing to show).

We proceed by contradiction, assuming that there exist $\varepsilon_n \to 0^+$ and (u_n,μ_n) as above and such that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n,\mu_n) = \liminf_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n \rho_n(x)\big)\,dx < +\infty.$$

However, in this case, it was shown in Theorem 1.1 that $(u, \mu) \in [BV(\Omega; \{\alpha, \beta\}) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ thus yielding the desired contradiction. □

We note that to complete the proof of Theorem 1.2 it suffices to show that

(a) Lower bound: For every $(u, \mu) \in [BV\big(\Omega; \{\alpha, \beta\}\big) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ and for all $\varepsilon_n \to 0^+$ and $\{(u_n, \rho_n)\} \subset [W^{1,p}(\Omega; \mathbb{R}^d_+) \times L^1\big(\Omega; [0, +\infty)\big)] \cap \mathcal{V}$ with $(u_n, \rho_n) \to (u, \mu)$ in $X(\Omega)$ then
$$F(u, \mu) \leq \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n, \rho_n).$$

(b) Upper bound: For every $(u, \mu) \in [BV\big(\Omega; \{\alpha, \beta\}\big) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ and every $\varepsilon_n \to 0^+$ there exists a sequence $\{(u_n, \mu_n)\} \subset [W^{1,p}(\Omega; \mathbb{R}^d_+) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ such that
$$(u_n, \mu_n) \to (u, \mu) \text{ in } X(\Omega) \qquad \text{and} \qquad \limsup_{n \to +\infty} F_{\varepsilon_n}(u_n, \mu_n) \leq F(u, \mu).$$

The proofs of properties i) and ii) can be found in Subsections 5.1 and 5.2 below.

5.1. **Lower bound.**

Proposition 5.2 (Lower bound)**.** *Let Ω be an open and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and assume that hypotheses (H1)-(H4) hold. Let $(u, \mu) \in [BV\big(\Omega; \{\alpha, \beta\}\big) \times \mathcal{M}^+(\Omega)] \cap \mathcal{W}$ and let $\varepsilon_n \to 0^+$ and $\{(u_n, \rho_n)\} \subset [W^{1,p}(\Omega; \mathbb{R}^d_+) \times L^1\big(\Omega; [0, +\infty)\big)] \cap \mathcal{V}$ be such that $(u_n, \rho_n) \to (u, \mu)$ in $X(\Omega)$, then*
$$F(u, \mu) \leq \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n, \rho_n). \tag{5.2}$$

Proof. Let (u, μ) and $\{(u_n, \rho_n)\}$ be as stated. If the right hand side of the inequality in (5.2) is infinite there is nothing to prove. Otherwise, we can extract subsequences, not relabeled, such that $u_n \to u$ $\mathcal{L}^N$-a.e. in Ω and
$$\liminf_{n \to +\infty} \frac{1}{\varepsilon_n} \int_\Omega f(u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n \rho_n(x))\, dx = \lim_{n \to +\infty} \frac{1}{\varepsilon_n} \int_\Omega f(u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n \rho_n(x))\, dx < +\infty.$$

Let E with $\mathrm{Per}_\Omega(E) < +\infty$ be such that $u = \beta \chi_E + \alpha\big(1 - \chi_E\big)$. We must show that
$$\lim_{n \to +\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n \rho_n(x)\big)\, dx \geq \int_{\Omega \cap \partial^* E} \sigma\big(\nu_u(x), \mu_0(x)\big)\, d\mathcal{H}^{N-1}(x), \tag{5.3}$$
where $\nu_u(x)$ is the inner unit normal to E at x (in the sense of Definition 2.6), and
$$\mu_0(x) := \frac{d\mu}{d\big(\mathcal{H}^{N-1} \llcorner (\Omega \cap \partial^* E)\big)}(x). \tag{5.4}$$

Set $f_n := \frac{1}{\varepsilon_n} f\big(u_n(\cdot), \varepsilon_n \nabla u_n(\cdot), \varepsilon_n \rho_n(\cdot)\big)$. Since the integrands f_n form a sequence of nonnegative functions which are bounded in $L^1\big(\Omega; [0, +\infty)\big)$, there exists a subsequence (not relabeled) and a nonnegative bounded Radon measure ζ such that
$$f_n \, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \zeta \quad \text{in } \mathcal{M}^+(\Omega). \tag{5.5}$$
Consider the nonnegative measure
$$\pi(A) := \mathcal{H}^{N-1}(A \cap \partial^* E)$$
defined over all Borel subsets $A \subset \Omega$, where $\partial^* E$ is the reduced boundary of E (see Definition 2.6). Since $\mathrm{Per}_\Omega(E) < +\infty$, we have that
$$\pi(\Omega) = \mathrm{Per}_\Omega(E) < +\infty,$$

so that π is a bounded Radon measure. Hence, using Theorem 2.3, we may decompose ζ as $\zeta = \zeta_a\pi + \zeta_s$, where ζ_a is a nonnegative π-integrable function and ζ_s is a nonnegative Radon measure with π and ζ_s mutually singular. We claim that

$$\zeta_a(x) \geq \sigma\big(\nu_u(x), \mu_0(x)\big), \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega \cap \partial^* E. \tag{5.6}$$

Assuming that (5.6) holds, we obtain

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n \rho_n(x)\big)\, dx &\geq \zeta(\Omega) \\ &\geq \int_\Omega \zeta_a(x)\, d\mathcal{H}^{N-1} \llcorner (\Omega \cap \partial^* E)(x) \\ &\geq \int_{\Omega\cap\partial^* E} \sigma\big(\nu_u(x), \mu_0(x)\big)\, d\mathcal{H}^{N-1}(x),\end{aligned}$$

which asserts (5.3).

It remains to show that (5.6) holds. Recall that, for every $x \in \partial^* E$, equalities (2.3), (2.4) and (2.5) hold, where ν_E is the generalised unit inner normal to E in the sense of Definition 2.6, which coincides on $S_u = \partial^* E$ with ν_u by Remark 2.7. Fix any such x and abbreviate $\nu := \nu_E(x) = \nu_u(x)$. In view of the Besicovitch Derivation Theorem 2.4 we can also assume that

$$\zeta_a(x) = \lim_{r\to 0^+} \frac{\zeta(Q_\nu(x,r))}{\mathcal{H}^{N-1}\big(Q_\nu(x,r)\cap \partial^* E\big)} < +\infty.$$

Choosing $r_k \to 0^+$ such that $\zeta\big(\partial Q_\nu(x, r_k)\big) = 0$ and $\mu\big(\partial Q_\nu(x, r_k)\big) = 0$, by (5.5), (2.3), we have (see, e.g., [13, Proposition 1.203]),

$$\begin{aligned}\zeta_a(x) &= \lim_{r\to 0^+} \frac{\zeta\big(Q_\nu(x,r)\big)}{r^{N-1}} \\ &= \lim_{k\to+\infty} \frac{1}{r_k^{N-1}} \lim_{n\to+\infty} \int_{Q_\nu(x,r_k)} \frac{1}{\varepsilon_n} f\big(u_n(z), \varepsilon_n \nabla u_n(z), \varepsilon_n \rho_n(z)\big)\, dz \\ &= \lim_{k,n\to+\infty} \int_{Q_\nu} \frac{r_k}{\varepsilon_n} f\big(u_n(x+r_k y), \varepsilon_n \nabla u_n(x+r_k y), \varepsilon_n \rho_n(x+r_k y)\big)\, dy \\ &= \lim_{k,n\to+\infty} \int_{Q_\nu} \frac{r_k}{\varepsilon_n} f\left(v_{n,k}(y), \frac{\varepsilon_n}{r_k} \nabla v_{n,k}(y), \frac{\varepsilon_n}{r_k} \lambda_{n,k}(y)\right) dy,\end{aligned} \tag{5.7}$$

where $v_{n,k} \in W^{1,p}(Q_\nu; \mathbb{R}^d_+)$ and $\lambda_{n,k} \in L^1(Q_\nu; [0,+\infty))$ are defined by

$$v_{n,k}(y) := u_n(x + r_k y), \qquad \lambda_{n,k}(y) := r_k \rho_n(x + r_k y).$$

Since $(u_n, \rho_n) \to (u, \mu)$ in $X(\Omega)$, we have that

$$\lim_{k,n\to+\infty} \|v_{n,k} - u_0\|_{L^1(Q_\nu;\mathbb{R}^d_+)} = 0 \qquad \text{and} \qquad \lim_{k,n\to+\infty} \int_{Q_\nu} \lambda_{n,k}(y)\, dy = \mu_0(x), \tag{5.8}$$

where μ_0 is given by (5.4) and

$$u_0(y) := \begin{cases} \beta, & \text{if } y\cdot\nu = y\cdot\nu_E(x) > 0, \\ \alpha, & \text{if } y\cdot\nu = y\cdot\nu_E(x) < 0. \end{cases}$$

Indeed,

$$\begin{aligned}
&\lim_{k,n\to+\infty}\int_{Q_\nu}|v_{n,k}(y)-u_0(y)|\,dy\\
&=\lim_{k,n\to+\infty}\left[\int_{Q_\nu\cap\{y:y\cdot\nu>0\}}|u_n(x+r_ky)-\beta|\,dy+\int_{Q_\nu\cap\{y:y\cdot\nu<0\}}|u_n(x+r_ky)-\alpha|\,dy\right]\\
&=\lim_{k\to+\infty}\left[\int_{Q_\nu\cap\{y:y\cdot\nu>0\}}|u(x+r_ky)-\beta|\,dy+\int_{Q_\nu\cap\{y:y\cdot\nu<0\}}|u(x+r_ky)-\alpha|\,dy\right]\\
&=\lim_{k\to+\infty}\frac{1}{r_k^N}\left[\int_{Q_\nu(x,r_k)\cap\{y:(y-x)\cdot\nu>0\}\setminus E}|\alpha-\beta|\,dy+\int_{Q_\nu(x,r_k)\cap\{y:(y-x)\cdot\nu<0\}\cap E}|\beta-\alpha|\,dy\right]\\
&=0,
\end{aligned}$$

where in the last equality we used (2.4) and (2.5). Furthermore, since $\rho_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+(\Omega)$, we have by (2.3)

$$\begin{aligned}
\lim_{k,n\to+\infty}\int_{Q_\nu}\lambda_{n,k}(y)\,dy &= \lim_{k,n\to+\infty}\int_{Q_\nu} r_k\rho_n(x+r_ky)\,dy\\
&=\lim_{k,n\to+\infty}\int_{Q_\nu(x,r_k)}\frac{1}{r_k^{N-1}}\rho_n(y)\,dy\\
&=\lim_{k\to+\infty}\frac{1}{r_k^{N-1}}\mu(Q_\nu(x,r_k))\\
&=\lim_{k\to+\infty}\frac{\mu\big(Q_\nu(x,r_k)\big)}{\mathcal{H}^{N-1}\big(Q_\nu(x,r)\cap\Omega\cap\partial^*E\big)}=\mu_0(x),
\end{aligned}$$

where we also used the fact that $\mu\big(\partial Q_\nu(x,r_k)\big)=0$.

By (5.7), (5.8), and using a diagonalisation argument, we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that, setting $\lambda_k := \lambda_{n_k,k}$,

$$t_k := \frac{\varepsilon_{n_k}}{r_k}\to 0,\qquad v_k := v_{n_k,k}\to u_0 \text{ in } L^1(Q_\nu;\mathbb{R}^d_+)\ ,\qquad \int_{Q_\nu}\lambda_k(y)\,dy\to\mu_0(x),$$

as $k\to+\infty$, and

$$\zeta_a(x)=\lim_{k\to+\infty}\int_{Q_\nu}\frac{1}{t_k}f\Big(v_k(y),t_k\nabla v_k(y),t_k\lambda_k(y)\Big)\,dy.$$

Applying Lemma 3.1 to the sequences $\{t_k\}$, $\{v_k\}$, and $\{\lambda_k\}$, with $\nu=\nu_E(x)=\nu_u$ and $\theta=\mu_0(x)$, we conclude that there exist a subsequence $\{k'\}$ of $\{k\}$ and a sequence $\{(w_{k'},\gamma_{k'})\}\in W^{1,p}(Q_\nu;\mathbb{R}^d_+)\times L^1\big(Q_\nu;[0,+\infty)\big)$ such that $w_{k'}\to u_0$ in $L^1(Q_\nu;\mathbb{R}^d_+)$, $(w_{k'},\gamma_{k'})\in\mathcal{A}(\nu,\mu_0(x))$, and

$$\begin{aligned}
\zeta_a(x)&=\lim_{k\to+\infty}\int_{Q_\nu}\frac{1}{t_k}f\Big(v_k(y),t_k\nabla v_k(y),t_k\lambda_k(y)\Big)\,dy\\
&\geq\limsup_{k'\to+\infty}\int_{Q_\nu}\frac{1}{t_{k'}}f\big(w_{k'}(y),t_{k'}\nabla w_{k'}(y),t_{k'}\gamma_{k'}(y)\big)dy.
\end{aligned}\tag{5.9}$$

Since $(w_{k'},\gamma_{k'})\in\mathcal{A}(\nu,\mu_0(x))$ (cf. Remark 3.2), (5.6) follows by (1.8) and (5.9). □

5.2. Upper bound.

Proposition 5.3 (Upper bound)**.** *Let Ω be an open and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Assume that hypotheses (H1)–(H6) hold. Let $(u,\mu)\in[BV\big(\Omega;\{\alpha,\beta\}\big)\times\mathcal{M}^+(\Omega)]\cap\mathcal{W}$ and let $\varepsilon_n\to0^+$.*

Then there exist $\{(u_n,\rho_n)\}\subset[W^{1,p}(\Omega;\mathbb{R}^d_+)\times L^1\big(\Omega;[0,+\infty)\big)]\cap\mathcal{V}$ *with* $(u_n,\rho_n)\to(u,\mu)$ *in* $X(\Omega)$ *such that*

$$F(u,\mu)\geq\limsup_{n\to+\infty}F_{\varepsilon_n}(u_n,\rho_n). \tag{5.10}$$

To prove the upper bound in Proposition 5.3 we begin by considering the case where, given a fixed direction $\nu\in\mathbb{S}^{N-1}$,

$$u(x):=\begin{cases}\beta & \text{if } (x-x_*)\cdot\nu>0\\ \alpha & \text{if } (x-x_*)\cdot\nu<0,\end{cases} \tag{5.11}$$

for some $x_*\in\mathbb{R}^N$, the domain is a rectangle of the form $\Omega_\nu:=\{x+t\nu\,:\,x\in H_\nu,\,|t|<r\}$ for some relatively open subset H_ν of a hyperplane orthogonal to ν and some $r>0$, and the measure $\mu\in\mathcal{M}^+(\Omega_\nu)$ has the form

$$\mu:=\theta\chi_K\mathcal{H}^{N-1}\llcorner H_\nu+a\delta_{x_0} \tag{5.12}$$

where $\theta\geq0$ is assumed to be constant, K is a relatively compact subset of H_ν, $a\in\mathbb{R}^+_0$ and $x_0\in\Omega_\nu\backslash H_\nu$.

Lemma 5.4. *Assume that (H1)–(H6) hold. Let* u *and* μ *be given as in* (5.11) *and* (5.12)*, respectively, and satisfy*

$$\int_{\Omega_\nu}u(x)\,dx=V_f,\qquad\mu(\Omega_\nu)=V_s.$$

Then, for every $\varepsilon_n\to0^+$*, there exist sequences* $\{u_n\}\subseteq W^{1,p}(\Omega_\nu;\mathbb{R}^d_+)$*,* $\{\rho_n\}\subseteq L^1\big(\Omega_\nu;[0,+\infty)\big)$ *such that*

$$\int_{\Omega_\nu}u_n(x)\,dx=V_f,\quad\int_{\Omega_\nu}\rho_n(x)\,dx=V_s,\quad(u_n,\rho_n)\to(u,\mu)\text{ in }X(\Omega_\nu)\text{ with }\|u_n-u\|_{L^1(\Omega_\nu;\mathbb{R}^d_+)}=O(\varepsilon_n) \tag{5.13}$$

and

$$\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_{\Omega_\nu}f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\rho_n(x))\,dx\leq\int_{H_\nu}\sigma\big(\nu,\theta\chi_K(x)\big)\,d\mathcal{H}^{N-1}(x)=F(u,\mu).$$

Proof. Since μ is given by (5.12), we must show that, given $\varepsilon_n\to0^+$ there exist sequences $\{u_n\}\subseteq W^{1,p}(\Omega_\nu;\mathbb{R}^d_+)$, $\{\rho_n\}\subseteq L^1\big(\Omega_\nu;[0,+\infty)\big)$ satisfying (5.13) and such that

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_{\Omega_\nu}f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\rho_n(x))\,dx\\&\qquad\leq\sigma(\nu,\theta)\mathcal{H}^{N-1}(K)+\sigma(\nu,0)\mathcal{H}^{N-1}(H_\nu\setminus K)=F(u,\mu).\end{aligned} \tag{5.14}$$

For simplicity, we assume that $x_*=0$, that $r=\frac{1}{2}$ and that $\nu=e_N$ and we denote Ω_ν by Ω, H_ν by H, and Q_ν by Q.

We fix $\eta>0$ and, by Proposition 3.3 and (1.7), choose $t_1,t_2>0$, $(w_1,\gamma_1)\in\mathcal{A}(e_N,\theta)$ and $(w_2,0)\in\mathcal{A}(e_N,0)$ such that $w_1,w_2\in L^\infty(S_{e_N};\mathbb{R}^d_+)$, $\gamma_1\in L^\infty(\mathbb{R}^N;[0,+\infty))$ and

$$\begin{aligned}&\int_Q\frac{1}{t_1}f\big(w_1(x),t_1\nabla w_1(x),t_1\gamma_1(x)\big)\,dx<\sigma(e_N,\theta)+\eta,\qquad\int_Q\gamma_1(x)\,dx=s\theta,\\&\int_Q\frac{1}{t_2}f\big(w_2(x),t_2\nabla w_2(x),0\big)\,dx<\sigma(e_N,0)+\eta.\end{aligned} \tag{5.15}$$

for some $s\in[0,1]$.

We extend w_1 and w_2 to the whole space $\mathbb{R}^N$ by setting $w_i(x)=\alpha$ if $x\cdot\nu=x_N\leq-1/2$ and $w_i(x)=\beta$ if $x\cdot\nu=x_N\geq1/2$, for $i=1,2$. We recall that $w_i(\cdot,x_N)$ and $\gamma_1(\cdot,x_N)$ are periodic functions with period one.

For every fixed $\delta > 0$, let $K_\delta \subset H$ be such that $\overline{K} \subset K_\delta$, $\mathcal{H}^{N-1}(K_\delta \setminus K) = O(\delta)$ and choose a cut-off function $\varphi \in C_0^\infty(H; [0,1])$ such that $\varphi \equiv 1$ in K, $\varphi \equiv 0$ in $H \setminus K_\delta$, and $\|\nabla\varphi\|_\infty \le \frac{C}{\delta}$ with C independent of δ. Setting $t_0 := \min\{t_1, t_2\}$, we now define $v_n \in W^{1,p}(\Omega; \mathbb{R}^d_+)$ and $\lambda_n \in L^1(\Omega; [0, +\infty))$ by

$$v_n(x) := \begin{cases} \varphi(x')w_1\left(\dfrac{t_1 x}{\varepsilon_n}\right) + \big(1 - \varphi(x')\big)w_2\left(\dfrac{t_2 x}{\varepsilon_n}\right) & \text{if } x = (x', x_N) \in H \times \left(-\dfrac{\varepsilon_n}{2t_0}, \dfrac{\varepsilon_n}{2t_0}\right), \\ u(x) & \text{if } x \in \Omega \text{ and } |x_N| \ge \dfrac{\varepsilon_n}{2t_0}, \end{cases} \tag{5.16}$$

where, denoting by ω_N the measure of the N-dimensional unit ball,

$$\lambda_n(x) := \begin{cases} \dfrac{t_1}{\varepsilon_n}\gamma_1\left(\dfrac{t_1 x}{\varepsilon_n}\right) & \text{if } x = (x', x_N) \in K \times \left(-\dfrac{\varepsilon_n}{2t_1}, \dfrac{\varepsilon_n}{2t_1}\right), \\ \dfrac{a}{\omega_N\sqrt{\varepsilon_n}} & \text{if } x \in B\big(x_0, \varepsilon_n^{\frac{1}{2N}}\big), \\ \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} & \text{if}\, x = (x', x_N) \in K \times \left(-\dfrac{\varepsilon_n + \sqrt{\varepsilon_n}}{2t_1}, \dfrac{\varepsilon_n + \sqrt{\varepsilon_n}}{2t_1}\right) \setminus \left(-\dfrac{\varepsilon_n}{2t_1}, \dfrac{\varepsilon_n}{2t_1}\right), \\ 0 & \text{elsewhere}, \end{cases}$$

with $B\big(x_0, \varepsilon_n^{\frac{1}{2N}}\big) \subset \{|x_N| > \frac{\varepsilon_n}{2t_0}\}$. Notice that $\|v_n\|_\infty \le \|w_1\|_\infty + \|w_2\|_\infty$, for all $n \in \mathbb{N}$.

We claim that

$$\lim_{n\to+\infty} \int_\Omega \lambda_n(x)\, dx = \mu(\Omega). \tag{5.17}$$

Indeed, by the Riemann-Lebesgue Lemma (see Lemma 2.1) and (5.15),

$$\begin{aligned} \lim_{n\to+\infty} \int_\Omega \lambda_n(x)\, dx &= \lim_{n\to+\infty} \int_{-\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n}{2t_1}} \int_K \frac{t_1}{\varepsilon_n}\gamma_1\left(\frac{t_1 x'}{\varepsilon_n}, \frac{t_1 x_N}{\varepsilon_n}\right) dx'\, dx_N + a + (1-s)\theta\mathcal{H}^{N-1}(K) \\ &= \lim_{n\to+\infty} \int_{-1/2}^{1/2} \int_K \gamma_1\left(\frac{t_1 y'}{\varepsilon_n}, y_N\right) dy'\, dy_N + a + (1-s)\theta\mathcal{H}^{N-1}(K) \\ &= \mathcal{H}^{N-1}(K) \int_{-1/2}^{1/2} \int_{Q'} \gamma_1(y', y_N)\, dy'\, dy_N + a + (1-s)\theta\mathcal{H}^{N-1}(K) \\ &= s\theta\mathcal{H}^{N-1}(K) + (1-s)\theta\mathcal{H}^{N-1}(K) + a = \mu(\Omega), \end{aligned}$$

where Q' is the projection of Q on $\mathbb{R}^{N-1}$, i.e., $Q' := \{x' \in \mathbb{R}^{N-1} : (x', 0) \in Q\}$. This proves (5.17). Therefore, it is possible to choose a normalization constant $c_n \to 1$ in such a way that, setting $\rho_n := c_n\lambda_n$, we have

$$\int_\Omega \rho_n(x)\, dx = \mu(\Omega), \qquad \text{for all } n \in \mathbb{N}.$$

We also claim that $\rho_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+(\Omega)$. Indeed, let $\psi \in C_0(\Omega)$ and $\varepsilon > 0$. A direct computation yields

$$\begin{aligned} \int_\Omega \psi(x)\lambda_n(x)\, dx \quad = \quad & \int_{-\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n}{2t_1}} \int_K \frac{t_1}{\varepsilon_n}\psi(x', x_N)\gamma_1\left(\frac{t_1 x'}{\varepsilon_n}, \frac{t_1 x_N}{\varepsilon_n}\right) dx'\, dx_N \\ & + \fint_{B\left(x_0, \varepsilon_n^{\frac{1}{2N}}\right)} a\psi(x)\, dx + \int_{-\frac{\varepsilon_n + \sqrt{\varepsilon_n}}{2t_1}}^{-\frac{\varepsilon_n}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \psi(x', x_N)\, dx'\, dx_N \end{aligned}$$

$$+\int_{\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \psi(x',x_N)\,dx'\,dx_N$$

$$= \int_{-1/2}^{1/2}\int_K \left[\psi\left(x',\frac{\varepsilon_n}{t_1}x_N\right)-\psi(x',0)\right]\gamma_1\left(\frac{t_1x'}{\varepsilon_n},x_N\right)dx'\,dx_N$$

$$+\int_{-1/2}^{1/2}\int_K \psi(x',0)\gamma_1\left(\frac{t_1x'}{\varepsilon_n},x_N\right)dx'\,dx_N + \fint_{B\left(x_0,\varepsilon_n^{\frac{1}{2N}}\right)} a\psi(x)\,dx$$

$$+\int_{-\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}}^{-\frac{\varepsilon_n}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \left[\psi(x',x_N)-\psi(x',0)\right]\,dx'\,dx_N$$

$$+\int_{\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \left[\psi(x',x_N)-\psi(x',0)\right]\,dx'\,dx_N$$

$$+\int_{-\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}}^{-\frac{\varepsilon_n}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \psi(x',0)\,dx'\,dx_N$$

$$+\int_{\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}} \frac{t_1(1-s)\theta}{\sqrt{\varepsilon_n}} \int_K \psi(x',0)\,dx'\,dx_N$$

$$=: I_n^1+I_n^2+I_n^3+I_n^4+I_n^5+I_n^6+I_n^7.$$

Choose $n_\varepsilon\in\mathbb{N}$ such that

$$\left|\psi\Big(x',\frac{\varepsilon_n}{t_1}x_N\Big)-\psi\big(x',0\big)\right|<\varepsilon \qquad \text{for every } x'\in H,\ |x_N|<r, \quad \text{and every } n>n_\varepsilon,$$

and such that

$$\left|\psi\Big(x',x_N\Big)-\psi\big(x',0\big)\right|<\varepsilon \qquad \text{for every } x'\in H,\ |x_N|<\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}, \quad \text{and every } n>n_\varepsilon.$$

Then, for every $n>n_\varepsilon$,

$$\left|I_n^1\right|<\varepsilon\int_{-1/2}^{1/2}\int_K\gamma_1\left(\frac{t_1x'}{\varepsilon_n},x_N\right)dx'\,dx_N.$$

By the Riemann-Lebesgue Lemma and (5.15) we get

$$\limsup_{n\to+\infty} I_n^1\le\varepsilon\, s\theta\mathcal{H}^{N-1}(K),$$

and also, since $K\subset\{x_N=0\}$,

$$\lim_{n\to+\infty} I_n^2 = s\theta\int_K\psi(x',0)\,dx' = s\theta\int_K\psi(x)\,d\mathcal{H}^{N-1}(x').$$

Similarly, we conclude that

$$\limsup_{n\to+\infty} I_n^4=O(\varepsilon),\qquad \limsup_{n\to+\infty} I_n^5=O(\varepsilon),$$

and that

$$\lim_{n\to+\infty} I_n^6+\lim_{n\to+\infty} I_n^7=(1-s)\theta\int_K\psi(x',0)\,dx'=(1-s)\theta\int_K\psi(x)\,d\mathcal{H}^{N-1}(x').$$

Hence, letting $\varepsilon \to 0^+$, we obtain

$$\lim_{n\to+\infty}\int_\Omega \psi(x)\lambda_n(x)\,dx = s\theta\int_K \psi(x)\,d\mathcal{H}^{N-1}(x') + (1-s)\theta\int_K \psi(x)\,d\mathcal{H}^{N-1}(x') + a\psi(x_0) = \int_\Omega \psi(x)\,d\mu(x),$$

which, since $c_n \to 1$, shows that $\rho_n = c_n\lambda_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+(\Omega)$.

We next prove the convergence in $L^1(\Omega;\mathbb{R}^d_+)$ of v_n to u. In fact, we can prove that

$$\|v_n - u\|_{L^1(\Omega;\mathbb{R}^d_+)} = O(\varepsilon_n). \tag{5.18}$$

Indeed, using a change of variable in the last coordinate and the periodicity of w_1 and w_2 in the first $N-1$ coordinates, we have for n sufficiently large

$$\begin{aligned}
&\|v_n\|_{L^1(\{x\in\Omega:\,|x_N|\le \frac{\varepsilon_n}{2t_0}\};\mathbb{R}^d_+)}\\
&= \frac{\varepsilon_n}{t_0}\int_{-1/2}^{1/2}\int_H \left|\varphi(y')w_1\left(\frac{t_1y'}{\varepsilon_n},\frac{t_1y_N}{t_0}\right) + \big(1-\varphi(y')\big)w_2\left(\frac{t_2y'}{\varepsilon_n},\frac{t_2y_N}{t_0}\right)\right| dy'\,dy_N\\
&\le \varepsilon_n\mathcal{H}^{N-1}(H)\left(\frac{1}{t_1}\|w_1\|_{L^1(\frac{t_1}{t_0}Q;\mathbb{R}^d_+)} + \frac{1}{t_2}\|w_2\|_{L^1(\frac{t_2}{t_0}Q;\mathbb{R}^d_+)} + 1\right),
\end{aligned}\tag{5.19}$$

and thus

$$\|v_n - u\|_{L^1(\Omega;\mathbb{R}^d_+)} \le \|v_n\|_{L^1(\{x\in\Omega:\,|x_N|\le\frac{\varepsilon_n}{2t_0}\};\mathbb{R}^d_+)} + C\,\mathcal{L}^N\left(\left\{x\in\Omega:\,|x_N|\le\frac{\varepsilon_n}{2t_0}\right\}\right) = O(\varepsilon_n).$$

To meet the first constraint in (5.13), we will modify the sequence $\{v_n\}$, defined above, to obtain a new sequence $\{u_n\}$ converging to u in $L^1(\Omega;\mathbb{R}^d_+)$ such that (5.13) and (5.14) are satisfied. We define

$$u_n := v_n + b_n \quad\text{where}\quad b_n := \fint_\Omega u(x) - v_n(x)\,dx.$$

It is clear that $b_n \to 0$, $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d_+)$ and that the first constraint in (5.13) holds. To prove (5.14), we first observe that by (5.18)

$$|b_n| \le \fint_\Omega \big|v_n(x) - u(x)\big|\,dx = O(\varepsilon_n). \tag{5.20}$$

It follows that $\|u_n - u\|_{L^1(\Omega;\mathbb{R}^d_+)} = O(\varepsilon_n)$. We now proceed by estimating $E_{\varepsilon_n}(u_n,\lambda_n)$. Since, by construction,

$$B\big(x_0,\varepsilon_n^{\frac{1}{2N}}\big) \subset \big\{|x_N| > \frac{\varepsilon_n}{2t_0}\big\}$$

and since, by definition, $v_n(x)\in\{\alpha,\beta\}$ whenever $|x_N|\ge\frac{\varepsilon_n}{2t_0}$, $v_n(x) = w_1\left(\frac{t_1x}{\varepsilon_n}\right)\in\{\alpha,\beta\}$ whenever $x'\in K$ and $|x_N|\ge\frac{\varepsilon_n}{2t_1}$, $v_n(x) = w_2\left(\frac{t_2x}{\varepsilon_n}\right)\in\{\alpha,\beta\}$ whenever $x'\in H\setminus K_\delta$ and $|x_N|\ge\frac{\varepsilon_n}{2t_2}$, and $\lambda_n(x) = 0$ whenever $x\notin\left(K\times\big(-\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1},\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}\big)\right)\cup B\big(x_0,\varepsilon_n^{\frac{1}{2N}}\big)$, in view of (H2) we can write,

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\int_\Omega f\big(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\lambda_n(x)\big)\,dx\\
&= \int_{-\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n}{2t_1}}\int_K \frac{1}{\varepsilon_n} f\left(w_1\left(\frac{t_1x'}{\varepsilon_n},\frac{t_1x_N}{\varepsilon_n}\right) + b_n, t_1\nabla w_1\left(\frac{t_1x'}{\varepsilon_n},\frac{t_1x_N}{\varepsilon_n}\right), t_1\gamma_1\left(\frac{t_1x'}{\varepsilon_n},\frac{t_1x_N}{\varepsilon_n}\right)\right) dx'\,dx_N\\
&\quad + \int_{-\frac{\varepsilon_n}{2t_2}}^{\frac{\varepsilon_n}{2t_2}}\int_{H\setminus K_\delta} \frac{1}{\varepsilon_n} f\left(w_2\left(\frac{t_2x'}{\varepsilon_n},\frac{t_2x_N}{\varepsilon_n}\right) + b_n, t_2\nabla w_2\left(\frac{t_2x'}{\varepsilon_n},\frac{t_2x_N}{\varepsilon_n}\right), 0\right) dx'\,dx_N
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}\frac{1}{\varepsilon_n}f\big(v_n\left(x',x_N\right)+b_n,\varepsilon_n\nabla v_n\left(x',x_N\right),0\big)\,dx'\,dx_N\\
&+\int_{B\left(x_0,\varepsilon_n^{\frac{1}{2N}}\right)}\frac{1}{\varepsilon_n}f\left(u(x)+b_n,0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N}\right)\,dx\\
&+\frac{1}{\varepsilon_n}\int_{-\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}}^{-\frac{\varepsilon_n}{2t_1}}\int_K f(u(x)+b_n,0,\sqrt{\varepsilon_n}t_1(1-s)\theta)\,dx'\,dx_N\\
&+\frac{1}{\varepsilon_n}\int_{\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n+\sqrt{\varepsilon_n}}{2t_1}}\int_K f(u(x)+b_n,0,\sqrt{\varepsilon_n}t_1(1-s)\theta)\,dx'\,dx_N\\
&=E_n^{(1)}+E_n^{(2)}+E_n^{(3)}+E_n^{(4)}+E_n^{(5)}+E_n^{(6)}. \qquad (5.21)
\end{aligned}
$$

By a change of variable, Corollary 2.2 (hypothesis (H3) and Fatou's Lemma guarantee that the sequence $f_n(y')=f\big((w_1+b_n)(y',y_N),t_1\nabla w_1(y',y_N),t_1\gamma_1(y',y_N)\big)$ satisfies the required conditions) and (5.15) we have

$$
\begin{aligned}
&\limsup_{n\to+\infty}E_n^{(1)}\\
&=\limsup_{n\to+\infty}\int_{-1/2}^{1/2}\int_K\frac{1}{t_1}f\left(w_1\left(\frac{t_1y'}{\varepsilon_n},y_N\right)+b_n,t_1\nabla w_1\left(\frac{t_1y'}{\varepsilon_n},y_N\right),t_1\gamma_1\left(\frac{t_1y'}{\varepsilon_n},y_N\right)\right)\,dy'\,dy_N\\
&=\mathcal{H}^{N-1}(K)\int_{-1/2}^{1/2}\int_{Q'}\frac{1}{t_1}f\left(w_1(y',y_N),t_1\nabla w_1(y',y_N),t_1\gamma_1(y',y_N)\right)\,dy'\,dy_N\\
&<\big(\sigma(e_N,\theta)+\eta\big)\,\mathcal{H}^{N-1}(K).
\end{aligned} \qquad (5.22)
$$

Analogously, a similar reasoning yields

$$
\begin{aligned}
\limsup_{n\to+\infty}E_n^{(2)}&=\limsup_{n\to+\infty}\int_{-1/2}^{1/2}\int_{H\setminus K_\delta}\frac{1}{t_2}f\left(w_2\left(\frac{t_2y'}{\varepsilon_n},y_N\right)+b_n,t_2\nabla w_2\left(\frac{t_2y'}{\varepsilon_n},y_N\right),0\right)\,dy'\,dy_N\\
&=\mathcal{H}^{N-1}(H\setminus K_\delta)\int_{-1/2}^{1/2}\int_{Q'}\frac{1}{t_2}f\left(w_2(y',y_N),t_2\nabla w_2(y',y_N),0\right)\,dy'\,dy_N\\
&<\big(\sigma(e_N,0)+\eta\big)\,\mathcal{H}^{N-1}(H\setminus K_\delta).
\end{aligned} \qquad (5.23)
$$

By (H3), the definition of v_n and the triangle inequality, we have

$$
\begin{aligned}
E_n^{(3)}&=\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}\frac{1}{\varepsilon_n}f\big(v_n\left(x',x_N\right)+b_n,\varepsilon_n\nabla v_n\left(x',x_N\right),0\big)\,dx'\,dx_N\\
&\leq\frac{C}{\varepsilon_n}\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}\big(h(v_n(x',x_N)+b_n,0)+\varepsilon_n^p|\nabla v_n(x)|^p\big)\,dx'\,dx_N\\
&\leq\frac{C}{\varepsilon_n}\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}\Bigg[h(v_n(x',x_N)+b_n,0)+t_1^p\left|\nabla w_1\left(\frac{t_1x}{\varepsilon_n}\right)\right|^p+t_2^p\left|\nabla w_2\left(\frac{t_2x}{\varepsilon_n}\right)\right|^p\\
&\qquad+\varepsilon_n^p|\nabla\varphi(x')|^p\left|w_1\left(\frac{t_1x}{\varepsilon_n}\right)-w_2\left(\frac{t_2x}{\varepsilon_n}\right)\right|^p\Bigg]\,dx'\,dx_N.
\end{aligned} \qquad (5.24)
$$

First notice that, since $w_1, w_2 \in W^{1,p}(Q;\mathbb{R}^d_+)$,

$$\begin{aligned}&\limsup_{n\to+\infty} C\,\varepsilon_n^{p-1}\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}|\nabla\varphi(x')|^p\left|w_1\left(\frac{t_1x}{\varepsilon_n}\right)-w_2\left(\frac{t_2x}{\varepsilon_n}\right)\right|^p dx'\,dx_N\\
&=\limsup_{n\to+\infty}\frac{C}{t_0}\,\varepsilon_n^p\int_{-\frac12}^{\frac12}\int_{K_\delta\setminus K}|\nabla\varphi(x')|^p\left|w_1\left(\frac{t_1x'}{\varepsilon_n},x_N\right)-w_2\left(\frac{t_2x'}{\varepsilon_n},x_N\right)\right|^p dx'\,dx_N=0.\end{aligned}$$

Also, since $b_n = O(\varepsilon_n)$ by (5.20), as v_n is uniformly bounded in L^∞ and $h \in L^\infty_{\rm loc}$, we have

$$\begin{aligned}\limsup_{n\to+\infty}\frac{C}{\varepsilon_n}\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}h(v_n(x)+b_n,0)\,dx &\le \limsup_{n\to+\infty} C\varepsilon_n^{-1}\mathcal{L}^N\left((K_\delta\setminus K)\times\left[-\frac{\varepsilon_n}{2t_0},\frac{\varepsilon_n}{2t_0}\right]\right)\\
&\le CH^{N-1}(K_\delta\setminus K)=O(\delta).\end{aligned}$$

The two remaining terms on the right hand side of the last equality in (5.24) are also $O(\delta)$ since they can be treated by the usual change of variables and the Riemann-Lebesgue Lemma. For example,

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{-\frac{\varepsilon_n}{2t_0}}^{\frac{\varepsilon_n}{2t_0}}\int_{K_\delta\setminus K}\frac{t_1^p}{\varepsilon_n}\left|\nabla w_1\left(\frac{t_1x'}{\varepsilon_n},\frac{t_1x_N}{\varepsilon_n}\right)\right|^p dx'\,dx_N\\
&=\limsup_{n\to+\infty}\int_{-\frac{\varepsilon_n}{2t_1}}^{\frac{\varepsilon_n}{2t_1}}\int_{K_\delta\setminus K}\frac{t_1^p}{\varepsilon_n}\left|\nabla w_1\left(\frac{t_1x'}{\varepsilon_n},\frac{t_1x_N}{\varepsilon_n}\right)\right|^p dx'\,dx_N\\
&=\limsup_{n\to+\infty} t_1^{p-1}\int_{-1/2}^{1/2}\int_{K_\delta\setminus K}\left|\nabla w_1\left(\frac{t_1y'}{\varepsilon_n},y_N\right)\right|^p dy'\,dy_N\\
&=\mathcal{H}^{N-1}(K_\delta\setminus K)\,t_1^{p-1}\int_Q|\nabla w_1(y)|^p\,dy=O(\delta).\end{aligned}$$

Thus, we obtain

$$\limsup_{n\to+\infty}E_n^{(3)}\le C\mathcal{H}^{N-1}(K_\delta\setminus K)=O(\delta),\tag{5.25}$$

where C depends on t_1, t_2, the L^∞-norms of w_1, w_2 and the L^p-norms of $\nabla w_1, \nabla w_2$. Using (H5) and (H6) we conclude that

$$\begin{aligned}\limsup_{n\to+\infty}E_n^{(4)}&=\limsup_{n\to+\infty}\int_{B\left(x_0,\varepsilon_n^{\frac{1}{2N}}\right)}\frac{1}{\varepsilon_n}f\left(u(x)+b_n,0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N}\right)dx\\
&\le\limsup_{n\to+\infty}\int_{B\left(x_0,\varepsilon_n^{\frac{1}{2N}}\right)}\frac{1}{\varepsilon_n}\left[f\left(u(x)+b_n,0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N}\right)-f\left(u(x),0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N}\right)\right]dx\\
&+\limsup_{n\to+\infty}\int_{B\left(x_0,\varepsilon_n^{\frac{1}{2N}}\right)}\frac{1}{\varepsilon_n}f\left(u(x),0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N}\right)dx\\
&\le\limsup_{n\to+\infty}\omega_N\sqrt{\varepsilon_n}\frac{C}{\varepsilon_n}b_n+\limsup_{n\to+\infty}\omega_N\sqrt{\varepsilon_n}\,\frac{1}{\varepsilon_n}f(u,0,\sqrt{\varepsilon_n}\,\frac{a}{\omega_N})=0,\end{aligned}\tag{5.26}$$

where $u=\alpha$ or $u=\beta$, depending on the location of the ball $B\big(x_0,\varepsilon_n^{\frac{1}{2N}}\big)$. By a similar reasoning, we conclude that

$$\limsup_{n\to+\infty}E_n^{(5)}=0\ \text{ and }\ \limsup_{n\to+\infty}E_n^{(6)}=0.\tag{5.27}$$

Combining (5.21), (5.22), (5.23), (5.25), (5.26) and (5.27), we obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\lambda_n(x))\,dx\\
&\le\big(\sigma(e_N,\theta)+\eta\big)\,\mathcal{H}^{N-1}(K)+\big(\sigma(e_N,0)+\eta\big)\,\mathcal{H}^{N-1}(H\setminus K)+O(\delta)<+\infty.\end{aligned}\tag{5.28}$$

By (H3) and (5.28)

$$\lim_{n\to+\infty}\int_\Omega \varepsilon_n^{p-1}|\nabla u_n(x)|^p\,dx < +\infty.$$

Thus, by (H4), using the fact that $c_n \to 1$ and since u_n and $\varepsilon_n\lambda_n$ are uniformly bounded in L^∞ and λ_n is bounded in L^1, we have

$$\lim_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\lambda_n(x))\,dx = \lim_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\rho_n(x))\,dx. \quad (5.29)$$

Hence, by (5.29) and (5.28), we obtain

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}&\int_\Omega f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\rho_n(x))\,dx\\ &\le \big(\sigma(e_N,\theta)+\eta\big)\,\mathcal{H}^{N-1}(K)+\big(\sigma(e_N,0)+\eta\big)\,\mathcal{H}^{N-1}(H\setminus K)+O(\delta),\end{aligned}$$

and thus, due to the arbitrariness of η and δ, we finally get (5.14), i.e,

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n\rho_n(x))\,dx &\le \sigma(e_N,\theta)\mathcal{H}^{N-1}(K)+\sigma(e_N,0)\mathcal{H}^{N-1}(H\setminus K)\\ &= F(u,\mu).\end{aligned}$$

□

Remark 5.5. The statement of Lemma 5.4 holds trivially in the case where u and Ω_ν are as before and

$$\mu := \theta\chi_K\mathcal{H}^{N-1}\llcorner H_\nu + \sum_{j=1}^k c_j\delta_{x_j},$$

with $x_j \in \Omega_\nu \setminus H_\nu$ and $c_j \in \mathbb{R}_0^+$. Following the procedure in [6], this result can be generalised for

$$u = \chi_E\beta + (1-\chi_E)\alpha,$$

where $E = E'\cap\Omega$ with E' a polyhedral set. The idea is to use an induction argument on the number of flat interfaces corresponding to $S_u\cap\Omega$, taking v_n to be a convolution of a convex combination of α and β and $\lambda_n = 0$ around the "edges", and (H2) and (H3). The compliance with the volume constraints follows from (H3) and (H6).

It is easy to show that the same result holds for u as above and

$$\mu := \theta\mathcal{H}^{N-1}\llcorner S_u + \sum_{j=1}^k c_j\delta_{x_j}$$

with $\theta \ge 0$ piecewise constant. In fact, by the assumption on μ, there exist a finite collection of pairwise disjoint relatively compact subsets $K_1,\dots,K_l \subset H_\nu$, and positive constants $\theta_1,\dots,\theta_l$ such that

$$\theta_{|K_i} = \theta_i \qquad\text{and}\qquad \theta = 0 \text{ on } H_\nu\setminus\bigcup_{i=1}^l K_i.$$

Since the sets K_i are pairwise disjoint relatively compact sets in H_ν, the construction of the recovery sequence can be localized near each set K_i so this case can be reduced to the one where θ is constant in a set $K \subset H_\nu$.

To complete the proof of the upper bound inequality (5.10) for the general case we will rely on a lower semicontinuity argument as in [14]. Namely, since $X(\Omega)$ is not metrisable and it is not clear a priori that the functional $\Gamma\big(X(\Omega)\big)-\limsup_{n\to+\infty} F_{\varepsilon_n}$ is lower semicontinuous with respect to the topology of this space, we introduce a new family of auxiliary functionals as follows. Given $M>0$ we consider

$$X_M(\Omega) := \{(u,\mu)\in X(\Omega):\ \mu(\Omega)\le M\}$$

endowed with the convergence inherited from $X(\Omega)$ and we define

$$\overline{F}_M(u,\mu) := \begin{cases} \Gamma\big(X_M(\Omega)\big)-\limsup_{n\to+\infty} F_{\varepsilon_n}(u,\mu) & \text{if } (u,\mu)\in X_M(\Omega),\\ \\ +\infty & (u,\mu)\in X(\Omega)\setminus X_M(\Omega), \end{cases} \tag{5.30}$$

where, for every $(u,\mu)\in X_M(\Omega)$,

$$\Gamma\big(X_M(\Omega)\big)-\limsup_{n\to+\infty} F_{\varepsilon_n}(u,\mu) := \inf\left\{\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n,\mu_n):\ (u_n,\mu_n)\to(u,\mu)\ \text{ in } X_M(\Omega)\right\}.$$

The advantage of considering this family of functionals is that from the metrisability of $X_M(\Omega)$ (see Theorem A.56 in [13]) $\overline{F}_M$ is sequentially lower semicontinuous with respect to the convergence in $X(\Omega)$.

Since it is clear that

$$\Gamma\big(X(\Omega)\big)-\limsup_{n\to+\infty} F_{\varepsilon_n} \le \overline{F}_M,\ \ \forall M>0,$$

to complete the proof of the upper bound (5.10) it suffices to show that

$$\overline{F}_M(u,\mu)\le F(u,\mu),\ \forall M>0,\ \forall (u,\mu)\in X_M(\Omega)\ \text{ such that }\ u\in BV(\Omega;\{\alpha,\beta\}). \tag{5.31}$$

We point out that (5.31) has already been proved for every pair (u,μ) satisfying the conditions of Lemma 5.4 or, more precisely, the conditions in Remark 5.5, and such that $\mu(\Omega)\le M$. We now address the general case.

Proof of Proposition 5.3.

Step 1. We begin by considering the case where $u=\chi_E\beta+(1-\chi_E)\alpha$, with E an open set such that $E=E'\cap\Omega$ where E' is a polyhedral set, and

$$\mu = g\mathcal{H}^{N-1}\llcorner S_u + \sum_{j=1}^k c_j\delta_{x_j},$$

with $g:\Omega\to\mathbb{R}$ continuous. Let $\{g_n\}$ be a sequence of piecewise constant functions that converge to g in $L^p(S_u;\mathcal{H}^{N-1})$, $\forall p\ge 1$ and such that

$$\int_{S_u} g_n(x)\,d\mathcal{H}^{N-1}(x) = \int_{S_u} g(x)\,d\mathcal{H}^{N-1}(x),\ \ \forall n\in\mathbb{N} \tag{5.32}$$

and set

$$\mu_n = g_n\mathcal{H}^{N-1}\llcorner S_u + \sum_{j=1}^k c_j\delta_{x_j}.$$

By (5.32) we clearly have that $\mu_n(\Omega)=\mu(\Omega)$, $\forall n\in\mathbb{N}$ and $\mu_n \overset{*}{\rightharpoonup} \mu$. Let $M\ge\mu(\Omega)$. By Remark 5.5, the lower semicontinuity of $\overline{F}_M$, the upper semicontinuity of σ (cf. Proposition 3.3) and Fatou's

Lemma, we have that

$$\begin{aligned}
\overline{F}_M(u,\mu) &\leq \liminf_{n\to+\infty} \overline{F}_M(u,\mu_n) \\
&\leq \liminf_{n\to+\infty} \int_{S_u} \sigma(\nu_u(x), g_n(x))\, d\mathcal{H}^{N-1}(x) \\
&\leq \int_{S_u} \limsup_{n\to+\infty} \sigma(\nu_u(x), g_n(x))\, d\mathcal{H}^{N-1}(x) \\
&\leq \int_{S_u} \sigma(\nu_u(x), g(x))\, d\mathcal{H}^{N-1}(x) = F(u,\mu).
\end{aligned}$$

Step 2. We consider now the case where $u = \beta\chi_{A\cap\Omega} + \alpha(1-\chi_{A\cap\Omega})$, for an arbitrary set A of finite perimeter in Ω and with μ as in the previous step. By Theorem 2.8 we can consider a sequence $\{A_n\}$ of polyhedral sets such that

$$\chi_{A_n} \to \chi_A \text{ in } L^1(\mathbb{R}^N), \ \ \mathrm{Per}_\Omega(A_n) \to \mathrm{Per}_\Omega(A) \ \text{ and } \ \mathcal{L}^N(\Omega\cap A_n) = \mathcal{L}^N(\Omega\cap A).$$

Define

$$\mu_n := g\mathcal{H}^{N-1} \llcorner \partial^* A_n + t_n \sum_{j=1}^k c_j \delta_{x_j} \ \text{ and } \ u_n = \beta\chi_{A_n\cap\Omega} + \alpha(1-\chi_{A_n\cap\Omega})$$

where t_n are chosen so that $\mu_n(\Omega) = \mu(\Omega)$ and notice that $\displaystyle\int_\Omega u_n(x)\,dx = \int_\Omega u(x)\,dx$, so the volume constraints are satisfied. Then $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d_+)$ and, by Reshetnyak's Theorem (see [3]), we have that, for every $\psi \in C(\Omega)$,

$$\int_\Omega \psi(x) g(x)\, d\mathcal{H}^{N-1} \llcorner \partial A_n^*(x) \to \int_\Omega \psi(x) g(x)\, d\mathcal{H}^{N-1} \llcorner \partial^* A(x),$$

and so we conclude that $\mu_n \overset{*}{\rightharpoonup} \mu$ and $t_n \to 1$.

As σ is upper semicontinuous and satisfies $\sigma(z,\theta) \leq C|z|$ for every $(z,\theta) \in \mathbb{R}^N \times [0,+\infty)$ (cf. (3.25)), there exists a non increasing sequence $\{\phi_m\}$ of continuous functions $\phi_m : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ such that, for every $(z,\theta) \in \mathbb{R}^N \times [0,+\infty)$,

$$\sigma(z,\theta) \leq \phi_m(z,\theta) \leq C|z| \quad \text{and} \quad \sigma(z,\theta) = \inf_{m\in\mathbb{N}} \phi_m(z,\theta). \tag{5.33}$$

Thus, from the previous step and again by Reshetnyak's Theorem, we obtain that

$$\begin{aligned}
\overline{F}_M(u,\mu) &\leq \liminf_{n\to+\infty} \overline{F}_M(u_n,\mu_n) \\
&\leq \liminf_{n\to+\infty} \int_\Omega \sigma(\nu_{u_n}(x), g(x))\, d\mathcal{H}^{N-1} \llcorner \partial A_n^*(x) \\
&\leq \liminf_{n\to+\infty} \int_\Omega \phi_m(\nu_{u_n}(x), g(x))\, d\mathcal{H}^{N-1} \llcorner \partial A_n^*(x) \\
&= \int_\Omega \phi_m(\nu_u(x), g(x))\, d\mathcal{H}^{N-1} \llcorner \partial^* A(x).
\end{aligned}$$

for any $m \in \mathbb{N}$. Passing to the limit in m, by Lebesgue's Monotone convergence Theorem and (5.33) we obtain

$$\overline{F}_M(u,\mu) \leq \int_\Omega \sigma(\nu_u(x), g(x))\, d\mathcal{H}^{N-1} \llcorner \partial^* A(x) = F(u,\mu).$$

Step 3. Finally we consider the general case where $u = \beta\chi_{A\cap\Omega} + \alpha(1-\chi_{A\cap\Omega})$ for an arbitrary set A of finite perimeter in Ω, and where $\mu \in \mathcal{M}^+(\Omega)$ is arbitrary. We define the sequence of measures

$$\mu_n := g_n \mathcal{H}^{N-1} \lfloor \partial^* A + \sum_{j=1}^{k_n} c_j^n \delta_{x_j^n},$$

where $g_n : \Omega \to \mathbb{R}$ are continuous,

$$g_n \to \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u} \text{ in } L^1(S_u; \mathcal{H}^{N-1}),$$

$$\sum_{j=1}^{k_n} c_j^n \delta_{x_j^n} \overset{*}{\rightharpoonup} \mu - \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u} \mathcal{H}^{N-1} \lfloor \partial^* A,$$

and $\mu_n(\Omega) = \mu(\Omega)$. Clearly $\mu_n \overset{*}{\rightharpoonup} \mu$ and, extracting a subsequence if necessary, we may assume that $g_n(x) \to \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}(x)$ for a.e. $x \in S_u$. Hence, by Step 2, the upper semicontinuity of σ and Fatou's Lemma, we conclude that

$$\begin{aligned}
\overline{F}_M(u,\mu) &\leq \liminf_{n\to+\infty} \overline{F}_M(u,\mu_n) \\
&\leq \liminf_{n\to+\infty} \int_{S_u} \sigma(\nu_u(x), g_n(x))\, d\mathcal{H}^{N-1}(x) \\
&\leq \int_{S_u} \limsup_{n\to+\infty} \sigma(\nu_u(x), g_n(x))\, d\mathcal{H}^{N-1}(x) \\
&\leq \int_{S_u} \sigma\left(\nu_u(x), \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}\right) d\mathcal{H}^{N-1}(x) = F(u,\mu).
\end{aligned}$$

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