A dimension reduction result in the framework of structured deformations

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Abstract

Structured deformations provide a model to non-classical deformations of continua suitable for the description of deformations of materials whose kinematics requires analysis at both the macroscopic and microscopic levels. In this work we apply dimension reduction techniques in order to derive models for thin structures in the framework of structured deformations of continua.

1 Introduction

Structured deformations were first introduced by Del Piero and Owen [15] and later generalized by Owen and Paroni [19]. The model introduced in [15] (first order structured deformations) provides a class of deformations which is appropriate to describe complicated processes of fracture at the macroscopic level and also permits to identify processes of microfracture that describe a continuum with structure. Choksi and Fonseca [8] extended the notion of first order structured deformation to the setting of special functions of bounded variation. Precisely, the authors defined a first order structured deformation as a pair (g, G) where the macroscopic deformation g is an element of $SBV(\Omega;\mathbb{R}^d)$ (the space of *special functions of bounded variation*, cf. section 2) and G is an integrable tensor field in Ω, and have proved that given such a pair there exist deformations u_n in $SBV(\Omega;\mathbb{R}^d)$ such that

$$u_n \xrightarrow{L^1} g \quad \text{and} \quad \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G.$$

Then the energy of (g, G) was defined as

$$\mathcal{I}(g,G) := \inf_{\{u_n\}\subset SBV(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n),\ \ u_n \xrightarrow{L^1} g,\ \ \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G \right\},$$

where

$$E(u) = \int_\Omega W(\nabla u)\,dx + \int_{S(u)} \psi([u],\nu(u))\,d\mathcal{H}^{N-1}$$

for any $u \in SBV(\Omega;\mathbb{R}^d)$, and an integral representation of $\mathcal{I}(g,G)$ was derived. Note that the energy of (g,G) corresponds to the most economical way to build up deformations using SBV- approximations.

In this work we consider a model for first order structured deformations departing from a different initial energy E which includes second order derivatives (see (1.1) below; see Carriero Leaci and Tomarelli [9] and [10] for other second order variational problems). Our goal is to derive a model for thin structures through dimensional reduction techniques. The need for second derivatives relies on the fact that, in order to avoid the formation of holes in the target lower dimensional domain, all the jumps in the approximating sequences must be properly aligned (see Remark 1.3 below).

Precisely, we consider the energy of three dimensional structures with vanishing thickness $\epsilon > 0$ as follows

$$E_\epsilon(v) := \int_{\Omega_\epsilon} W(\nabla v, \nabla^2 v)dy + \int_{S_v} \Psi_1([v],\nu(v))d\mathcal{H}^2 + \int_{S_{\nabla v}} \Psi_2([\nabla v],\nu(\nabla v))d\mathcal{H}^2 \tag{1.1}$$

for $v \in SBV^2(\Omega_\epsilon;\mathbb{R}^3)$, where $\Omega_\epsilon = w \times (0,\epsilon)$ and $\omega \subset \mathbb{R}^2$ is an open bounded set. We assume the following hypothesis in the energy densities

(H_1) : there exists $C > 0$ such that

$$\frac{1}{C}|B| - C \le W(A,B) \le C\big(1+|B|\big)$$

for all $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 3\times 3}$;

(H_2) : there exists $C > 0$ such that

$$|W(A_1,B_1) - W(A_2,B_2)| \le C\big(|A_1 - A_2| + |B_1 - B_2|\big)$$

for all $A_i \in \mathbb{R}^{3\times 3}$ and $B_i \in \mathbb{R}^{3\times 3\times 3}$, $i = 1,2$;

(H_3) : there exists $0 < \alpha < 1$ and $L > 0$ such that

$$\left|W^\infty(A,B) - \frac{W(A,tB)}{t}\right| \le \frac{C}{t^\alpha}$$

for all $t > L$, $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^{3\times 3\times 3}$ B with $|B| = 1$, where W^∞ denotes, as usual, the *recession function* of W in the variable B, i.e.,

$$W^\infty(A,B) = \limsup_{t\to+\infty} \frac{W(A,tB)}{t};$$

(H_4) : there exist $c_1 > 0$, $C_1 > 0$ such that

$$c_1|\lambda| \le \Psi_1(\lambda,\nu) \le C_1|\lambda|,$$

for all $\lambda \in \mathbb{R}^3$ and $\nu \in S^2$;

(H_5): there exist $c_2 > 0$, $C_2 > 0$ such that

$$c_2|\Lambda| \le \Psi_2(\Lambda,\nu) \le C_2|\Lambda|,$$

for all $\Lambda \in \mathbb{R}^{3\times 3}$ and $\nu \in S^2$;

(H_6): (homogeneity of degree one)

$$\Psi_1(t\lambda, \nu) = t\Psi_1(\lambda, \nu), \qquad \Psi_2(t\Lambda, \nu) = t\Psi_2(\Lambda, \nu)$$

for all $\nu \in S^2$, $\lambda \in \mathbb{R}^3$, $\Lambda \in \mathbb{R}^{3\times 3}$ and $t > 0$;

(H_7): (sub-additivity)

$$\Psi_1(\lambda_1 + \lambda_2, \nu) \le \Psi_1(\lambda_1, \nu) + \Psi_1(\lambda_2, \nu),$$
$$\Psi_2(\Lambda_1 + \Lambda_2, \nu) \le \Psi_2(\Lambda_1, \nu) + \Psi_2(\Lambda_2, \nu)$$

for all $\nu \in S^2$, $\lambda_i \in \mathbb{R}^3$, $\Lambda_i \in \mathbb{R}^{3\times 3}$, $i = 1, 2$;

(H_8):

$$\Psi_2(\Lambda, \nu_\alpha, \nu_3) = \Psi_2(\Lambda, \nu_\alpha, -\nu_3),$$

for all $\nu \in S^2$ (written as $\nu = (\nu_\alpha, \nu_3)$), $\Lambda \in \mathbb{R}^{3\times 3}$.

Remark 1.1 *The hypotheses $(H_1) - (H_7)$ were considered in the relaxation result [7] which is used in order to derive the lower bound inequality in this work. They generalize to this setting the ones considered by Choksi and Fonseca in [8]. Hypothesis (H_8) is only used in order to derive the upper bound inequality and is a property of invariance under the particular reflection associated with the plane in the reference configuration occupied by the two-dimensional "reduced" continuum. Most lattices and submacroscopic geometries indeed have such a plane of symmetry.*

As usual in dimensional reduction problems we change variables in order to have a fixed domain. Precisely, let $y = (y_\alpha, y_3) \in \Omega_\epsilon$ and define $x = (x_\alpha, x_3) \in \Omega := \omega \times (0, 1)$ through $x_\alpha = y_\alpha$ and $x_3 = \frac{y_3}{\epsilon}$. Then

$$u(x_\alpha, x_3) := v(x_\alpha, \epsilon x_3)$$

is clearly a function in $SBV^2(\Omega; \mathbb{R}^3)$ and the integral in (1.1) becomes

$$\begin{aligned} E_\epsilon(u) \quad = \quad & \epsilon \Bigg[\int_\Omega W(\nabla_\alpha u, \frac{1}{\epsilon}\nabla_3 u, \nabla^2_{\alpha,\beta} u, \frac{1}{\epsilon}\nabla^2_{\alpha 3} u, \frac{1}{\epsilon}\nabla^2_{3\beta} u, \frac{1}{\epsilon^2}\nabla^2_{33} u)\, dx \\ & + \int_{S_u} \Psi_1([u], \nu_\alpha(u), \frac{1}{\epsilon}\nu_3(u))\, d\mathcal{H}^2 \\ & + \int_{S_{\nabla u}} \Psi_2([\nabla_\alpha u], \frac{1}{\epsilon}[\nabla_3 u], \nu_\alpha(\nabla u), \frac{1}{\epsilon}\nu_3(\nabla u))\, d\mathcal{H}^2 \Bigg] \end{aligned}$$

where $\alpha, \beta \in \{1, 2\}$.

We introduce now the ϵ-scaled $3D$ energies $J_\epsilon := \frac{E_\epsilon}{\epsilon}$ and our aim is to derive the asymptotic behaviour as $\epsilon \to 0^+$ in the sense of Γ-convergence (see [12], [13], [3] and [14]). More precisely we consider

$$I(g, b, G) := \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} J_{\epsilon_n}(u_n),\ u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow{L^1} G \right\},$$

where $(g, G) \in BV^2(\omega; \mathbb{R}^3) \times BV(\omega; \mathbb{R}^{3\times 2})$ and $b \in BV(w; \mathbb{R}^3)$. The remark 1.3 gives the motivation for the definition of I, in particular the convergence and the appearance of fields independent of the transversal variable x_3 in the limit space.

The main result of this paper is the following representation result for I (see section 2.1 for notation).

Theorem 1.2 *The functional I does not depend on the sequence $\{\epsilon_n\}$ and admits an integral representation of the form*

$$I = I_1 + I_2$$

where

$$I_1(g, G) = \int_w W_1(G - \nabla g)\, dx_\alpha + \int_{S_g} \Gamma_1([g], \nu(g))\, d\mathcal{H}^1 + \int_\omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right)\, d|D^c g|,$$

and

$$\begin{aligned} I_2(b, G) \ &= \int_\omega W_2(b, G, \nabla b, \nabla G)\, dx + \int_{\omega \cap S((b,G))} \Gamma_2((b, G)^+, (b, G)^-, \nu((b, G)))\, d\mathcal{H}^1 \\ &\quad + \int_\omega W_2^\infty\left(b, G, \frac{dD^c(b, G)}{d|D^c(b, G)|}\right) d|D^c(b, G)|. \end{aligned}$$

The energy densities of I_1 are obtained as follows

$$W_1(A) = \inf_{u \in SBV(Q'; \mathbb{R}^3)} \left\{ \int_{Su \cap Q'} \overline{\Psi}_1([u], \nu(u))\, d\mathcal{H}^1, u|_{\partial Q} = 0, \nabla u = A \text{ a.e. in } Q' \right\},$$

$$\Gamma_1(\lambda, \nu) = \inf_{u \in SBV(Q'_\nu; \mathbb{R}^3)} \left\{ \int_{Q'_\nu \cap S_u} \overline{\Psi}_1([u], \nu(u))\, d\mathcal{H}^1, u|_{\partial Q'_\nu} = \tau_{(\lambda,\nu)}, \nabla u = 0 \right\},$$

with

$$\tau_{(\lambda,\nu)}(x_\alpha) := \begin{cases} \lambda & \text{if } x_\alpha.\nu > 0 \\ 0 & \text{if } x_\alpha.\nu < 0 \end{cases}$$

and

$$\overline{\Psi}_1(\lambda, \nu_\alpha) := \inf \left\{ \frac{1}{\sqrt{|\nu_\alpha|^2 + t^2}} \Psi_1(\lambda, \nu_\alpha, t) : t \in \mathbb{R} \right\}.$$

The energy densities for I_2 are as follows

$$\begin{aligned} W_2(A, B_\alpha) := \inf \Bigg\{ &\int_{Q'} \overline{W}(A, \nabla u)\, dy + \int_{Q' \cap S_u} \overline{\Psi}_2([u], \nu(u))\, d\mathcal{H}^1 : \\ &u \in SBV(Q'; \mathbb{R}^3), u|_{\partial Q'} = B_\alpha y \Bigg\}, \end{aligned}$$

$$\begin{aligned} \Gamma_2(\lambda, \theta, \nu) := \inf \Bigg\{ &\int_{Q'_\nu} \overline{W}^\infty(u, \nabla u)\, dy + \int_{Q'_\nu \cap S(u)} \overline{\Psi}_2([u], \nu(u)) d\mathcal{H}^1 : \\ &u \in SBV(Q'_\nu; \mathbb{R}^3), u|_{\partial Q'_\nu} = u_{\lambda,\theta,\nu} \Bigg\}, \end{aligned}$$

where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y \cdot \nu > 0, \\ \\ \theta & \text{otherwise} \end{cases}$$

and with $\overline{W}$ *and* $\overline{\Psi}_2$ *as follows: decomposing the pair* $(A, B) \in \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3\times3}$ *into* $(A, B_\alpha, B_3) \in \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3\times2} \times \mathbb{R}^{3\times3\times1}$ *define*

$$\overline{W}(A, B_\alpha) := \inf_{B_3 \in \mathbb{R}^{3\times3\times1}} W(A, B_\alpha, B_3),$$

and for $\lambda \in \mathbb{R}^{3\times3}$, $\nu_\alpha \in S^1$, *let*

$$\overline{\Psi}_2(\lambda, \nu_\alpha) := \inf\left\{ \frac{1}{\sqrt{|\nu_\alpha|^2 + t^2}} \Psi_2(\lambda, \nu_\alpha, t) : t \in \mathbb{R} \right\}.$$

Remark 1.3 *Suppose we have a sequence of deformations clamped in the boundary and with finite total energy. Thus, for a given sequence* $\{\epsilon_n\}$, *we have a sequence* $\{v_n\} \subset SBV^2(\Omega_{\epsilon_n}; \mathbb{R}^3)$ *such that* $v_n = x$ *in a neighborhood of* $\partial\omega \times (0, \epsilon_n)$. *After rescaling we obtain a new sequence* $\{u_n\} \subset SBV^2(\Omega; \mathbb{R}^3)$ *such that* $u_n = (x_\alpha, \epsilon_n x_3)$ *and*

$$\sup_n J_{\epsilon_n}(u_n) < \infty.$$

From the growth conditions $(H1)$, $(H4)$ *and* $(H5)$, *we obtain*

$$\sup_n \left(|D(\nabla u_n)|(\Omega) + \int_{S_{u_n}} |[u_n]| \right) < \infty,$$

which, together with the boundary condition, implies the boundedness of u_n *and* ∇u_n *in the* BV*-norm. Thus, up to a subsequence, we have* $u_n \xrightarrow{L^1} g$ *and* $\nabla u_n \xrightarrow{L^1} G$. *Now, defining* $b_n := \frac{\nabla_3 u_n}{\epsilon_n}$, *and using* $(H1)$ *and* $(H5)$ *we get that*

$$\sup_n |D(b_n)|(\Omega) < \infty,$$

which, together with the boundary condition $b_n = (0, 0, 1)$ *implies the boundedness of* b_n *in* BV*-norm and consequently the existence of a subsequence such that* $b_n \to b$ *in* L^1. *The field* g *represents the deformation of the mid-surface and the field* b *represents the rotation and compression of the normal sections. On the other hand, using the same growth conditions, we have*

$$\sup_n \left(|D_3(\nabla u_n)|(\Omega) + \int_{S_{u_n}} |[u_n]\nu_3| + |D_3(b_n)|(\Omega) \right) < C\epsilon_n,$$

which, together with boundary conditions, implies that the limit fields g, G *and* b *do not depend on* x_3.

The overall plan of this work in the ensuing sections will be as follows. In section 2 we collect the main notations and results used troughout. In section 3 we prove theorem 1.2.

2 Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. Almost all these results are stated without proofs as they can be readily found in the references given below.

2.1 Notation

Throughout the text $w \subset \mathbb{R}^2$ will denote an open bonded set and for $\epsilon > 0$, $\Omega_\epsilon = w \times (0, \epsilon)$. We denote simply by Ω the subset of $\mathbb{R}^3$ corresponding to $\Omega_1 = w \times (0,1) = w \times I$. If $x \in \mathbb{R}^3$ then $x_\alpha := (x_1, x_2) \in \mathbb{R}^2$ is the vector of the first two components of x.

We wil use the following notations:

- $\mathcal{A}(\Omega)$ (resp. $\mathcal{A}(w)$) is the family of all open subsets of Ω (resp. w),
- $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}(w)$) is the set of finite Radon measures on Ω (resp. w),
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ stand, respectively, for the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.
- $||\mu||$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$ (resp. $\mathcal{M}(w)$),
- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$,
- Q denotes the unit cube of $\mathbb{R}^3$ centered at the origin with one side orthogonal to e_3,
- $Q(x, \delta)$ denotes a cube in $\mathbb{R}^3$ centered at $x \in \Omega$ with side length δ and with one side orthogonal to e_3,
- $Q_\nu(x, \delta)$ is the cube centered at $x \in \Omega$ with side length δ and with one side orthogonal to $\nu \in S^2$,
- when related to $\mathbb{R}^2$ and w we use the previous notations with the obvious adaptations with Q' in place of Q,
- C represents a generic constant,
- $\lim\limits_{n,m\to\infty} := \lim\limits_{n\to\infty} \lim\limits_{m\to\infty}$ while $\lim\limits_{m,n\to\infty} := \lim\limits_{m\to\infty} \lim\limits_{n\to\infty}$.

2.2 BV-functions

We start by recalling some facts on functions of bounded variation which will be used afterwards. We refer to Ambrosio, Fusco and Pallara [1], Evans and Gariepy [16], Federer [17], Giusti [18] and Ziemer [20] for a detailed theory on this subject.

A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i \in \mathcal{M}(\Omega)$ for $i = 1, ..., d$ and $j = 1, ..., N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du. The space $BV(\Omega; \mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + ||Du||(\Omega).$$

By the Lebesgue Decomposition theorem Du can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$ (the absolutely continous part and singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^N$). By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u.$$

Let Ω_u be the set of points where the approximate limits of u exists and S_u the *jump set* of this function, i.e., the set of points $x \in \Omega \setminus \Omega_u$ for which there exists a, $b \in \mathbb{R}^N$ and a unit vector $\nu \in S^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_\nu(x,\varepsilon) : (y-x)\cdot\nu > 0\}} |u(y) - a| \, dy = 0 \tag{2.2}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{\{y\in Q_\nu(x,\varepsilon):(y-x)\cdot\nu<0\}}|u(y)-b|\,dy=0. \tag{2.3}$$

The triple (a,b,ν) uniquely determined by (2.2) and (2.3) up to permutation of (a,b), and a change of sign of ν and is denoted by $(u^+(x),u^-(x),\nu_u(x))$.

If $u\in BV(\Omega)$ it is well known that S_u is countably $N-1$ rectifiable, i.e.

$$S_u=\bigcup_{n=1}^{\infty}K_n\cup E,$$

where $\mathcal{H}^{N-1}(E)=0$ and K_n are compact subsets of C^1 hypersurfaces. Furthermore, $\mathcal{H}^{N-1}((\Omega\setminus\Omega_u)\setminus S_u)=0$ and the following decomposition holds

$$Du=\nabla u\mathcal{L}^N\lfloor\Omega+[u]\otimes\nu_u\mathcal{H}^{N-1}\lfloor S_u+D^cu,$$

where $[u]:=u^+-u^-$ and D^cu is the Cantor part of the measure Du, i.e., $D^cu=D^su\lfloor(\Omega_u)$.

We next recall some properties of BV functions used in the sequel. We start with the following Lemma whose proof can be found in [8]:

Lemma 2.1 *Let $u\in BV(\Omega;\mathbb{R}^d)$. Then there exist piecewise constant functions u_n such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^d)$ and*

$$||Du||(\Omega)=\lim_{n\to\infty}||Du_n||(\Omega)=\lim_{n\to\infty}\int_{S_{u_n}}|[u_n](x)|\,d\mathcal{H}^{N-1}.$$

The space of *special functions of bounded variation*, $SBV(\Omega;\mathbb{R}^d)$, introduced by De Giorgi and Ambrosio in [11] to study free discontinuity problems, is the space of functions $u\in BV(\Omega;\mathbb{R}^d)$ such that $C_u=0$, i.e. for which

$$Du=\nabla u\mathcal{L}^N+[u]\otimes\nu_u\mathcal{H}^{N-1}\lfloor S_u.$$

The next result is a Lusin type theorem for gradients due to Alberti [2] and is essential to our arguments.

Theorem 2.2 *Let $f\in L^1(\Omega;\mathbb{R}^{d\times N})$. There exists $u\in SBV(\Omega;\mathbb{R}^d)$ and a Borel function $g:\Omega\to\mathbb{R}^{d\times N}$ such that*

$$Du=f\mathcal{L}^N+g\mathcal{H}^{N-1}\lfloor S_u,$$

$$\int_{S_u}|g|\,d\mathcal{H}^{N-1}\le C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

Remark 2.3 *From the proof of Theorem 2.2 it also follows that*

$$||u||_{L^1(\Omega)}\le 2C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

Following Carriero, Leaci and Tomarelli (see [9] and [10]) we define

$$SBV^2(\Omega;\mathbb{R}^d)=\{v\in SBV(\Omega;\mathbb{R}^d),\ \nabla v\in SBV(\Omega;\mathbb{R}^{d\times N})\}.$$

If $u\in SBV^2(\Omega;\mathbb{R}^d)$ we use the notation $\nabla^2u=\nabla(\nabla u)$, that is, ∇^2u is the absolutely continuous part of $D(\nabla u)$ with respect to Lebesgue measure.

We will also denote by

$$BV^2(\Omega;\mathbb{R}^d)=\{v\in BV(\Omega;\mathbb{R}^d),\ \nabla v\in BV(\Omega;\mathbb{R}^{d\times N})\}.$$

2.3 Integral representation results

In this section we recall Theorem 3.12 in [5] and apply it to an auxiliary functional which will be used in order to derive the upper bound inequality. Let:

$$\mathcal{F} : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty)$$

satisfying:

i) $\mathcal{F}(u;.)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,

ii) $\mathcal{F}(.;A)$ is $L^1(A;\mathbb{R}^d)$- lower semicontinuous,

iii) $\frac{1}{C}|Du|(A) \leq \mathcal{F}(u,A) \leq C\left(\mathcal{L}^N(A) + |Du|(A)\right)$ for some $C > 0$,

iv) There exists a modulus of continuity $\phi(t)$ satisfying

$$|\mathcal{F}(u(.-z)+b; z+A) - \mathcal{F}(u,A)| \leq \phi(|b|+|z|)\left(\mathcal{L}^N(A) + |Du|(A)\right).$$

Define the set function:

$$m(u;A) := \inf\left\{\mathcal{F}(v;A), v|_{\partial A} = u|_{\partial A},\, v \in BV(\Omega;\mathbb{R}^d)\right\},$$

and let

$$f(x_0,a,\zeta) := \limsup_{\epsilon\to 0^+} \frac{m(a+\zeta(.-x_0);Q(x_0,\epsilon))}{\epsilon^N}, \tag{2.4}$$

$$g(x_0,\lambda,\theta,\nu) := \limsup_{\epsilon\to 0^+} \frac{m(u_{\lambda,\theta,\nu}(.-x_0);Q_\nu(x_0,\epsilon))}{\epsilon^{N-1}} \tag{2.5}$$

for all $x_0 \in \Omega, a, \theta, \lambda \in \mathbb{R}^d, \zeta \in \mathbb{R}^{d\times N}$, where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \theta & \text{otherwise.} \end{cases}$$

Then the following full representation result of $\mathcal{F}$ on $BV(\Omega;\mathbb{R}^d)$ holds:

Theorem 2.4 *Under hypotheses i), ii), iii) and iv),*

$$\begin{aligned} \mathcal{F}(u;A) &= \int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u\cap A} g(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \\ &\quad + \int_A f^\infty\left(x,u,\frac{dD^c u}{d|D^c u|}\right)\,d|D^c u|, \end{aligned}$$

where f and g are defined by (2.4) and (2.5) respectively and f^∞ denotes the recession function of f in the last variable, defined by

$$f^\infty(x_0,u_0,\xi) := \limsup_{t\to\infty} \frac{f(x_0,u_0,t\xi)}{t}.$$

Next we apply the Theorem above to the functional

$$\hat{I}_{1,\delta}^{(G,b)}(g,A) = \inf_{\{g_n\}} \Big\{ \liminf_{\epsilon_n\to 0} \delta \int_{A\times I} \left| \left(\nabla_\alpha g_n | \frac{\nabla_3 g_n}{\epsilon_n} \right) \right| \, dx + \int_{S_{g_n}\cap(A\times I)} \Psi_1 \left([g_n], \nu_\alpha(g_n), \frac{1}{\epsilon_n}\nu_3(g_n) \right) \, d\mathcal{H}^2$$
$$g_n \xrightarrow{L^1} g, \ \frac{1}{\epsilon_n}\nabla_3 g_n \xrightarrow{L^1} b, \ ||\nabla_\alpha g_n - G||_{L^1} \le C{\epsilon_n}^2 \Big\}, \tag{2.6}$$

defined in $\mathcal{A}(\omega)$ where G, b are fixed piecewise constant L^∞ functions. We prove that

Proposition 2.5 *The functional $\hat{I}_{1,\delta}^{(G,b)}(g,A)$ admits an integral representation of the form:*

$$\hat{I}_{1,\delta}^{(G,b)}(g,A) = \int_A f_\delta^{(G,b)}(\nabla g)\, dx + \int_{S_g\cap A} h_\delta^{(G,b)}([g],\nu_g)\, d\mathcal{H}^2,$$

where $A \in \mathcal{A}(\omega)$.

Proof. In order to comply with the conditions in Theorem 2.4 we first prove that $\hat{I}_{1,\delta}^{(G,b)}(g,.)$ is the restriction of a Radon measure in $\mathcal{M}(\mathbb{R}^2)$ to $\mathcal{A}(\omega)$. For each point $a \in \omega$ with rational coordinates consider balls $B(a,r_i)$ with radius r_i (defined for $i \in \mathbb{N}$ large enough, depending on a) such that

$$|r_i - \frac{1}{i}| \le \frac{1}{i^2}, \ \ \overline{B(a,r_i)} \subset \omega \ \text{ and } \ ||D^s g||(\partial B(a,r_i)) = 0.$$

We denote by $\mathcal{B}(\omega)$ the set of all such balls and their finite unions (it consists of a numerable number of sets). The set of all closed balls $\overline{B(a,r_i)}$ is a fine cover to ω. We can take an appropriate subsequence of $\{\epsilon_n\}$, which we denote by ϵ_{n_k}, such that for each element in $\mathcal{B}$ (which we denote by B) there exists a sequence $g_k^{\{B\}} \subset SBV_2(\Omega;\mathbb{R}^d)$ (which we denote by g_k for simplicity) such that

$$g_k \xrightarrow{L^1} g, \quad \frac{1}{\epsilon_{n_k}}\nabla_3 g_k \xrightarrow{L^1} b, \quad ||\nabla_\alpha g_k - G||_{L^1} \le C{\epsilon_{n_k}}^2,$$

and

$$\hat{I}_{1,\delta}^{(G,b)}(g;B) = \lim_{k\to\infty} \delta \int_{B\times I} \left| \left(\nabla_\alpha g_k, \frac{\nabla_3 g_k}{\epsilon_{n_k}} \right) \right| \, dx + \int_{S_{g_k}\cap(B\times I)} \Psi_1 \left([g_k], \nu_\alpha(g_k), \frac{1}{\epsilon_{n_k}}\nu_3(g_k) \right) \, d\mathcal{H}^2. \tag{2.7}$$

In order to prove that $\hat{I}_{1,\delta}^{(G,b)}(g,.)$ is a Radon measure we need first the following subadditivity Lemma:

Lemma 2.6 *Let A, B, C be open sets in $\mathcal{A}(\omega)$ such that $A \subset\subset B \subset C$. Then we have that*

$$\hat{I}_{1,\delta}^{(G,b)}(g,C) \le \hat{I}_{1,\delta}^{(G,b)}(g,B) + \hat{I}_{1,\delta}^{(G,b)}(g,C\setminus \overline{A}).$$

Proof. We first derive an upper bound for $\hat{I}_{1,\delta}^{(G,b)}$. Let

$$g_k := g + h - h_k + \epsilon_{n_k} x_3 b, \tag{2.8}$$

where $h \in SBV_2(\omega;\mathbb{R}^3)$ is such that $\nabla h = G - \nabla g$ (see Theorem 2.2) and h_k is piecewise constant function such that $h_k \to h$ in L^1 (see Lemma 2.1). Using the sequence above we get the upper bound as follows

$$\hat{I}_{1,\delta}^{(G,b)}(g,A) \le \delta \int_A |(G,b)|\, dx + C\int_A |G - \nabla g|\, dx + ||D^s g||(A), \tag{2.9}$$

for every open set $A \in \mathcal{A}(\omega)$.

For each Borel set B define the bounded Radon measure

$$\Delta(B) := \delta \int_B |(G,b)|\,dx + C\int_B |G - \nabla g|\,dx + ||D^s g||(B).$$

For each $\rho > 0$ consider an open set B_ρ in $\mathcal{B}(\omega)$ such that $B_\rho \subset B$. Using Besicovitch's Covering Theorem we can find a set $A_\rho \in \mathcal{B}$ such that $A_\rho \subset C \setminus \overline{A}$ and

$$\Delta((C \setminus \overline{A}) \setminus \overline{A}_\rho) < \rho.$$

Note that we can choose the sets above in such a way that there exist an open set with Lipschitz boundary $\hat{A}$, with $A \subset\subset \hat{A} \subset\subset B_\rho$, with $\partial\hat{A} \subset A_\rho$.

Now we can find sequences $g_k^1 \subset SBV_2(A_\rho;\mathbb{R}^3)$ and $g_k^2 \subset SBV_2(B_\rho;\mathbb{R}^3)$ verifying (2.7). We can define a new sequence

$$\hat{g}_k := \begin{cases} g_k^1 \text{ in } A_\rho \setminus \hat{A}, \\ g_k^2 \text{ in } \hat{A}, \\ g_k \ \text{ otherwise in } C \end{cases} \tag{2.10}$$

where g_k is defined in (2.8). We then have that

$$\begin{aligned} \hat{I}_{1,\delta}^{(G,b)}(g,C) \quad \leq \quad & \lim_{k\to\infty} \delta \int_{C\times I} \left| \left(\nabla_\alpha \hat{g}_k, \frac{\nabla_3 \hat{g}_k}{\epsilon_{n_k}} \right) \right| \,dx && (2.11) \\ & + \int_{S_{\hat{g}_k} \cap (C\times I)} \Psi_1\left([\hat{g}_k], \nu_\alpha(\hat{g}_k), \frac{1}{\epsilon_{n_k}} \nu_3(\hat{g}_k) \right)\, d\mathcal{H}^2 && (2.12) \\ & \leq \hat{I}_{1,\delta}^{(G,b)}(g,B_\rho) + \hat{I}_{1,\delta}^{(G,b)}(g,A_\rho) + \Delta((C \setminus \overline{A}) \setminus \overline{A}_\rho) && (2.13) \\ & \leq \hat{I}_{1,\delta}^{(G,b)}(g,B_\rho) + \hat{I}_{1,\delta}^{(G,b)}(g,A_\rho) + \rho, && (2.14) \end{aligned}$$

and the result follows from letting $\rho \to 0$. ■

We may suppose, without loss of generality, that the equality (2.7) also holds for the set ω, with an appropriate sequence g_k^ω (which we again denote by g_k). Define the sequence of bounded Radon measures as follows

$$\Lambda_k(B) := \delta \int_{B\times I} \left| \left(\nabla_\alpha g_k, \frac{\nabla_3 g_k}{\epsilon_{n_k}} \right) \right| \,dx + \int_{S_{g_k} \cap (B\times I)} \Psi_1\left([g_k], \nu_\alpha(g_k), \frac{1}{\epsilon_{n_k}} \nu_3(g_k) \right)\, d\mathcal{H}^2$$

where $B \subset \mathbb{R}^2$ is an arbitrary Borel set. We may extract a subsequence such that $\Lambda_k \overset{*}{\rightharpoonup} \Lambda$. Then the following holds

Lemma 2.7 *For every open set $A \in \mathcal{A}(\omega)$ we have that*

$$\hat{I}_{1,\delta}^{(G,b)}(g;A) = \Lambda(A).$$

Proof. First note that for any open set $A \in \mathcal{A}(\omega)$ we have that

$$\hat{I}_{1,\delta}^{(G,b)}(g;A) \leq \Lambda(\overline{A}). \tag{2.15}$$

Given $V \in \mathcal{A}(\omega)$, let $\rho > 0$ and take $W \subset\subset V$ such that $\Lambda(V \backslash W) < \rho$. It follows that

$$\begin{array}{rcl} \Lambda(V) & \leq & \Lambda(W) + \rho \\ & = & \Lambda(\omega) - \Lambda(\omega \backslash W) + \rho \\ & \leq & \hat{I}_{1,\delta}^{(G,b)}(g;\omega) - \hat{I}_{1,\delta}^{(G,b)}(g;\omega \backslash \bar{W}) + \rho \\ & \leq & \hat{I}_{1,\delta}^{(G,b)}(g;V) + \rho \end{array}$$

where we have used the equality $\Lambda(\omega) = \Lambda(\overline{\omega})$, (2.15) and Lemma 2.6. Thus, letting $\rho \to 0$, we get

$$\Lambda(V) \leq \hat{I}_{1,\delta}^{(G,b)}(g;V). \tag{2.16}$$

Let us see now the reverse inequality. Let $K \subset\subset V$ be a compact set such that $\Delta(V \backslash K) < \rho$ (see 2.9), and choose an open set W such that $K \subset\subset W \subset\subset V$. Using again Lemma 2.6 we have

$$\begin{aligned} \hat{I}_{1,\delta}^{(G,b)}(g;V) \quad & \leq \hat{I}_{1.\delta}^{(G,b)}(g;W) + \hat{I}_{1,\delta}^{(G,b)}(g;V \backslash K) \\ & \leq \Lambda(\bar{W}) + \Delta(V \backslash K) \\ & \leq \Lambda(V) + C\rho, \end{aligned}$$

which, together with (2.16), yields the statement after letting $\rho \to 0$. ∎

From Lemma 2.7 we have that $\hat{I}_{1,\delta}^{(G,b)}(g,A)$ satisfies the point *i*) of Theorem 2.4. The points *ii*) and *iv*) are easy to verify. The bounds in point *iii*) follow from the upper bound given by Δ in (2.9) (note that we are assuming G to be a fixed L^∞ function) and the lower semicontinuity of the total variation with respect to weak* convergence (for the lower bound). Thus, applying Theorem 2.4, we get the existence of Borel functions $f_\delta^{(G,b)}$ (only dependent of ∇u in this case) and $h_\delta^{(G,b)}$ (only dependent on $([u], \nu_u)$) such that

$$\hat{I}_{1,\delta}^{(G,b)}(g;A) = \int_A f_\delta^{(G,b)}(\nabla g)\, dx + \int_{A \cap S_g} h_\delta^{(G,b)}([g], \nu_g)\, d\mathcal{H}^2. \tag{2.17}$$

∎

2.4 Relaxation results for structured deformations

In [7] we studied the relaxation

$$I(g,G) = \inf_{\{u_n \in SBV^2(\Omega;\mathbb{R}^d)\}} \left\{ \liminf_{n\to\infty} E(u_n) \colon\; u_n \in SBV^2(\Omega;\mathbb{R}^d),\; u_n \xrightarrow{L^1} g,\; \nabla u_n \xrightarrow{L^1} G \right\}, \tag{2.18}$$

under hypotheses $(H_1) - (H_7)$, of the energy

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\, dx + \int_{S_u} \Psi_1([u], \nu(u)) d\mathcal{H}^{N-1} + \int_{S(\nabla u)} \Psi_2([\nabla u], \nu(\nabla u)) d\mathcal{H}^{N-1},$$

where $\Omega \subset \mathbb{R}^N$, $u \in SBV^2(\Omega;\mathbb{R}^d)$.

Remark 2.8 *We extend $\Psi_i, i = 1,2$ as homogeneous functions of degree one in the second variable to all of $\mathbb{R}^N$ (respectively $\mathbb{R}^{d\times N}$).*

Under the hypotheses $(H_1)-(H_7)$, an integral representation of the energy $I(g,G)$ was derived for $g \in BV^2(\Omega;\mathbb{R}^d)$ and $G \in BV(\Omega;\mathbb{R}^{d\times N})$. Namely, given $A, B \in \mathbb{R}^{d\times N}$ and $C \in \mathbb{R}^{d\times N\times N}$ and defining

$$W_1(A)= \inf_{u\in SBV(Q;\mathbb{R}^d)} \left\{ \int_{Su\cap Q} \Psi_1([u],\nu(u))\, d\mathcal{H}^{N-1}, u|_{\partial Q}=0, \nabla u = A \text{ a.e. in } Q \right\},$$

$$\begin{aligned} W_2(B,C) &= \inf_{v\in SBV(Q;\mathbb{R}^{d\times N})} \left\{ \int_Q W(B,\nabla v(x))\, dx \right. \\ &\quad \left. + \int_{Q\cap S_v} \Psi_2([v],\nu(v))\, d\mathcal{H}^{N-1}, v|_{\partial Q} = C\,x \right\}, \end{aligned}$$

$$\gamma_1(\lambda,\nu) = \inf_{u\in SBV(Q_\nu;\mathbb{R}^d)} \left\{ \int_{Q_\nu\cap S_u} \Psi_1([u],\nu(u))\, d\mathcal{H}^{N-1}, u|_{\partial Q_\nu} = \gamma_{(\lambda,\nu)}, \nabla u = 0 \right\},$$

where

$$\gamma_{(\lambda,\nu)}(x) := \begin{cases} \lambda & \text{if } x.\nu > 0 \\ 0 & \text{if } x.\nu < 0, \end{cases}$$

$$\begin{aligned} \gamma_2(\Lambda,\Gamma,\nu)= \inf_{v\in SBV(Q_\nu;\mathbb{R}^{d\times N})} & \left\{ \int_{Q_\nu} W^\infty(v,\nabla v)\, dx + \int_{Q_\nu\cap S_v} \Psi_2([v],\nu(v))\, d\mathcal{H}^{N-1}, \right. \\ & \left. v|_{\partial Q_\nu} = \gamma_{(\Lambda,\Gamma,\nu)} \right\} \end{aligned}$$

where

$$\gamma_{(\Lambda,\Gamma,\nu)}(x) := \begin{cases} \Lambda & \text{if } x.\nu > 0 \\ \Gamma & \text{if } x.\nu < 0, \end{cases}$$

and

$$\begin{aligned} W_2^\infty(A,B) &= \inf_{v\in SBV(Q;\mathbb{R}^{d\times N})} \left\{ \int_Q W^\infty(A,\nabla v(x))\, dx \right. \\ &\quad \left. + \int_{Q\cap S_v} \Psi_2([v],\nu(v))\, d\mathcal{H}^{N-1}, v|_{\partial Q} = Bx \right\}, \end{aligned}$$

the following result was proved:

Theorem 2.9 *Under hypotheses $(H_1)-(H_7)$, for all $(g,G) \in BV^2(\Omega;\mathbb{R}^d) \times BV(\Omega;\mathbb{R}^{d\times N})$, we have that*

$$I(g,G) = I_1(g,G) + I_2(g,G)$$

where

$$I_1(g,G) = \inf_{u_n\in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n)) d\mathcal{H}^{N-1} \colon\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\},$$

and

$$I_2(g,G) = \inf_{u_n\in SBV^2(\Omega;\mathbb{R}^3)}\left\{\liminf_{n\to\infty}\int_\Omega W(\nabla u_n,\nabla^2 u_n)\,dx + \int_{S_{\nabla u_n}}\Psi_2([\nabla u_n],\nu(\nabla u_n))d\mathcal{H}^{N-1}\colon\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G\right\},$$

and the functionals above admit an integral representation as follows

$$I_1(g,G) = \int_\Omega W_1(G-\nabla g)\,dx + \int_\Omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right)\,d|D^c g| + \int_{S_g}\gamma_1([g],\nu(g))d\mathcal{H}^{N-1},$$

$$I_2(g,G) = \int_\Omega W_2(G,\nabla G) + \int_\Omega W_2^\infty\left(G,\frac{dD^c G}{d|D^c G|}\right)\,d|D^c G| + \int_{S_G}\gamma_2(G^+,G^-,\nu(G))\,d\mathcal{H}^{N-1}.$$

3 Proofs

In this section we prove Theorem 1.2.

We start by noting that for any $(g,b,G)\in BV^2(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^{3\times 2})$ and for any fixed sequence $\epsilon_n\to 0$ there exists $u_n\in SBV^2(\Omega;\mathbb{R}^3)$ such that $u_n \xrightarrow{L^1} g$, $\frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b$, and $\nabla u_n \xrightarrow{L^1}(G,0)$. In fact, given $(g,G)\in BV(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^{3\times 2})$ and $b\in BV(\omega;\mathbb{R}^3)$, by Theorem 2.2 there exists $h\in SBV(\omega;\mathbb{R}^3)$ such that $\nabla h(x_\alpha)=G(x_\alpha)$ a.e. $x_\alpha\in w$ and

$$||D^s h||(\omega)\le C_1||G||_{L^1(\omega;\mathbb{R}^3)} \tag{3.19}$$

for some $C_1\equiv C_1(N)>0$. By Lemma 2.1, there exist $\{v_n\}$ piecewise constant such that

$$v_n \overset{L^1(\omega;\mathbb{R}^3)}{\longrightarrow} g-h \quad\text{and}\quad ||Dv_n||(\omega)=||D^s v_n||(\omega)\to||Dg-Dh||(\omega).$$

Define now $u_n\in SBV(\Omega;\mathbb{R}^3)$ by $u_n(x_\alpha,x_3):=v_n(x_\alpha)+h(x_\alpha)+\epsilon_n b(x_\alpha)x_3$. Clearly we have that

$$\nabla_\alpha u_n(x) \overset{L^1(\Omega;\mathbb{R}^{3\times 3})}{\longrightarrow} G(x_\alpha),\quad u_n \overset{L^1(\Omega;\mathbb{R}^3)}{\longrightarrow} g,\quad \frac{\nabla_3 u_n}{\epsilon_n} \overset{L^1(\Omega;\mathbb{R}^3)}{\longrightarrow} b.$$

3.1 Decomposition

In order to get an integral representation for $I(g,b,G)$ we bound it by two first order functionals and then use integral representation results from Bouchitté, Fonseca and Mascarenhas (see Theorem 2.4). In this section we derive appropriate first order lower (see 3.20) and upper bounds (see 3.21).

For $(g,b,G)\in BV^2(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^{3\times 2})$ write

$$\begin{aligned} I(g,b,G) = \inf_{u_n\in SBV^2(\Omega;\mathbb{R}^3)}\Bigg\{&\liminf_{\epsilon_n\to 0}\Bigg[\int_\Omega W(\nabla_\alpha u_n,b_n,\nabla^2_{\alpha,\beta}u_n,\nabla_\alpha b_n,\frac{1}{\epsilon_n}\nabla^2_{3\beta}u_n,\frac{1}{\epsilon_n}\nabla_3 b_n)dx \\ &+\int_{S_{u_n}}\Psi_1([u_n],\nu_\alpha(u_n),\frac{1}{\epsilon_n}\nu_3(u_n))\,d\mathcal{H}^2 \\ &+\int_{S_{\nabla u_n}}\Psi_2([\nabla_\alpha u_n],[b_n],\nu_\alpha(\nabla u_n),\frac{1}{\epsilon_n}\nu_3(\nabla u_n))\,d\mathcal{H}^2\Bigg] \\ &u_n \xrightarrow{L^1} g,\ b_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow{L^1} G\Bigg\}, \end{aligned}$$

where

$$b_n := \frac{\nabla_3 u_n}{\epsilon_n}.$$

Then the functional $I(g,b,G)$ has the lower bound

$$I(g,b,G) \geq I_1(g,b,G) + I_2(b,G) \tag{3.20}$$

with

$$\begin{aligned} I_1(g,b,G) &= \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n), \frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2 \right. \\ & \left. u_n \xrightarrow{L^1} g, \ \ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b, \ \ \nabla_\alpha u_n \underset{n\to\infty}{\xrightarrow{L^1}} G \right\}, \end{aligned}$$

and

$$\begin{aligned} I_2(b,G) &= \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \left\{ \liminf_{\epsilon_n \to 0} \left[\int_\Omega W(h_n, \nabla_\alpha h_n, \frac{1}{\epsilon_n}\nabla_3 h_n)\, dx \right.\right. \\ & \left. + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n), \frac{1}{\epsilon_n}\nu_3(h_n))\, d\mathcal{H}^2 \right] \\ & \left. h_n \underset{n\to\infty}{\xrightarrow{L^1}} (G,b) \right\}. \end{aligned}$$

Indeed if we put $v_n = \nabla_\alpha u_n$ and $h_n = (v_n, b_n)$ it is immediate to see that

$$I(g,b,G) \geq I_1(g,b,G) + I_2(b,G).$$

Next we prove and upper bound for I

$$I(g,G,b) \leq \hat{I}_1(g,G,b) + \hat{I}_2(b,G) \tag{3.21}$$

where

$$\begin{aligned} \hat{I}_1(g,b,G) &= \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n) | \frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2 \right. \\ & \left. u_n \xrightarrow{L^1} g, \ \ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b, \ \ ||\nabla_\alpha u_n - G||_{L^1} \leq C{\epsilon_n}^2 \right\}, \end{aligned}$$

and

$$\begin{aligned} \hat{I}_2(b,G) &= \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \left\{ \liminf_{\epsilon_n \to 0} \left[\int_\Omega W(h_n, \nabla_\alpha h_n | \frac{1}{\epsilon_n}\nabla_3 h_n)\, dx \right.\right. \\ & \left. + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n) | \frac{1}{\epsilon_n}\nu_3(h_n))\, d\mathcal{H}^2 \right] \\ & \left. h_n \underset{n\to\infty}{\xrightarrow{L^1}} (G,b), \ \ ||v_n - G||_{L^1} \leq C{\epsilon_n}^2 \right\}. \end{aligned}$$

Fix a sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$. From the definition of $\hat{I}_1$ we can find a sequence $u_n \in SBV^2(\Omega;\mathbb{R}^d)$ with

$$\left(u_n, \nabla_\alpha u_n, \frac{1}{\epsilon_n}\nabla_3 u_n\right) \xrightarrow{L^1} (g, G, b)$$

and

$$||\nabla_\alpha u_n - G||_{L^1} \leq C\epsilon_n^2 \tag{3.22}$$

such that

$$\hat{I}_1(g,b,G) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2(x).$$

Moreover, from the definition of $\hat{I}_2$, we can find sequences $v_n \in SBV(\Omega;\mathbb{R}^{3\times 2})$, $b_n \in SBV(\Omega;\mathbb{R}^3)$ with

$$(v_n, b_n) \xrightarrow{L^1} (G, b),$$

and

$$||v_n - G||_{L^1} \leq C{\epsilon_n}^2 \tag{3.23}$$

in such a way that (setting $h_n = (v_n, b_n)$) the equality bellow holds

$$\begin{aligned}
\hat{I}_2(b,G) &= \lim_{n\to\infty} \left[\int_\Omega W(h_n, \nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\, dx \right.\\
&+ \left. \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n), \frac{1}{\epsilon_n}\nu_3(h_n))\, d\mathcal{H}^2\right].
\end{aligned}$$

Remark 3.1 *In fact the equalities above for $\hat{I}_1$ and $\hat{I}_2$ only hold if we pass to a subsequence of $\{\epsilon_n\}$ (for which we still use the same notation). In the next sections we will prove that $\hat{I}_1$ and $\hat{I}_2$ are independent of the sequence ϵ_n.*

Now we can construct a sequence w_n as follows

$$w_n := u_n + \rho_n - \hat{\rho}_n$$

where $\nabla\rho_n = (v_n - \nabla_\alpha u_n, \epsilon_n b_n - \nabla_3 u_n)$ is obtained from Theorem 2.2 and $\hat{\rho}_n$ is a piecewise constant function such that $\rho_n - \hat{\rho}_n \xrightarrow{L^1} 0$ and $||D\rho_n||(\Omega) - ||D\hat{\rho}_n||(\Omega) \to 0$ (see Lemma 2.1). Note that

$$||D\rho_n||(\Omega) \leq C\int_\Omega |v_n - \nabla_\alpha u_n|\, dx + C\epsilon_n \int_\Omega \left|b_n - \frac{\nabla_3 u_n}{\epsilon_n}\right|\, dx. \tag{3.24}$$

It is easy to check that w_n is an admissible sequence for I. Indeed we have

$$\nabla_\alpha w_n = v_n,\ \frac{1}{\epsilon_n}\nabla_3 w_n = b_n$$

and thus $\left(w_n, \nabla_\alpha w_n, \frac{1}{\epsilon_n}\nabla_3 w_n\right) \xrightarrow{L^1} (g, G, b)$. Then, setting $h_n = (v_n, b_n)$ and using H_4, (3.22), (3.23) and (3.24), we have that

$$\begin{aligned} I(g,b,G) &\leq \liminf \int_{\Omega} W\left(h_n, \nabla_\alpha h_n, \frac{1}{\epsilon_n} h_n\right) dx + \int_{Sh_n} \Psi_2\left([h_n], \nu_\alpha(h_n), \frac{1}{\epsilon_n}\nu_3(h_n)\right) d\mathcal{H}^2 \\ &+ \int_{Sw_n} \Psi_1\left([w_n], \nu_\alpha(w_n), \frac{1}{\epsilon_n}\nu_3(w_n)\right) d\mathcal{H}^2 \\ &\leq \hat{I}_2(b,G) + \hat{I}_1(g,b,G) + C\int_{S\rho_n} |[\rho_n]| \left|\left(\nu_\alpha(\rho_n), \frac{1}{\epsilon_n}\nu_3(\rho_n)\right)\right| d\mathcal{H}^2 \\ &+ C\int_{S\hat{\rho}_n} |[\hat{\rho}_n]| \left|\left(\nu_\alpha(\hat{\rho}_n), \frac{1}{\epsilon_n}\nu_3(\hat{\rho}_n)\right)\right| d\mathcal{H}^2 \\ &\leq \hat{I}_2(b,G) + \hat{I}_1(g,b,G) + \frac{C}{\epsilon_n}||D\rho_n||(\Omega) \\ &\leq \hat{I}_2(b,G) + \hat{I}_1(g,b,G) + \frac{C}{\epsilon_n}\int_\Omega |v_n - \nabla_\alpha u_n|\, dx + C\int_\Omega \left|b_n - \frac{\nabla_3 u_n}{\epsilon_n}\right| dx \\ &\leq \hat{I}_2(b,G) + \hat{I}_1(g,b,G) + C\epsilon_n + C\int_\Omega \left|b_n - \frac{\nabla_3 u_n}{\epsilon_n}\right| dx. \end{aligned}$$

Letting $n \to \infty$, (3.21) follows.

3.2 Lower bounds

We first derive a lower bound for I^1. Fix $\{\epsilon_n\}$ and denote by

$$I_1^{\{\epsilon_n\}}(g,b,G) = \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n), \frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2 \right. \\ \left. u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow{L^1} G \right\}.$$

Given an arbitrary sequence $u_n \xrightarrow{L^1} g$ with $\nabla_\alpha u_n \xrightarrow{L^1} G$ and $\frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b$, define

$$L_1 := \liminf_{n\to\infty} \int_{S_{u_n}\cap\Omega} \Psi_1\left([u_n], \nu_\alpha(u_n), \frac{1}{\epsilon_n}\nu_3(u_n)\right) d\mathcal{H}^2.$$

Then clearly

$$L_1 \geq \liminf_{n\to\infty} \int_{S_{u_n}\cap\Omega} \overline{\Psi}_1([u_n], \nu_\alpha(u_n))\, d\mathcal{H}^2.$$

For x_3 fixed set now $u_n^{x_3}(x_\alpha) := u_n(x_\alpha, x_3)$. Then, by Theorem 3.1.1 in [6](Slicing Theorem),

$$\begin{aligned} L_1 &\geq \liminf_{n\to\infty} \int_{S_{u_n}\cap\Omega} \overline{\Psi}_1([u_n], \nu_\alpha(u_n))\, d\mathcal{H}^2 \\ &\geq \liminf_{n\to\infty} \int_0^1 \int_{S_{u_n^{x_3}}\cap\omega} \overline{\Psi}_1([u_n^{x_3}], \nu_\alpha(u_n^{x_3}))\, d\mathcal{H}^1(x_\alpha)\, dx_3. \end{aligned}$$

Note that

$$u_n^{x_3} \xrightarrow{L^1} g, \qquad \nabla u_n^{x_3} \xrightarrow{L^1} G.$$

Hence, by Fatou's Lemma and the integral representation in Theorem 2.9 (see also Remark 3.2 and 3.3 below), and using the arbitrariness of the sequence u_n we arrive at the inequality below

$$I_1^{\{\epsilon_n\}}(g,b,G) \geq \int_\omega W_1(G-\nabla g)\, dx_\alpha + \int_{S_g} \Gamma_1([g],\nu(g))\, d\mathcal{H}^1 + \int_\omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right) d|D^c g|$$

where

$$W_1(A) = \inf_{u\in SBV(Q';\mathbb{R}^3)} \left\{ \int_{Su\cap Q'} \overline{\Psi}_1([u],\nu(u))\, d\mathcal{H}^1, u|_{\partial Q} = 0, \nabla u = A \text{ a.e. in } Q' \right\},$$

and

$$\Gamma_1(\lambda,\nu) = \inf_{u\in SBV(Q'_\nu;\mathbb{R}^3)} \left\{ \int_{Q'_\nu\cap S_u} \overline{\Psi}_1([u],\nu(u))\, d\mathcal{H}^1, u|_{\partial Q'_\nu} = \tau_{(\lambda,\nu)}, \nabla u = 0 \right\},$$

with

$$\tau_{(\lambda,\nu)}(x_\alpha) := \begin{cases} \lambda & \text{if } x_\alpha.\nu > 0 \\ 0 & \text{if } x_\alpha.\nu < 0. \end{cases}$$

Remark 3.2 *We note that $\overline{\Psi}_1$ is continuous. In fact, if $(\lambda_n,\nu_n)\to(\lambda,\nu)$ in $\mathbb{R}^3\times S^1$ and if we assume that* $\lim_{n\to\infty}\overline{\Psi}_1(\lambda_n,\nu_n) = \liminf_{n\to\infty}\overline{\Psi}_1(\lambda_n,\nu_n)$ *we get that*

$$\lim_{n\to\infty}\overline{\Psi}_1(\lambda_n,\nu_n) = \lim_{n\to\infty}\Psi_1(\lambda_n,\tau_n) = \Psi_1(\lambda,\tau)$$

where $\tau_n = \frac{(\nu_n,t_n)}{\sqrt{|\nu_n|^2+t_n^2}}$ *is a sequence in S^2 which we assume to be convergent to a point $\tau\in S^2$. On the other hand if we write* $\tau = \lim_{n\to\infty}\frac{(\nu,t_n)}{\sqrt{|\nu|^2+t_n^2}}$ *and use the definition of $\overline{\Psi}_1$ we get that*

$$\Psi_1(\lambda,\tau) = \lim_{n\to\infty}\frac{\Psi_1(\lambda,\nu,t_n)}{\sqrt{|\nu|^2+t_n^2}} \geq \overline{\Psi}_1(\lambda,\nu),$$

which together with the previous equality gives the lower semicontinuity of $\overline{\Psi}_1$. For the upper semi-continuity of $\overline{\Psi}_1$ we consider again a sequence $(\lambda_n,\nu_n)\to(\lambda,\nu)$ in $\mathbb{R}^3\times S^1$ and t_n such that $\overline{\Psi}_1(\lambda,\nu) = \lim_{n\to\infty}\frac{\Psi_1(\lambda_n,\nu_n,t_n)}{\sqrt{|\nu_n|^2+t_n^2}}$ *and thus*

$$\limsup_{n\to\infty}\overline{\Psi}_1(\lambda_n,\nu_n) \leq \lim_{n\to\infty}\frac{\Psi_1(\lambda_n,\nu_n,t_n)}{\sqrt{|\nu_n|^2+t_n^2}} = \lim_{n\to\infty}\frac{\Psi_1(\lambda,\nu,t_n)}{\sqrt{|\nu|^2+t_n^2}} = \overline{\Psi}_1(\lambda,\nu).$$

Remark 3.3 *We do not know if $\overline{\Psi}_1$ inherits the subadditivity from Ψ_1 but it keeps the Lipschitz continuity and the growth conditions from Ψ_1 which are also sufficient to apply Theorem 2.9.*

Now we derive the lower bound for I_2. Again, for a given sequence $\{\epsilon_n\}$ and $(b, G) \in BV(w; \mathbb{R}^3) \times BV(w; \mathbb{R}^{3\times 2})$, we define

$$
\begin{aligned}
I_2{}^{\{\epsilon_n\}}(b, G) \quad = \quad & \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \Bigg\{ \liminf_{\epsilon_n \to 0} \Bigg[\int_\Omega W(h_n, \nabla_\alpha h_n, \frac{1}{\epsilon_n} \nabla_3 h_n)\, dx \\
& + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n), \frac{1}{\epsilon_n} \nu_3(h_n))\, d\mathcal{H}^2(x) \Bigg] \\
& h_n \xrightarrow{L^1} (G, b) \Bigg\}.
\end{aligned}
$$

Let $h_n \xrightarrow{L^1} (G, b)$ be an arbitrary sequence and for fixed $\{\epsilon_n\}$ define

$$
\begin{aligned}
L_2 : \quad = \quad & \liminf_{n\to\infty} \Bigg[\int_\Omega W(h_n, \nabla_\alpha h_n, \frac{1}{\epsilon_n} \nabla_3 h_n)\, dx \\
& + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n), \frac{1}{\epsilon_n} \nu_3(h_n))\, d\mathcal{H}^2 \Bigg].
\end{aligned}
$$

Clearly we have that

$$
\begin{aligned}
L_2 \quad \geq \quad & \liminf_{n\to\infty} \Bigg[\int_\Omega \overline{W}(h_n, \nabla_\alpha h_n)\, dx \\
& + \int_{S_{h_n}} \overline{\Psi}_2([h_n], \nu_\alpha(h_n))\, d\mathcal{H}^2 \Bigg].
\end{aligned}
$$

Let now x_3 be fixed and set $h_n^{x_3}(x_\alpha) = h_n(x_\alpha, x_3)$. Then, by Theorem 3.1.1 of [6],

$$
\begin{aligned}
L_2 \quad \geq \quad & \liminf_{n\to\infty} \Bigg[\int_\Omega \overline{W}(h_n, \nabla_\alpha h_n)\, dx \\
& + \int_{S_{h_n}} \overline{\Psi}_2([h_n], \nu_\alpha(h_n))\, d\mathcal{H}^2(x) \Bigg] \\
& \geq \liminf_{n\to\infty} \Bigg[\int_0^1 \int_w \overline{W}(h_n, \nabla_\alpha h_n^{x_3})\, dx_\alpha\, dx_3 \\
& + \int_0^1 \int_{S_{h_n^{x_3}} \cap w} \overline{\Psi}_2([h_n^{x_3}], \nu_\alpha(h_n^{x_3}))\, d\mathcal{H}^1(x_\alpha)\, dx_3 \Bigg].
\end{aligned}
$$

Noting that $h_n^{x_3} \xrightarrow{L^1} (b, G)$, by Fatou's Lemma a lower bound for $I_2{}^{\{\epsilon_n\}}(b, G)$ will be given by Theorem 4.2.2 in [5].

3.3 Upper bounds

We first derive an upper bound for $\hat{I}_1(g,b,G)$. We recall the family of functionals (depending on a parameter $\delta > 0$) defined in (2.6) for G and b fixed piecewise constant functions:

$$\begin{aligned}\hat{I}_{1,\delta}^{(G,b)}(g,A) &= \inf_{\{g_n\}}\Bigg\{\liminf_{\epsilon_n\to 0}\delta\int_{A\times I}\left|\left(\nabla_\alpha g_n|\frac{\nabla_3 g_n}{\epsilon_n}\right)\right|\,dx \\ &\quad + \int_{S_{g_n}\cap(A\times I)}\Psi_1\left([g_n],\nu_\alpha(g_n),\frac{1}{\epsilon_n}\nu_3(g_n)\right)\,d\mathcal{H}^2 \\ &\quad g_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 g_n \xrightarrow{L^1} b,\ ||\nabla_\alpha g_n - G||_{L^1}\le C{\epsilon_n}^2\Bigg\}.\end{aligned}$$

Using Proposition 2.5 we get that for $A \in \mathcal{A}(w)$

$$\hat{I}_{1,\delta}^{(G,b)}(g,A) = \int_A f_\delta^{(G,b)}(\nabla g)\,dx_\alpha + \int_{S_g\cap A} h_\delta^{(G,b)}([g],\nu_g)\,d\mathcal{H}^1.$$

Given $g \in SBV^2(\omega;\mathbb{R}^3)$ with ∇g an L^∞ piecewise constant function we fix $G = \nabla g$ and construct an admissible sequence for $\hat{I}_{1,\delta}^{(\nabla g,b)}$ as follows

$$g_n := g + \epsilon_n x_3 b.$$

Using the sequence above we get

$$\int_A f_\delta^{(\nabla g,b)}(\nabla g) + \int_{S_g\cap A} h_\delta^{(\nabla g,b)}([g],\nu_g)d\mathcal{H}^1 \le \delta\int_A |(\nabla g,b)| + \int_{S_g\cap A}\Psi_1([g],\nu_g)d\mathcal{H}^1,$$

and the by taking the Radon-Nikodým derivative with respect to the Lebesgue measure it follows that

$$f_\delta^{(\nabla g,b)}(\nabla g) \le \delta|(\nabla g,b)|$$

for a.e. $x\in\omega$.

Now we construct a sequence in order to get an estimate for $h_\delta^{(\nabla g,b)}$. Fix μ in $\mathbb{R}^3$ and $\nu_\alpha \in S^1$, and let $\tau\in\mathbb{R}$ be such that

$$\overline{\Psi}_1(\mu,\nu_\alpha) = \frac{1}{\sqrt{|\nu_\alpha|^2+\tau^2}}\Psi_1(\mu,\nu_\alpha,\tau)$$

Consider the sequence

$$\rho_n := \begin{cases}\mu & \text{if } \nu_\alpha.(x_\alpha - x_0) + \epsilon_n x_3\tau > 0\\ 0 & \text{if } \nu_\alpha.(x_\alpha - x_0) + \epsilon_n x_3\tau < 0\end{cases}$$

and let

$$g_n := g - h_n + \rho_n + \epsilon_n x_3 b,$$

where h_n is a piecewise constant function such that $h_n \to g$ in L^1, and $||Dh_n||(\Omega)\to||Dg||(\Omega)$ (see Lemma 2.1). We then have that

$$g_n \xrightarrow{L^1}\begin{cases}\mu & \text{if } \nu_\alpha.(x_\alpha - x_0) > 0,\\ 0 & \text{if } \nu_\alpha.(x_\alpha - x_0) < 0.\end{cases}$$

Note that $\nabla g_n = \nabla g$ and $\frac{1}{\epsilon_n}\nabla_3 g_n = b$. Thus

$$\begin{aligned}\hat{I}_{1,\delta}^{(\nabla g,b)}(g,A) &= \int_A f_\delta^{(\nabla g,b)}(0)\,dx + \int_{A\cap\{\nu_\alpha.(x_\alpha-x_0)=0\}} h_\delta^{(\nabla g,b)}(\mu,\nu_\alpha)\,d\mathcal{H}^1 \\ &\le \delta\int_A |(\nabla g,b)| + \Psi_1(\mu,\nu_\alpha,\tau)\mathcal{L}^1\left(A\cap\{\nu_\alpha.(x_\alpha-x_0)=0\}\right) + C||Dg||(A).\end{aligned}$$

If we choose $A = Q'_{\nu_\alpha}(x_0;s)$ and let $s\to 0$ we get

$$h_\delta^{(\nabla g,b)}(\mu,\nu_\alpha) \le \overline{\Psi}_1(\mu,\nu_\alpha).$$

Using the bounds above we get the estimates

$$\hat{I}_{1,\delta}^{(\nabla g,b)}(g,A) \le \delta\int_A |(\nabla g,b)|\,dx + \int_{Sg}\overline{\Psi}_1([g],\nu_g)\,d\mathcal{H}^1$$

and letting $\delta\to 0$ we have that

$$\hat{I}_1(g,\nabla g,b) \le \int_{S_g}\overline{\Psi}_1([g],\nu_g)\,d\mathcal{H}^1. \tag{3.25}$$

Next we extend the upper bound above to a general $g\in SBV^2(\omega;\mathbb{R}^3)$. Let h_n be a L^∞ piecewise constant function such that $h_n\to\nabla g$ in L^1 (see Lemma 2.1), $H_n\in SBV_2(\omega;\mathbb{R}^3)$ such that $\nabla H_n = h_n - \nabla g$ and $\int_{S_{H_n}}|[H_n]|d\mathcal{H}^1\to 0$ (see Theorem 2.2) and $\tilde{H}_n$ a sequence of piecewise constant functions such that $H_n - \tilde{H}_n\to 0$ and $|DH_n|(\omega) - |D\tilde{H}_n|(\omega)\to 0$. Using the sequence

$$g_n = g + H_n - \tilde{H}_n,$$

the bounds of $\overline{\Psi}_1$ and the lower semicontinuity of $\hat{I}_1$ we get (3.25). Note that the estimate (3.25) is independent of b, so by using again the lower-semicontinuity of $\hat{I}_1$ if follows that holds for general b.

Using now Theorem 2.9 it follows that

$$\hat{I}_1(g,G,b) \le \int_\omega W_1(G-\nabla g)\,dx + \int_{S_g}\Gamma_1([g],\nu_g)d\mathcal{H}^1 + \int_\omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right)\,d|D^c g|.$$

Finally we derive the upper bound for $\hat{I}_2$. By a straightforward application of Theorem 2.4 we obtain that $\hat{I}_2$ is independent of the sequence $\{\epsilon_n\}$ and has an integral representation of the form (see section 2.3 where we have used similar arguments for $\hat{I}_1$)

$$\hat{I}_2(h;A) = \int_A W_0(h,\nabla h)\,dx + \int_{A\cap S(h)}\Psi_0([h],\nu(h))\,d\mathcal{H}^1 + \int_A W_0^\infty(h,\frac{dD^c h}{d|D^c h|})d|D^c h|.$$

We next prove that

$$W_0(a,F_\alpha) \le \overline{W}(a,F_\alpha),\ \forall a\in\mathbb{R}^{3\times 3},\ \forall F_\alpha\in\mathbb{R}^{3\times 3\times 2}, \tag{3.26}$$

and that

$$\Psi_0(\lambda,\nu_\alpha) \le \overline{\Psi}_2(\lambda,\nu_\alpha), \forall\lambda\in\mathbb{R}^{3\times 3},\ \forall\nu_\alpha\in S^1. \tag{3.27}$$

In order to prove (3.26) consider

$$h_n = a + F_\alpha(x_\alpha - x_0) + \epsilon_n F_3 x_n,$$

where $F_3 \in \mathbb{R}^3$ is such that

$$\overline{W}(a, F_\alpha) = W(a, F_\alpha, F_3).$$

Notice that, by (H_1) the infimum in the definition fo $\overline{W}$ is attained. Since

$$h_n \overset{L^1}{\longrightarrow} h = a + F_\alpha(x_\alpha - x_0),$$

by the definition of $\hat{I}_2$, we have that

$$\begin{aligned}\hat{I}_2(a + F_\alpha(x_\alpha - x_0); Q(x_0, \delta)) &\leq \liminf_{\epsilon_n \to 0} \int_{Q(x_0,\delta)} W(a + F_\alpha(x_\alpha - x_0) + \epsilon_n F_3 x_3, F_\alpha | F_3)\, dx \\ &\leq \int_{Q(x_0,\delta)} W(a, F_\alpha, F_3)\, dx + C \int_{Q(x_0,\delta)} F_\alpha(x_\alpha - x_0)\, dx,\end{aligned}$$

where we used (H_2) in the last inequality. Dividing both terms in the previous inequality by δ^N and letting $\delta \to 0^+$, we end up with (3.26).

We proceed similarly to prove (3.27). Define h_n (see figure 1 below) by:

$$h_n = \begin{cases} \lambda & \text{if } \ \nu_\alpha \cdot (x_\alpha - x_0) + \epsilon_n (x_3 - i\epsilon_n)\mu > 0, \ \ x_3 \in [i\epsilon_n, (i+1)\epsilon_n] \qquad (\mathrm{i}\, even) \\ \lambda & \text{if } \ \nu_\alpha \cdot (x_\alpha - x_0) - \epsilon_n (x_3 - (i+1)\epsilon_n)\mu > 0, \ x_3 \in [i\epsilon_n, (i+1)\epsilon_n] \ (\mathrm{i}\, odd) \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \mathbb{N}$.

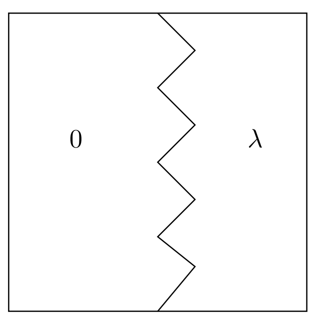

Figure 1:

Clearly $h_n \overset{L^1(Q(x_0;\delta);\mathbb{R}^3)}{\longrightarrow} u_{x_0}^{\lambda,\nu_\alpha}(x)$ given by

$$u_{x_0}^{\lambda,\nu_\alpha}(x) = \begin{cases} \lambda & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) > 0, \\ 0 & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) \leq 0. \end{cases}$$

and $||h_n - u_{x_0}^{\lambda,\nu_\alpha}||_{L^1(Q(x_0;\delta);\mathbb{R}^3)} \leq C\epsilon_n^2$. Thus h_n is admissible for $\hat{I}_2$ and we have that

$$\begin{aligned}
\hat{I}_2(u_{x_0}^{\lambda,\nu_\alpha}(x); Q(x_0,\delta)) \quad \leq \quad & \liminf_{n\to\infty} \int_{Q(x_0,\delta)} W(h_n, 0)\, dx \\
& + \frac{1}{2}\Psi_2(\lambda, \nu_\alpha, \mu)\mathcal{L}^2\left(\{\nu_\alpha \cdot (x_\alpha - x_0) = 0 \cap (Q(x_0,\delta)\}\right) \\
& + \frac{1}{2}\Psi_2(\lambda, \nu_\alpha, -\mu)\mathcal{L}^2\left(\{\nu_\alpha \cdot (x_\alpha - x_0) = 0 \cap (Q(x_0,\delta)\}\right) \\
& = \liminf_{n\to\infty} \int_{Q(x_0,\delta)} W(h_n, 0)\, dx + \Psi_2(\lambda, \nu_\alpha, \mu)\mathcal{L}^2\left(\{\nu_\alpha \cdot (x_\alpha - x_0) = 0 \cap (Q(x_0,\delta)\}\right)
\end{aligned}$$

where we used (H_8) in the last equality. Here $\mu \in \mathbb{R}$ is such that $\bar{\Psi}_2(\lambda, \nu_\alpha) := \frac{1}{\sqrt{|\nu_\alpha|^2+\mu^2}}\Psi_2(\lambda, \nu_\alpha, \mu)$. The existence of such μ results from the coercivity condition (H_5). Dividing both terms by δ^{N-1} letting $\delta \to 0^+$ (and for x_0 a point in S_h) we get (3.27). The upper bound results now of relaxing from SBV to BV, together with an application of Theorem 4.2.2 in [5].

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