Mathematical measures of fairness in legislative districting

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Abstract

Gerrymandering is the term used to describe the drawing of legislative districts to favor one party over another. Recently, many mathematicians have tried to develop mathematical tools to decide if legislative districts are gerrymandered, and define fair methods of districting. For example, Landau, Reid, and Yershov [A Fair Division Solution to the Problem of Redistricting, *Social Choice and Welfare*, 2008] propose a protocol for districting based on a two-player fair-division process, where each player is entitled to draw the districts for a portion of the state. We call this the LRY protocol. Landau and Su [Fair Division and Redistricting, arXiv:1402.0862, 2014] propose a measure of the fairness of a districting called the geometric target. In this paper we prove that the number of districts a party can win under the LRY protocol can be at most two fewer than their geometric target, assuming no geometric constraints on the districts. This is the first quantitative bound of this type and we provide examples to prove this bound is tight. The main tools involved in the proof are identifying optimal strategies for each player in the protocol, and analyzing the number of districts they win using these strategies. We also show that the protocol on a state drawn with geometric constraints can produce an unbounded difference from the geometric target. Lastly, we explore ways to generalize the LRY protocol to more than two players, and define a similar protocol with lower bounds on number of victories.

1 Introduction

Gerrymandering is the act of drawing legislative districts to favor one group over another. A common form of Gerrymandering is known as *Partisan Gerrymandering* where a political party draws districts to favour their party over opposing parties. Recent work has been done on designing protocols to prevent gerrymandering or to detect if a state has been gerrymandered. Some protocols for districting are [8, 9]. Different methods of detecting gerrymandering are discussed in [2, 11, 4]. Redistricting of states can also be thought of as what is known as a *Fair Division* problem. *Fair Division* is the question of how to divide an object or set of objects among parties so that all parties receive a portion that is considered *fair* by their own evaluation or some metric. The classic *fair-division* problem of *I cut you choose* for cake cutting has been heavily expanded upon such as in [3]. Many other ideas have been explored in the field of *Fair Division* such as Sperner's lemma for Cake cutting and Sperner applications in rental division, [10, 5] and envy-free necklace cutting [1].

The main *Fair Division* protocol we analyze in this paper, which we call the LRY protocol, is a two-player fair-division procedure described in [6]. In the LRY protocol the two parties are presented with a sequence of splits of a given state, and asked to submit preferences for which side of the split they would prefer to draw the districts. In the end, a split is chosen and one party draws the districts on one side, and the other party draws the districts on the other. The protocol is designed so that the competing interests of the two parties prevent either one from gaining a significant advantage.

In [7], Landau and Su define a measure of fairness for a given district map called the geometric target which is the average of the best and worst cases for a party's measure of success. In Section 2 we fully explain this protocol using our notation. In Theorem 2.13, we show that the number of districts won by a party under the LRY protocol can differ from that party's geometric target by at most 2 and by a partys k-split geometric target by at most $\frac{3}{2}$.

In section 3 we show that using geometric constraints, it is possible to create a situation where the protocol can return a result that arbitrarily far from the geometric target.

In section 4 we discuss the difficulties of directly applying the LRY protocol to an arbitrary m party case. We define a new protocol which we first apply to three partys is of a similar essence of the LRY protocol, and we prove a minimum number of districts a party is guaranteed relative to their support in Theorem 4.3. This insight allows us to then generalize to m parties and prove a lower bound for the number of districts a party can win by playing certain strategies in Theorem 4.4.

2 The LRY Protocol

2.1 Notation for parties, support and splits

We first define the notation we will use to describe the LRY protocol. We suppose there are two parties, denoted A and B. We will use P to denote a generic party, $P \in \{A, B\}$ and $\overline{P}$ to denote the party opposing P. The goal of the protocol is to draw districts for a state, and we let $n \in \mathbb{N}$ denote the number of districts.

For a party P, let x_P denote the total support of player P in the state, and we assume $x_A + x_B = n$, so since there are n districts, the support of the parties in each district should sum to 1.

For $0 \leq k \leq n$ define a k-split to be a division of the state so that on one side of the split (by convention, we call this the left side) the two parties' support sums to k, i.e. there are k districts' worth of population on the left side of the k-split, and $n - k$ districts' worth on the right side. For a k-split let L_k denote the area on the left of the split, and R_k denote the area on the right. We denote by S a side $S \in \{L, R\}$. Given a k-split, Let S_k denote the area on the specified side of the k-split, and $\overline{S_k}$ denote the area on the other side.

Call a sequence of k-splits, $0 \leq k \leq n$ *nested* if for every $k \geq 1$, $L_{k-1} \subseteq L_k$. For any nested sequence of k-splits, denote by $x_P(S_k)$ the total support for party P in S_k, and let $x_P(k) = x_P(L_k) - x_P(L_{k-1})$, i.e. $x_P(k)$ is the support for party P between the $(k-1)$-split and the k-split. Note that for all k, $x_A(k) + x_B(k) = 1$

We assume that whichever candidate has more support in a district will win the district, and we adopt the convention that for all k, $x_P(S_k)$ is not an integer multiple of .5. Therefore, we assume a party can always win a district with a strict majority, and there is no need to consider tied districts.

2.2 The LRY Protocol

Following Landau and Su [7], we now describe the LRY protocol. In the protocol, an administrator (someone not affiliated with either party) makes a sequence of nested k-splits.

For all k, each party indicates which of the following options they prefer:

1. Party A districts L_k and Party B districts R_k

2. Party B districts L_k and Party A districts R_k

A party may also indicate that they are indifferent between the two options.

The outcome of the protocol is as follows: If there exists some k such that Party A and Party B both prefer that Party A district L_k or they both prefer that Party B district L_k, then a map is created using that assignment of areas to district. If there is a k such that a party is indifferent but the other is not, the preferences of the non-indifferent party are chosen. If there exists a k such that both parties are indifferent then one of Option 1 and Option 2 is randomly chosen. If any of these options are chosen, then both parties will district a side they prefer or are indifferent to. Therefore, they will district a side in which they desire to.

Since both sides prefer to district the right side in a 0-split and the left side in an n-split, if none of the above scenarios occur, it must be the case that there exists a k such that for the $(k-1)$-split Party A prefers Option 2 and Party B prefers Option 1 but for the k-split they switch their preferences. We call this scenario the *coin flip scenario.* In this case, the protocol randomly returns one of the following four options:

- Option 1 for the $(k-1)$-split
- Option 2 for the $(k-1)$-split
- Option 1 for the k-split
- Option 2 for the k-split.

We analyze the four options of the *coin flip* with respect to our measure of fairness the *geometric target.* This will show that the protocol contains underlying inherent properties to all fair division problems. These properties are as follows: The LRY protocol represents properties inherent to all fair - division protocols.

1. *Multilateral evaluation.* A party defines fairness by their own preferences and metric.
2. *Procedural fairness.* All parties understand the procedure, are involved in the process and understand the way fairness is measured. The parties are more likely to feel that the process is fair as they are involved in the process by stating their preferences and drawing the districtings of the side they are given as they please
3. *Fairness guarantee.* If you are honest with your preferences, you will be guaranteed to obtain a portion that you deem fair under your own evaluation.

Parties define their preferences based upon the number of districts won given by their optimal strategies. Since each party is involved in the process of districting this allows the party's to openly state and try to attain what they believe is fair according to the number of districts they can win given their optimal strategy.

2.3 Optimal Strategies

We now turn our attention to the optimal strategies that each player should use in the LRY protocol. In this section, we assume there are no geometric constraints on how the districts are drawn other than the k-splits. We assume throughout that each player is simply trying to maximize the number of districts they win according to a *voting model* V they create. A *voting model* is a prediction of how the people in the state will vote. Each party may have their own private voting model which they base their preferences upon. Notice, however, that since each party may have their own voting model that differs for one another that If both parties share the same voting model then it is impossible for them to prefer the same option. Additionally, if one party is indifferent then both parties are indifferent.

We define $P_i(S_k, P_j)$ to be the number of districts won by party P_i on side S_k when party P_j draws districts on S_k. Let $P(S_k)$ be the total number of wins for P when they district S_k and $\overline{P}$ districts $\overline{S}_k$. That is,

$$P_i(S_k) = P_i(S_k, P_i) + P_i(\overline{S}_k, \overline{P}_i).$$

since the number of districts in the state is the sum of the districts won by both parties

$$P_i(S_k) + \overline{P}_i(\overline{S}_k) = n.$$

Proposition 2.1 shows the maximum number of wins a party can win when they are in control of the districting. Let $|S_k|$ be the number of districts in S_k, i.e. $|L_k| = k$ and $|R_k| = n - k$.

Proposition 2.1. $P(S_k, P) = \min\{\lfloor 2x_P(S_k)\rfloor,\ |S_k|\}$.

Proof. There are two cases, depending whether party P has a majority or not.

Case 1: Suppose $x_P(S_k) > \frac{|S_k|}{2}$, and thus $|S_k| = \min\{\lfloor 2x_P(S_k)\rfloor,\ |S_k|\}$. In this case, player P can create $|S_k|$ districts where in each district their support is $\frac{x_P(S_k)}{|S_k|} > \frac{1}{2}$. Thus they can win $|S_k|$ districts, and this is the best possible.

Case 2: Suppose $x_P(S_k) < \frac{|S_k|}{2}$, and thus $\lfloor 2x_P(S_k)\rfloor = \min\{\lfloor 2x_P(S_k)\rfloor,\ |S_k|\}$. For any district i to be a victory for P then $x_P(i) > 0.5$. Thus player P can create $\lfloor 2x_P(S_k)\rfloor$ districts with support just over 0.5, and can win these districts, but their remaining support is less than .5, so they cannot win any more. $\square$

Next, we show how many districts P will win if their opponent draws the districts on one side of a k-split. First we need to prove a property of floors and ceilings.

Lemma 2.2. *Assume $x, y \in \mathbb{R}$ and $x + y = k$ with $k \in \mathbb{N}$. Then* $\min\{\lfloor 2x\rfloor, k\} + \max\{\lceil y - x\rceil, 0\} = k$.

Proof. Since $y = k - x$,

$$\min\{\lfloor 2x\rfloor, k\} + \max\{\lceil y - x\rceil, 0\} = \min\{\lfloor 2x\rfloor, k\} + \max\{\lceil k - 2x\rceil, 0\}.$$

Case 1: Assume $x > \frac{k}{2}$. Then

$$\min\{\lfloor 2x\rfloor, k\} + \max\{\lceil k - 2x\rceil, 0\} = k + 0 = k.$$

Case 2: Assume $x \leq \frac{k}{2}$. Then

$$\begin{aligned} &\min\{\lfloor 2x\rfloor, k\} + \max\{\lceil k - 2x\rceil, 0\} \\ &= \lfloor 2x\rfloor + \lceil k - 2x\rceil \\ &= \lfloor 2x\rfloor + k + \lceil -2x\rceil. \end{aligned}$$

Since $\lceil -z\rceil = -\lfloor z\rfloor$ for all $z \in \mathbb{R}$, this quantity is equal to $\lfloor 2x\rfloor + k - \lfloor 2x\rfloor = k$. $\square$

Corollary 2.3 shows the maximum number of wins on a side by a party when the other party is districting that side.

Corollary 2.3. $P(S_k, \overline{P}) = \max\left\{\lceil x_P(S_k) - x_{\overline{P}}(S_k)\rceil, 0\right\}.$

Proof. Since $x_P(S_k) + x_{\overline{P}}(S_k) = |S_k|$ and $|S_k| \in \mathbb{N}$, by Lemma 2.2 it follows that

$$\min\{\lfloor 2x_{\overline{P}}(S_k)\rfloor,\ |S_k|\} + \max\{\lceil x_P(S_k) - x_{\overline{P}}(S_k)\rceil, 0\} = |S_k|.$$

Additionally,

$$|S_k| = \overline{P}(S_k, \overline{P}) + P(S_k, \overline{P}).$$

By Proposition 2.1, $\overline{P}(S_k, \overline{P}) = \min\{\lfloor 2x_{\overline{P}}(S_k)\rfloor,\ |S_k|\}$, so

$$P(S_k, \overline{P}) = \max\{\lceil x_P(S_k) - x_{\overline{P}}(S_k)\rceil, 0\}.$$

□

2.4 The Geometric Target

The geometric target provides a measure of fairness for a districting protocol. The geometric target is defined as the average of the parties best case scenario and worst case scenario with respect to there voting model V. That is the average of the number of districts a party wins when they district the state and the number of districts a party wins when the opposing party districts the state.

We now wish to show that the geometric target for P is at most $\frac{1}{2}$ away from any k-split geometric target. To do this, we first prove a result about the difference between floors and ceilings.

Lemma 2.4. *Let $r, s \in R^+$, and $t = r + s$. Then*

i. $|\lceil t\rceil - (\lceil r\rceil + \lceil s\rceil)| \leq 1$

ii. $|\lceil t\rceil - (\lceil r\rceil + \lfloor s\rfloor)| \leq 1$

iii. $|\lfloor t\rfloor - (\lceil r\rceil + \lfloor s\rfloor)| \leq 1$

iv. $|\lfloor t\rfloor - (\lfloor r\rfloor + \lfloor s\rfloor)| \leq 1$.

Proof. Parts i. and ii. follow from the observations that

$$\begin{aligned} \lceil r\rceil + \lceil s\rceil &\geq \lceil r+s\rceil \\ \lceil r+s\rceil &\geq \lceil r\rceil + \lfloor s\rfloor \\ (\lceil r\rceil + \lceil s\rceil) - (\lceil r\rceil + \lfloor s\rfloor)| &\leq 1 \end{aligned}$$

Similarly, Parts iii. and iv. follow from the observations that

$$\begin{aligned} \lfloor r\rfloor + \lfloor s\rfloor &\leq \lfloor r+s\rfloor \\ \lfloor r+s\rfloor &\leq \lceil r\rceil + \lfloor s\rfloor \\ (\lceil r\rceil + \lfloor s\rfloor) - (\lfloor r\rfloor + \lfloor s\rfloor)| &\leq 1 \end{aligned}$$

□

We would like to give formula for the geometric target for a party in terms of that party's support.

Lemma 2.5. *If $x_P > n/2$, then* $\text{geo}(P) = \frac{\lceil 2(x_P)\rceil}{2}$. *If $x_P < n/2$, then* $\text{geo}(P) = \frac{\lfloor 2x_P\rfloor}{2}$.

Proof. In both cases, the party A will win the most districts when it draws all districts, and will win the fewest when B draws all the districts. Thus, by Propositions 2.1 and 2.3, when $x_P > n/2$,

$$\begin{aligned}\operatorname{geo}(A) &= \frac{A(L_n) + A(L_0)}{2} \\ &= \frac{n + \lceil x_A - x_B \rceil}{2} \\ &= \frac{n + \lceil x_A - (n - x_A) \rceil}{2} \\ &= \frac{\lceil 2(x_A) \rceil}{2}.\end{aligned}$$

Similarly, when $x_P < n/2$,

$$\begin{aligned}\operatorname{geo}(A) &= \frac{A(L_n) + A(L_0)}{2} \\ &= \frac{\lfloor 2x_A \rfloor + 0}{2} \\ &= \frac{\lfloor 2x_A \rfloor}{2}.\end{aligned}$$

□

Proposition 2.6. $\operatorname{geo}_k(P) = \frac{P(L_k)+P(R_k)}{2}$.

Proof. For every k- split,

$$\begin{aligned}\operatorname{geo}_k(P) &= \frac{\text{Best Case} + \text{Worst case}}{2} \\ &= \frac{P(L_k, P) + P(R_k, P) + P(L_k, \overline{P}) + P(R_k, \overline{P})}{2} \\ &= \frac{P(L_k) + P(R_k)}{2}.\end{aligned}$$

□

Now we can use Lemma 2.4, Lemma 2.5, and Proposition 2.6 to show that the geometric target for P is at most $\frac{1}{2}$ away from any k-split geometric target.

Lemma 2.7. $|\operatorname{geo}(P) - \operatorname{geo}_k(P)| \leq \frac{1}{2}$.

Proof. There are 4 cases.

Case 1: Assume $x_A > \frac{n}{2}$, $x_A(L_k) > \frac{k}{2}$, and $x_A(R_k) > \frac{n-k}{2}$. Then, since A will win the most districts when it districts both sides, and will win the fewest when B districts both sides,

$$\begin{aligned}\operatorname{geo}_k(A) &= \frac{A(L_k, B) + A(R_k, B) + A(L_k, A) + A(R_k, A)}{2} \\ &= \frac{\lceil 2x_A(L_k) - k \rceil + \lceil 2x_A(R_k) - (n-k) \rceil + k + (n-k)}{2} \\ &= \frac{\lceil 2x_A(L_k) \rceil + \lceil 2x_A(R_k) \rceil}{2}.\end{aligned}$$

By Lemma 2.5, $\text{geo}(A) = \frac{\lceil 2(x_A) \rceil}{2}$. Thus,

$$|\text{geo}(A) - \text{geo}_k(A)| = \left| \frac{\lceil 2x_A \rceil}{2} - \frac{\lceil 2x_A(L_k) \rceil + \lceil 2x_A(R_k) \rceil}{2} \right|.$$

Since $x_A(L_k) + x_A(R_k) = x_A$, Lemma 2.4 Part i. implies this quantity is at most 1/2.
Case 2: Assume $x_A > \frac{n}{2}$, $x_A(L_k) > \frac{k}{2}$ and $x_A(R_k) \leq \frac{n-k}{2}$. Then

$$\begin{aligned}
\text{geo}_k(A) &= \frac{A(L_k, B) + A(R_k, B) + A(L_k, A) + A(R_k, A)}{2} \\
&= \frac{\lceil x_A(L_k) - x_B(L_k) \rceil + 0 + k + \lfloor 2x_A(R_k) \rfloor}{2} \\
&= \frac{\lceil 2x_A(L_k) - k \rceil + k + \lfloor 2x_A(R_k) \rfloor}{2} \\
&= \frac{\lceil 2x_A(L_k) \rceil + \lfloor 2x_A(R_k) \rfloor}{2}.
\end{aligned}$$

By Lemma 2.5, $\text{geo}(A) = \frac{\lceil 2(x_A) \rceil}{2}$. Thus,

$$|\text{geo}(A) - \text{geo}_k(A)| = \left| \frac{\lceil 2x_A \rceil}{2} - \frac{\lceil 2x_A(L_k) \rceil + \lfloor 2x_A(R_k) \rfloor}{2} \right|.$$

Since $x_A(L_k) + x_A(R_k) = x_A$, Lemma 2.4 Part ii.implies this quantity is at most 1/2.
Case 3: Assume $x_A \leq \frac{n}{2}$, $x_A(L_k) > \frac{k}{2}$ and $x_A(R_k) \leq \frac{n-k}{2}$. Then

$$\begin{aligned}
\text{geo}_k(A) &= \frac{A(L_k, B) + A(R_k, B) + A(L_k, A) + A(R_k, A)}{2} \\
&= \frac{\lceil x_A(L_k) - x_B(L_k) \rceil + 0 + k + \lfloor 2x_A(R_k) \rfloor}{2} \\
&= \frac{\lceil 2x_A(L_k) - k \rceil + k + \lfloor 2x_A(R_k) \rfloor}{2} \\
&= \frac{\lceil 2x_A(L_k) \rceil + \lfloor 2x_A(R_k) \rfloor}{2}.
\end{aligned}$$

By Lemma 2.5, $\text{geo}(A) = \frac{\lceil 2(x_A) \rceil}{2}$.
Thus,

$$|\text{geo}(A) - \text{geo}_k(A)| = \left| \frac{\lfloor 2x_A \rfloor}{2} - \frac{\lceil 2x_A(L_k) \rceil + \lfloor 2x_A(R_k) \rfloor}{2} \right|.$$

Since $x_A(L_k) + x_A(R_k) = x_A$, Lemma 2.4 Part iii. implies this quantity is at most 1/2.
Case 4: Assume $x_A \leq \frac{n}{2}$, $x_A(L_k) \leq \frac{k}{2}$ and $x_A(R_k) \leq \frac{n-k}{2}$. Then

$$\begin{aligned}
\text{geo}_k(A) &= \frac{A(L_k, B) + A(R_k, B) + A(L_k, A) + A(R_k, A)}{2} \\
&= \frac{0 + 0 + \lfloor 2x_A(L_k) \rfloor + \lfloor 2x_A(R_k) \rfloor}{2} \\
&= \frac{\lfloor 2x_A(L_k) \rfloor + \lfloor 2x_A(R_k) \rfloor}{2}.
\end{aligned}$$

By Lemma 2.5, $\mathrm{geo}(A) = \frac{\lceil 2(x_A) \rceil}{2}$. Thus,

$$|\,\mathrm{geo}(A) - \mathrm{geo}_k(A)| = \left| \frac{\lfloor 2x_A \rfloor}{2} - \frac{\lfloor 2x_A(L_k) \rfloor + \lfloor 2x_A(R_k) \rfloor}{2} \right|$$

. Since $x_A(L_k) + x_A(R_k) = x_A$, Lemma 2.4 Part iv. implies this quantity is at most 1/2. $\square$

2.5 Analyzing the fairness of the LRY Protocol

Here we prove that the number of districts a party wins under the LRY protocol is at most 2 less than its geometric target.

Lemma 2.8. *Suppose $x_P(k) > 0.5$. Then $P(L_k, P) - P(L_{k-1}, P) \geq 1$ and $P(R_{k-1}, \overline{P}) - P(R_k, \overline{P}) \leq 1$.*

Proof. Since $x_k(A) > 0.5$, there will always exist a districting of L_k by A such that the number of wins for A, which we indicate as $A_-(L_k, A)$, obeys $A_-(L_k, A) = A(L_{k-1}, A) + 1$. Namely, A chooses the same districting for the $k-1$ area on the left and makes k^{th} area worth of people a district. Since A plays optimally, they maximize their wins. Thus, $A(L_k, A) \geq A_-(L_k, A)$ and then

$$A(L_k, A) \geq A_-(L_k, A) = A(L_{k-1}, A) + 1.$$

Since $x_k(A) > 0.5$, there will always exist a districting of R_{k-1} by B such that the number of wins for A, which we indicate as $A_-(R_{k-1}, B)$, obeys $A_-(R_{k-1}, B) = A(R_k, B) + 1$. Namely, B chooses the same districting for the k area on the right and makes the $(k-1)^{th}$ area worth of people a district. Since B playing optimally, they minimize A's wins. Thus, $A(R_{k-1}, B) \leq A_-(R_{k-1}, B)$ and then

$$A(R_{k-1}, B) \leq A_-(R_{k-1}, B) = A(R_k, B) + 1.$$

$\square$

We have shown that when the support of the k^{th} district is a majority for P then P will win at least one more district when optimally districting the left side for k-split compared to the $k-1$-split. Coinciding with this, since $\overline{P}$ districts the right side then P will lose at most 1 district. However, when the support of P is a minority in the k^{th} district, when P is districting the left they will always win at least as much for the k-split as for the $k-1$ split. For the right side they will always will as least as much for the k-split as for the $k-1$ split when $\overline{P}$ is districting.

Lemma 2.9. *Suppose $x_P(k) < 0.5$. Then $P(L_{k-1}, P) \leq P(L_k, P)$ and $P(R_k, \overline{P}) \geq P(R_{k-1}, \overline{P})$.*

Proof. Since $x_k(A) < 0.5$, there will always exist a districting of L_k by A such that the number of wins for B, which we indicate as $B_-(L_k, A)$, obeys $B_-(L_k, A) = B(L_{k-1}, A) + 1$. Namely, A chooses the same districting for the $k-1$ area on the left and makes the k^{th} area worth of people a district. Since A is playing optimally, they minimize B's wins. Thus, $B(L_k, A) \leq B_-(L_k, A)$ and then

$$B(L_k, A) \leq B_-(L_k, A) = B(L_{k-1}, A) + 1.$$

Additionally, since $B(L_k, A) \leq B(L_{k-1}, A) + 1$,

$$\begin{aligned} B(L_k, A) &\leq B(L_{k-1}, A) + 1 \\ k - A(L_k, A) &\leq (k-1) - A(L_{k-1}, A) + 1 \\ A(L_{k-1}, A) &\leq A(L_k, A). \end{aligned}$$

Since $x_k(A) < 0.5$, there will always exist a districting of R_{k-1} by B such that the number of wins for B, which we indicate as $B_-(R_{k-1}, B)$, obeys $B_-(R_{k-1}, B) = B(R_k, B) + 1$. Namely, B chooses the same districting for the k area on the right and makes the $(k-1)^{th}$ area worth of people a district. Since B is playing optimally, they maximize their wins. Thus, $B(R_{k-1}, B) \geq B_-(R_{k-1}, B)$ and then

$$B(R_{k-1}, B) \geq B_-(R_{k-1}, B) = B(R_k, B) + 1.$$

Additionally, since $B(R_{k-1}, B) \geq B(R_k, B) + 1$,

$$\begin{aligned} B(R_{k-1}, B) &\geq B(R_k, B) + 1 \\ n - k + 1 - A(R_{k-1}, B) &\geq n - k - A(R_k, B) + 1 \\ A(R_{k-1}, B) &\leq A(R_k, B). \end{aligned}$$

□

Lemma 2.8 and Lemma 2.9 now allow us to prove that in the coin flip scenario, when P is districting the left side for the k-split, P will win at least as much as when P is districting the left side for the $k-1$-split and at most as much as 2 greater than P districting the left side for the $k-1$-split.

Theorem 2.10. $P(L_{k-1}) \leq P(L_k) \leq P(L_{k-1}) + 2$.

Proof. We will first prove that $A(L_{k-1}) \leq A(L_k)$.

Case 1: $x_A(k) > 0.5$,

$$A(L_{k-1}) = A(L_{k-1}, A) + A(R_{k-1}, B) = A(L_{k-1}, A) + A(R_{k-1}, B) + 1 - 1$$

by Lemma 2.8

$$\leq A(L_{k-1}, A) + A(R_{k-1}, B) + A(L_K, A) - A(L_{k-1}, A) + A(R_K, B) - A(R_{k-1}, B) = A(L_k).$$

Case 2: $x_A(k) < 0.5$,

$$A(L_{k-1}) = A(L_{k-1}, A) + A(R_{k-1}, B)$$

by Lemma 2.9

$$\leq A(L_k, A) + A(R_k, B) = A(L_k).$$

Now we will prove: $A(L_k) - A(L_{k-1}) \leq 2$.

Case 1: $x_A(k) < 0.5$.

We first wish to prove that $B(R_{k-1}, B) \leq B(R_k, B) + 2$ and $B(L_{k-1}, A) \leq B(L_k, A)$. If these are true then,

$$\begin{aligned} B(R_{k-1}, B) &\leq B(R_k, B) + 2 \\ n - k + 1 - A(R_{k-1}, B) &\leq n - k - A(R_k, B) + 2 \\ A(R_{k-1}, B) + 1 &\geq A(R_k, B) \end{aligned} \tag{2.10.1}$$

and

$$
\begin{aligned}
B(L_{k-1}, A) &\leq B(L_k, A) \\
k - 1 - A(L_{k-1}, A) &\leq k - A(L_k, A) \\
A(L_{k-1}, A) + 1 &\geq A(L_k, A).
\end{aligned} \tag{2.10.2}
$$

Using (2.10.1) and (2.10.2) we can conclude that:

$$
\begin{aligned}
A(L_k) - A(L_{k-1}) &= A(L_k, A) + A(R_k, B) - A(L_{k-1}, A) - A(R_{k-1}, B) \\
&\leq 1 + 1 = 2.
\end{aligned}
$$

Now we prove our claims:
We know that $x_B(R_{k-1}) \leq x_B(R_k) + 1$,

$$
\begin{aligned}
B(R_{k-1}, B) &= \min\{\lfloor 2x_B(R_{k-1}) \rfloor, k\} \\
&\leq \min\{\lfloor 2x_B(R_k) \rfloor + 2, k\} \\
&= \min\{\lfloor 2x_B(R_k) \rfloor, k - 2\} + 2 \\
&\leq \min\{\lfloor 2x_B(R_k) \rfloor, k\} + 2 \\
&= B(R_k, B) + 2.
\end{aligned}
$$

Thus $B(R_{k-1}, B) \geq B(R_k, B) + 2$.
We know that since $x_A(k) < 0.5$, $x_B(k) - x_A(k) \geq 0$,

$$
\begin{aligned}
B(L_{k-1}, A) &= \max\{\lceil x_B(L_{k-1}) - x_A(L_{k-1}) \rceil, 0\} \\
&\leq \max\{\lceil x_B(L_{k-1}) - x_A(L_{k-1}) + x_B(k) - x_A(k) \rceil, 0\} \\
&= \max\{\lceil x_B(L_k) - x_A(L_k) \rceil, 0\} \\
&= B(L_k, A).
\end{aligned}
$$

Thus $B(L_{k-1}, A) \leq B(L_k, A)$.
Case 2: $x_A(k) > 0.5$
We first wish to prove that $A(L_k, A) \leq A(L_{k-1}, A) + 2$ and $A(R_k, B) \leq A(R_{k-1}, B)$. If these are true then

$$
\begin{aligned}
A(L_k) - A(L_{k-1}) &= A(L_k, A) + A(R_k, B) - A(L_{k-1}, A) - A(R_{k-1}, B) \\
&\leq 2 + 0 = 2.
\end{aligned}
$$

Now we prove our claims.
Firstly, we know that $x_A(L_k) \leq x_A(L_{k-1}) + 1$,

$$
\begin{aligned}
A(L_k, A) &= \min\{\lfloor 2x_A(L_k) \rfloor, k\} \\
&\leq \min\{\lfloor 2x_A(L_{k-1}) \rfloor + 2, k\} \\
&= \min\{\lfloor 2x_A(L_{k-1}) \rfloor, k - 2\} + 2 \\
&\leq \min\{\lfloor 2x_A(L_{k-1}) \rfloor, k - 1\} + 2 \\
&= A(L_{k-1}, A) + 2.
\end{aligned}
$$

For our second claim, by Corollary 2.3 we know that,

$$\begin{aligned} A(R_k, B) &= \max\{\lceil x_A(R_k) - x_B(R_k)\rceil, 0\} \\ &= \max\{\lceil x_A(R_{k-1}) - (x_B(R_{k-1}) + 1 - 2x_A(k))\rceil, 0\} \\ &= \max\{\lceil x_A(R_{k-1}) - x_B(R_{k-1})\rceil + \lceil -2x_A(k)\rceil + 1 + \lceil x_A(R_{k-1}) - x_B(R_{k-1}) \bmod 1+ \\ &(-2x_A(k) \mod 1)\rceil - [x_A(R_{k-1}) - x_B(R_{k-1}) \notin \mathbb{Z}] - [-2x_A(K) \notin \mathbb{Z}], 0\}. \end{aligned}$$

If $0.5 < x_A(k) < 1$ then $\lceil -2x_A(k)\rceil + 1 = 0$ and $[x_A(R_{k-1}) - x_B(R_{k-1}) \notin \mathbb{Z}] + [-2x_A(K) \notin \mathbb{Z}] = 2$. Thus since $\lceil x_A(R_{k-1}) - x_B(R_{k-1}) \mod 1 + (-2x_A(k) \mod 1)\rceil \leq 2$ we have that

$$\begin{aligned} &\max\{\lceil x_A(R_{k-1}) - x_B(R_{k-1})\rceil + \lceil -2x_A(k)\rceil + 1 + \lceil x_A(R_{k-1}) - x_B(R_{k-1}) \mod 1+ \\ &(-2x_A(k) \bmod 1)\rceil - [x_A(R_{k-1}) - x_B(R_{k-1}) \notin \mathbb{Z}] - [-2x_A(K) \notin \mathbb{Z}], 0\} \\ &\leq \max\{\lceil x_A(R_{k-1}) - x_B(R_{k-1})\rceil + 2 - 2, 0\} \\ &= A(R_{k-1}, B). \end{aligned}$$

Additionally if $x_A(k) = 1$ then $\lceil -2x_A(k)\rceil + 1 = -1$ and $[x_A(R_{k-1}) - x_B(R_{k-1}) \notin \mathbb{Z}] + [-2x_A(K) \notin \mathbb{Z}] \geq 1$. Thus since $\lceil x_A(R_{k-1}) - x_B(R_{k-1}) \bmod 1 + (-2x_A(k) \bmod 1)\rceil \leq 2$ we get the same inequality as before. Thus $A(R_k, B) \leq A(R_{k-1}, B)$. □

Theorem 2.5 allows us to prove the coinciding corollary for R_k and R_{k-1}.

Corollary 2.11. $P(R_{k-1}) \geq P(R_k) \geq P(R_{k-1}) + 2$.

Proof. From the first part of Theorem 2.5,

$$\begin{aligned} B(L_{k-1}) &\leq B(L_k) \\ n - A(R_{k-1}) &\leq n - A(R_k) \\ A(R_{k-1}) &\geq A(R_k). \end{aligned}$$

From the second part of Theorem 2.5,

$$\begin{aligned} B(L_k) &\leq B(L_{k-1}) + 2 \\ n - A(R_k) &\leq n - A(R_{k-1}) + 2 \\ A(R_k) &\geq A(R_{k-1}) + 2. \end{aligned}$$

□

The coin flip occurs for the same change in preferences of sides for the $k-1$ versus k-split. Meaning when both parties prefer option 1 for the $k-1$-split such that P districts L_{k-1} and $\overline{P}$ districts R_{k-1} and both parties prefer option 2 for the k-split where P districts R_k and $\overline{P}$ districts L_k. In order to prove the fairness of the protocol we prove that the greatest difference between any two options of the coin flip can be at most 3. This will allow us to relate the party's geometric targets and k-split geometric targets to their coin flip options.

Proposition 2.12. *If the protocol returns a coin flip, then*

$$|P(L_{k-1}) - P(L_k)| \leq 2,$$

$$|P(L_{k-1}) - P(R_{k-1})| \leq 3,$$

$$|P(L_{k-1}) - P(R_k)| \leq 1,$$
$$|P(L_k) - P(R_{k-1})| \leq 1,$$
$$|P(L_k) - P(R_k)| \leq 3,$$
$$|P(R_k) - P(R_{k-1})| \leq 2.$$

Thus the greatest difference between any two option from the coin flip is 3.

Proof. In the *coin flip scenario*, there exists 4 options which correspond to $A(L_{k-1}),\ A(L_k),\ A(R_{k-1}),\ A(R_k)$ From Theorem 2.5 and Corollary 2.11, we can see that we obtain the 4 inequalities:

$$A(L_{k-1}) \leq A(L_k) \tag{i}$$

$$A(R_k) \leq A(R_{k-1}) \tag{ii}$$

$$A(L_{k-1}) + 2 \geq A(L_k) \tag{iii}$$

$$A(R_k) + 2 \geq A(R_{k-1}). \tag{iv}$$

In the *coin flip scenario* we know that all parties switch their preference from either L to R or R to L. Thus it must be the case that either

$$A(L_{k-1}) < A(R_{k-1}) \textbf{ and } A(L_k) > A(R_k)$$

or

$$A(L_{k-1}) > A(R_{k-1}) \textbf{ and } A(L_k) < A(R_k).$$

Note that an equality is not possible here as it would indicate indifference between the two options and indifference is not possible in the *coin flip scenario*.
If

$$A(L_{k-1}) > A(R_{k-1}) \textbf{ and } A(L_k) < A(R_k),$$

then

$$A(R_k) \leq A(R_{k-1}) \leq A(L_{k-1}) < A(L_k) < A(R_k).$$

This is a contradiction, as we must have that

$$A(L_{k-1}) < A(R_{k-1}), \tag{v}$$

and

$$A(L_k) > A(R_k). \tag{vi}$$

From now on when we are in the *coin flip scenario* (v) and (vi) hold. Thus, given the inequalities (i), (ii), (iii), (iv), (v), (vi) we prove the following differences between the 4 *coin flip* options:

1. $|A(L_{k-1}) - A(L_k)| \leq 2$

$$\begin{aligned} |A(L_{k-1}) - A(L_k)| &= A(L_k) - A(L_{k-1}) && \text{(i)} \\ &\leq A(L_{k-1}) + 2 - A(L_{k-1}) && \text{(iii)} \\ &= 2 \end{aligned}$$

2. $|A(L_{k-1}) - A(R_{k-1})| \leq 3$

$$\begin{aligned}
|A(L_{k-1}) - A(R_{k-1})| &= A(R_{k-1}) - A(L_{k-1}) && \text{(v)}\\
&\leq A(R_{k-1}) - A(L_k) + 2 && \text{(iii)}\\
&< A(R_{k-1}) - A(R_k) + 2 && \text{(vi)}\\
&\leq 2 + 2 && \text{(iv)}\\
&= 4
\end{aligned}$$

Since $|A(L_{k-1}) - A(R_{k-1})| < 4$ and $|A(L_{k-1}) - A(R_{k-1})| \in \mathbb{N}$, we must have $|A(L_{k-1}) - A(R_{k-1})| \leq 3$.

3. $|A(L_{k-1}) - A(R_k)| \leq 1$

$$\begin{aligned}
\text{case 1: } |A(L_{k-1}) - A(R_k)| &= A(L_{k-1}) - A(R_k)\\
&< A(R_{k-1}) - A(R_k) && \text{(v)}\\
&\leq A(R_k) + 2 - A(R_k) && \text{(iv)}\\
&= 2\\
\text{case 2: } |A(L_{k-1}) - A(R_k)| &= A(R_k) - A(L_{k-1})\\
&\leq A(R_k) - A(L_k) + 2 && \text{(iii)}\\
&< A(L_k) + 2 - A(L_k) && \text{(vi)}\\
&= 2
\end{aligned}$$

Since $|A(L_{k-1}) - A(R_k)| < 2$ and $|A(L_{k-1}) - A(R_k)| \in \mathbb{N}$, we must have $|A(L_{k-1}) - A(R_k)| \leq 1$.

4. $|A(L_k) - A(R_{k-1})| \leq 1$

$$\begin{aligned}
\text{case 1: } |A(L_k) - A(R_{k-1})| &= A(L_k) - A(R_{k-1})\\
&\leq A(L_{k-1}) + 2 - A(R_{k-1}) && \text{(iii)}\\
&< A(R_{k-1}) + 2 - A(R_{k-1}) && \text{(v)}\\
&= 2\\
\text{case 2: } |A(L_k) - A(R_{k-1})| &= A(R_{k-1}) - A(L_k)\\
&\leq A(R_k) + 2 - A(L_k) && \text{(iv)}\\
&< A(L_k) + 2 - A(L_k) && \text{(vi)}\\
&= 2
\end{aligned}$$

Since $|A(L_k) - A(R_{k-1})| < 2$ and $|A(L_k) - A(R_{k-1})| \in \mathbb{N}$, we must have $|A(L_k) - A(R_{k-1})| \leq 1$

5. $|A(L_k) - A(R_k)| \leq 3$

$$\begin{aligned}
|A(L_k) - A(R_k)| &= A(L_k) - A(R_k) && \text{(vi)}\\
&\leq A(L_{k-1}) + 2 - A(R_k) && \text{(iii)}\\
&< A(R_{k-1}) + 2 - A(R_k) && \text{(v)}\\
&\leq A(R_k) + 2 + 2 - A(R_k) && \text{(iv)}\\
&= 4
\end{aligned}$$

Since $|A(L_k) - A(R_k)| < 4$ and $|A(L_k) - A(R_k)| \in \mathbb{N}$, we must have $|A(L_k) - A(R_k)| \leq 3$

6. $|A(R_k) - A(R_{k-1})| \leq 2$

$$\begin{aligned}
|A(R_k) - A(R_{k-1})| &= A(R_{k-1}) - A(R_k) && \text{(ii)}\\
&\leq A(R_k) + 2 - A(R_k) && \text{(iv)}\\
&= 2
\end{aligned}$$

□

Using the Proposition 2.12, we can now show the maximum possible distance from the geometric target by all Coin Flip options is at most 2.

Theorem 2.13. *Suppose that k and $k-1$ are the two splits in the coin flip scenario for a given state. Then*

i. $|P(L_k) - \text{geo}_k(P)| \leq \frac{3}{2}$

ii. $|P(L_{k-1}) - \text{geo}_{k-1}(P)| \leq \frac{3}{2}$

iii. $|P(R_k) - \text{geo}_k(P)| \leq \frac{3}{2}$

iv. $|P(R_{k-1}) - \text{geo}_{k-1}(P)| \leq \frac{3}{2}$

v. $\forall S$ *and for* $i \in \{k, k-1\}$, $|\,\text{geo}(P) - P(S_i)| \leq 2$

Proof of i.

$$|A(L_k) - \text{geo}_k(A)| = \left| A(L_k) - \frac{A(L_k) + A(R_k)}{2} \right|$$

by Proposition 2.12

$$= \left| \frac{A(L_k) - A(R_k)}{2} \right| \leq \frac{3}{2}.$$

□

Proof of ii.

$$|A(L_{k-1}) - \text{geo}_{k-1}(A)| = \left| A(L_{k-1}) - \frac{A(L_{k-1}) + A(R_{k-1})}{2} \right|$$

by Proposition 2.12

$$= \left| \frac{A(L_{k-1}) - A(R_{k-1})}{2} \right| \leq \frac{3}{2}.$$

□

Proof of iii.
$$|A(R_k) - \text{geo}_k(A)| = \left| A(R_k) - \frac{A(L_k) + A(R_k)}{2} \right|$$

by Proposition 2.12
$$= \left| \frac{A(R_k) - A(L_k)}{2} \right| \le \frac{3}{2}.$$

□

Proof of iv.
$$|A(R_{k-1}) - \text{geo}_{k-1}(A)| = \left| A(R_{k-1}) - \frac{A(L_{k-1}) + A(R_{k-1})}{2} \right|$$

by Proposition 2.12

$$= \left| \frac{A(R_{k-1}) - A(L_{k-1})}{2} \right| \le \frac{3}{2}.$$

□

Proof of v. Since by Lemma 2.7
$$|\,\text{geo}(A) - \text{geo}_i(A)| \le \frac{1}{2}$$

and
$$|\,\text{geo}_i(A) - A(S_i)| \le \frac{3}{2},$$

it follows that:
$$\begin{aligned} |\,\text{geo}(A) - A(S_i)| &= |\,\text{geo}(A) + \text{geo}_i(A) - \text{geo}_i(A) - A(S_i)| \\ &\le |\,\text{geo}_i(A) - A(S_i)| + |\,\text{geo}(A) - \text{geo}_i(A)| \\ &\le 2. \end{aligned}$$

□

Therefore we have shown that for the available coin flip options that can be selected for P, the difference from the k-split geometric target can be at most $\frac{3}{2}$ and the difference from the geometric target can be at most 2. We provide an example to that this bound is tight.

Example 2.14. *Let*

- $x_A(L_5) = 1.9$, $x_B(L_5) = 3.1$
- $x_A(D_6) = 1$, $x_B(D_6) = 0$
- $x_A(R_6) = 1.4$, $x_B(R_6) = 2.6$.

Then by Proposition 2.1 and Corollary 2.3, $A(R_6, A) = 2$, $A(R_6, B) = 0$, $A(L_6, A) = 5$, $A(L_6, B) = 0$, $A(R_5, A) = 4$, $A(R_5, B) = 0$, $A(L_5, A) = 3$, $A(L_5, B) = 0$, $A(L_6, A) = 5$, and $A(L_6, B) = 0$.
Thus $A(R_5) = 4$, $A(L_5) = 3$, $A(R_6) = 2$, and $A(L_6) = 5$. Additionally since $x_A = 4.3$ by Lemma 2.5, $\text{geo}_A = 4$. Lastly by Proposition 2.6 $\text{geo}_5 = \frac{A(R_5)+A(L_5)}{2} = 3.5$ and $\text{geo}_6 = \frac{A(R_6)+A(L_6)}{2} = 3.5$ Thus there exists a *coin flip* option 1.5 away from geo_k and 2 away from geo. Namely $A(R_6) = 2$ and $\text{geo}_6 = 3.5$, geo $= 4$.

0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

3 The LRY Protocol with Geometric Constraints

We now wish to show that given some geometric constraints, it is possible to obtain a result from the *coin flip scenario* that is arbitrarily far from the geometric target.

First let us define our geometric constraints. We model our state through a square lattice with $l \in \mathbb{N}$ squares. Each square represents an indivisible area of population. We call the number of squares that constitutes the population of one district, $\phi \in \mathbb{N}$. Each square has a number inside it which represents the fraction of support for party A in that square, i.e a square with 0 has no support for party A and a square with 0.4 has 40% support for party A.

We also stipulate that districts must be drawn contiguously, i.e squares in the same district must share a side with another square in that district. Lastly, our districts must be compact, which we define as each distinct fitting inside a z by z square, where $z = \lfloor 2\sqrt{\phi} \rfloor$. Our first example is the following state with $\phi = 100$ and $l = 10000$, meaning there are 20 total districts. We consider the k-splits as vertical lines moving across the state from left to right. We will analyze the *coin flip scenario* between L_{50}, L_{51}, R_{50}, and R_{51}, that is between the 50-split and the 51-split. We will refer to the squares in the state by their corresponding matrix coordinate $a_{i,j}$, i.e the square in the top left corner is $a_{1,1}$. Using these constraints we can define an example with a *coin flip* option that is 2.5 away from the geometric target.

Example 3.1. *Construct a state with the following distribution:*

$$a_{i,j} = \begin{cases} 1 & \textit{if } j = 51 \textit{ or } j = 50, 49, 48, 47, \textit{or } 46 \textit{ and } i \equiv 1, 2, 3, 4, 5, \textit{or } 6 \bmod 20 \\ 0.1 & j = 50 \textit{ and } i \equiv 7 \bmod 20 \\ 0 & \textit{otherwise.} \end{cases}$$

For the 50-split and 51-split, there is a *coin flip scenario* where three *coin flip* options, $A(L_{50})$, $A(R_{51})$, and $A(L_{51})$, differ from the geometric target by 2.5.

Proof. The figure above is a representation of this state and it starts with its top left corner as $a_{1,44}$. This pattern repeats down the rows of the grid, and this repeating pattern is the only non-zero elements of the grid.

We can see that $A(L_{50}) = 0$, $A(R_{50}) = 2$, $A(R_{51}) = 0$.
The only non-trivial *coin flip* option is $A(L_{51})$. $A(L_{51}) = 5$, and the winning 5 districts can be made in the following manner. For the q^{th} winning district where $q \in \mathbb{N}$ and $q \in [1, 5]$, include all $a_{i,j}$ such that $i = (1, 2, ..., \text{ or } 16)+20(q-1)$ and $j = 51, 50, 49, 48, 47, \text{ or } 46$ or $i = (17, 18, 19 \text{ or } 20)+20(q-1)$ and $j = 51$.

We can see that the worst case for A overall for this state is 0 victories and the best case is 5 victories, giving a geometric target of $\text{geo}(A) = \frac{0+5}{2} = 2.5$. Since $A(L_{50}) = 0$, $A(R_{51}) = 0$ and $A(L_{51}) = 5$, the *coin flip* options $A(L_{50})$, $A(R_{51})$, and $A(L_{51})$ are 2.5 away from the geometric target.

□

We can now use this example to construct a state with the same geometric constraints and a *coin flip* result arbitrarily far from the geometric target. Let Δ' be an arbitrary number of districts from the geometric target. We will construct a state such that there is a *coin flip scenario* with *coin flip* options at least Δ' away from the geometric target. Set $\Delta = \lceil \Delta' \rceil$. We will construct a state with a geometric target of Δ and two *coin flip* options, $A(L_{\phi/2})$ and $A(L_{\phi/2+1})$, that are 0 districts won and 2Δ districts won, respectively. From our geometric constraint from before, districts must fit inside z by z square, where $z = \lfloor 2\sqrt{\phi} \rfloor$. We wish to have 2Δ districts border in $A(L_{\phi/2+1})$ on the left side of the $\phi/2$-split, which will be of height ϕ. We must allot an area for each district in $A(L_{\phi/2+1})$ of size z by z. To prevent creating a winning district by districting support from two potential district areas in $A(L_{\phi/2})$, we must fill out the z by z square with 0. Thus $2\Delta = \frac{\phi}{2\sqrt{\phi}}$ meaning $\phi = 16\Delta^2$. Additionally, in order to have enough support to win, the total support in our allotted area for each district in $A(L_{\phi/2+1})$ must be $\frac{\phi}{2} + \epsilon$ for $\epsilon > 0$. Since our final winning district will also include squares from the $\phi/2$ split, which will be all 1's, our potential district area will need an additional support of $\frac{\phi}{2} - \frac{\phi}{2\Delta} + \epsilon$. We also stipulate that this additional support be close to the $\phi/2$ split, to allow for contiguity. Thus we construct a rectangle of squares that are all 1 such that the area of this rectangle is $\frac{\phi}{2} - \frac{\phi}{2\Delta} = \frac{\phi\Delta - \phi}{2\Delta}$. Additionally since our district must be within a $2\sqrt{\phi}$ by $2\sqrt{\phi}$ square, we shall make the height of this rectangle $2\sqrt{\phi}$, meaning that the length is $\frac{\phi\Delta-\phi}{2\Delta} = \frac{\sqrt{\phi}}{4} - 1$. This will give a total support of $\phi/2$ in each district, so we must also include a single square with 0.1 in each $2\sqrt{\phi}$ by $2\sqrt{\phi}$ square.

Example 3.2. *Thus we construct a state as follows. Create a ϕ by ϕ square grid of size $l = \phi^2$ with the following distribution:*

$$a_{i,j} = \begin{cases} 1 & \text{if } j = \phi/2 \text{ or } \forall\, q, (q-1)2\sqrt{\phi}+1 \le i \le q2\sqrt{\phi},\ \phi/2 - \frac{\sqrt{\phi}}{4} + 1 \le j \le \phi/2 - 1 \\ 0.1 & \forall q,\ i = (q-1)2\sqrt{\phi}+1,\ j = \phi/2 - \frac{\sqrt{\phi}}{4} \\ 0 & \text{otherwise.} \end{cases}$$

For the $\phi/2$-split and $\phi/2+1$-split, there is a coin flip scenario where three coin flip options, $A(L_{\phi/2})$, $A(R_{\phi/2+1})$, and $A(L_{\phi/2+1})$, differ from the geometric target by Δ.

Proof. We can see that $A(L_{\phi/2}) = 0$, $A(R_{\phi/2}) = 2$, $A(R_{\phi/2+1}) = 0$. The only non-trivial *coin flip* option is $A(L_{\phi/2+1})$. $A(L_{\phi/2+1}) = 2\Delta$ and to create the winning q^{th} district for $A(L_{\phi/2+1})$, include all $a_{i,j}$ such that $(q-1)2\sqrt{\phi}+1 \leq i \leq q2\sqrt{\phi}$, $\phi/2 - \frac{\sqrt{\phi}}{2} + 1 \leq j \leq \phi/2$. This will include all squares in the $2\sqrt{\phi}$ by $2\sqrt{\phi}$ square such that $a_{i,j} = 1$ or 0.1 and it will be of size ϕ, the size of one district.

We can see that the worst case for A overall for this state is 0 victories and the best case is 2Δ victories, giving a geometric target of $\text{geo}(A) = \frac{0+2\Delta}{2} = \Delta$. Since Δ is arbitrary, $A(L_{\phi/2})$, $A(R_{\phi/2+1}) = 0$, and $A(L_{\phi/2+1})$ will be an arbitrary distance away from the geometric target. $\square$

4 Generalizing the Protocol

4.1 3 Party Protocol

When adapting the LRY protocol to 3 parties we found that a minority party can district where they do not have support and will decide who wins between the two other parties. Evidently, the minority party can give a strong advantage to one of the parties who's ideals align with theirs or who they make an agreement with. An additional problem we found with generalizing the LRY protocol occurs due to a coin flip scenario. When the k-split lines create two highly desirable areas for districting where moving 1 k split changes preferences of the parties, it is possible that the result of the coin flip would give a party an area that they have no preference for. Thus since we found that the LRY protocol does generalize well to 3 parties, we defined a new protocol with many similarities to that of the LRY protocol which adapts itself more efficiently to 3 or more parties. We make the following assumptions:

(1) *Agreed upon Voting model.* There exists some agreed upon voting model V_{all}. For example V_{all} could be the results of the last elections.

(2) *V_{all} is true.* Suppose V_{all} actually represents the support of each party in the state.

(3) *No Geometric Constraints.* As before we do not consider geometric constraints in the state.

(4) *Drawing winning districts.* Parties can only draw districts which they win under V_{all}.

(5) *Party Drop Out.* If a party does not have enough support to make a winning district during their next turn, they cannot choose another districting.

In the protocol parties play a game where they take turns in a circular fashion where the first party in the first round is chosen at random. A *round* consists of 1 turn from each party. A *turn* consists of a party drawing one district that will be in the final districting. If it is a party's turn but they cannot create a winning district under V_{all}, start a new game with the remaining parties and remaining area from the districting created by the previous map. The first turn goes to the following party from the previous game. The game concludes when all districts are drawn. In this protocol there exists two possible optimal strategies. Both strategies are generalizations of the optimal strategy for I cut I choose protocol for 2 players.

Strategy 4.1. *Efficient: A player uses as little support as possible to win a district. For example, in the 3 player case with parties A, B, and C for A playing the efficient strategy, A would create a district with $x_A = \frac{1}{3} + 2\epsilon$, $x_B = \frac{1}{3} - \epsilon$ and $x_c = \frac{1}{3} - \epsilon$ for $\epsilon > 0$.*

Strategy 4.2. *Cracking: A players uses the maximum amount of support from 1 other player while still winning the district. This strategy does not depend on the number of parties. For example, in the 3 player case with parties A, B, and C when A is playing the cracking strategy against B it follows that* $x_A = \frac{1}{2} + \epsilon$*,* $x_B = \frac{1}{2} - \epsilon$ *and* $x_C = 0$ *for* $\epsilon > 0$*.*

Note: The *cracking* and the *efficient* strategies are the same for 2 players.

There exists a third logical strategy called *Packing*, Such that a party is creates a district with only the support of 1 opposing party. This strategy is disqualified under assumption (4) in our protocol. However, this strategy is less effective for the party implementing it compared with the other two strategies. In essence when a party is packed, they gain an extra win for that round, but they only have to spend an extra $\frac{1}{2}$ more support than if they were cracked. This gives the party that is packed a higher wins per support spent overall than the cracking strategy.

We define new notation for the new protocol. Let n be the number of districts and $P(3)$ be the total number of wins by a party P when there exists 3 parties at the beginning of the protocol. As before, x_P is the total support of a player in the state .

We can now define a lower bound for the number of wins that a party can guarantee themselves to win using each strategy.

Theorem 4.3. *For a party P in the 3 party protocol playing*

i. efficient strategy, $P(3) \geq \frac{3}{4}x_P - \frac{3}{4}$,

ii. cracking strategy, $P(3) \geq \frac{2}{3}x_P - \frac{2}{3}$.

Proof of i. Using induction on the number of districts n, we will prove $A(3) \geq \frac{3}{4}x_A - \frac{3}{4}$ for a party A using the efficient strategy for n districts. Let $x_{A,n}$ be the amount of support in for A when n districts remain.

Base cases: Assume $n = 0$, then $x_{A,0} = 0$ and $A(3) = 0$. It follows that $A(3) = 0 \geq \frac{3}{4}(0) - \frac{3}{4} = -\frac{3}{4}$, so the inequality holds.

Assume $n = 1$, then $x_{A,1} \leq 1$ and it follows that $\frac{3}{4}x_{A,1} - \frac{3}{4} \leq \frac{3}{4}(1) - \frac{3}{4} = 0$. Since $A(3) \geq 0$ it follows that $A(3) \geq \frac{3}{4}x_{A,1} - \frac{3}{4}$.

Assume $n = 2$, then $x_{A,2} \leq 2$. If $1 < x_{A,2} \leq 2$ then $\frac{3}{4}x_{A,2} - \frac{3}{4} \leq \frac{3}{4}(2) - \frac{3}{4} = \frac{3}{4}$. If $1 < x_{A,2} \leq 2$ then $A(3) \geq 1$ and thus $A(3) \geq \frac{3}{4}x_{A,2} - \frac{3}{4}$.
If $1 \geq x_{A,2}$ then $\frac{3}{4}x_{A,2} - \frac{3}{4} \leq \frac{3}{4}(1) - \frac{3}{4} = 0$. Since $A(3) \geq 0$ it follows that $A(3) \geq \frac{3}{4}x_{A,1} - \frac{3}{4}$.

Inductive Hypothesis: Suppose for districts $n \geq 3$ $A(3) \geq \frac{3}{4}x_{A,n-3} - \frac{3}{4}$. In the worst case scenario for A, both of A's opponents crack A. Thus, in order for A to win 1 district per round, A uses $\frac{4}{3}$ of their support. Then for n districts it follows that A has 1 win and a support of $x_{A,n-3} \geq x_{A,n} - \frac{4}{3}$.

By inductive hypothesis,

$$\begin{aligned} A(3) &\geq \frac{3}{4}x_{A,n-3} - \frac{3}{4} \\ &\geq \frac{3}{4}(x_{A,n} - \frac{4}{3}) - \frac{3}{4} + 1 \\ &= \frac{3}{4}x_{A,n} - 1 - \frac{3}{4} + 1 \\ &= \frac{3}{4}x_{A,n} - \frac{3}{4}. \end{aligned}$$

Therefore, by induction

$$A(3) \geq \frac{3}{4}x_A - \frac{3}{4}.$$

□

Proof of ii. Using induction on the number of districts n, we will prove $A(3) \geq \frac{2}{3}x_A - \frac{2}{3}$ for a party A using the cracking strategy for n districts.

Base cases: Assume $n = 0$, then $x_{A,0} = 0$ and $A(3) = 0$. It follows that $A(3) = 0 \geq \frac{2}{3}(0) - \frac{2}{3} = -\frac{2}{3}$, so the inequality holds.

Assume $n = 1$, then $x_{A,1} \leq 1$ and it follows that $\frac{2}{3}x_{A,1} - \frac{2}{3} \leq \frac{2}{3}(1) - \frac{2}{3} = 0$. Since $A(3) \geq 0$ it follows that $A(3) \geq \frac{2}{3}x_{A,1} - \frac{2}{3}$.

Assume $n = 2$, then $x_{A,2} \leq 2$. If $1 < x_{A,2} \leq 2$ then $\frac{2}{3}x_{A,2} - \frac{2}{3} \leq \frac{2}{3}(2) - \frac{2}{3} = \frac{2}{3}$. If $1 < x_{A,2} \leq 2$ then $A(3) \geq 1$ and thus $A(3) \geq \frac{2}{3}x_{A,2} - \frac{2}{3}$.
If $1 \geq x_{A,2}$ then $\frac{2}{3}x_{A,2} - \frac{2}{3} \leq \frac{2}{3}(1) - \frac{2}{3} = 0$. Since $A(3) \geq 0$ it follows that $A(3) \geq \frac{2}{3}x_{A,1} - \frac{2}{3}$.

Inductive Hypothesis: Suppose for districts $n \geq 3$ $A(3) \geq \frac{2}{3}x_{A,n-3} - \frac{2}{3}$. In the worst case scenario for A, A is being cracked by both opponents. Thus, since A is playing the cracking strategy, in order for A to win 1 district per round, A uses $\frac{3}{2}$ of their support. Then for n districts it follows that A has 1 win and a support of $x_{A,n-3} \geq x_{A,n} - \frac{3}{2}$.
By inductive hypothesis,

$$\begin{aligned} A(3) &\geq \frac{2}{3}x_{A,n-3} - \frac{2}{3} \\ &\geq \frac{2}{3}(x_{A,n} - \frac{3}{2}) - \frac{2}{3} + 1 \\ &= \frac{2}{3}x_{A,n} - 1 - \frac{2}{3} + 1 \\ &= \frac{2}{3}x_{A,n} - \frac{2}{3}. \end{aligned}$$

Therefore, by induction

$$A(3) \geq \frac{2}{3}x_A - \frac{2}{3}.$$

□

4.2 Generalizing the 3 Party Protocol to m Parties

We can now generalize the 3 party protocol to m parties. The generalization of the protocol extends directly with the same assumptions. Let $P(m)$ be the total number of wins by a party P when there exists m parties at the beginning of the protocol.

Theorem 4.4. *For a party P in the m party protocol playing the*

i. efficient strategy, $P(m) \geq \frac{2m}{m^2-m+2}x_P - 1$.

ii. cracking strategy, $P(m) \geq \frac{2}{m}x_P - 1$.

Proof of i. Using induction on the number of districts n, we will prove $A(m) \geq \frac{2m}{2+m^2-m}x_A - 1$ for a party A using the efficient strategy for n districts. Let $\gamma \in [0, m-1]$. and $\gamma \in \mathbb{N}$.
Base Case: We consider all base cases where $n \leq m-1$. Therefore it follows that the support of $x_{A,\gamma}$ is $0 \leq x_{A,\gamma} \leq m-1$. Thus

$$\begin{aligned} x_{A,\gamma} &\leq \frac{m^2-m}{m} \\ &< \frac{m^2-m+2}{m} \\ &= \frac{2(m^2-m+2)}{2m}. \end{aligned}$$

Therefore,

$$x_{A,\gamma} < \frac{2(m^2-m+2)}{2m}$$
$$0 \leq \frac{2m}{m^2-m+2}x_{A,\gamma} < 2.$$

Therefore there are two cases:
Case 1: Assume $2 > \frac{2m}{m^2-m+2}x_{A,\gamma} > 1$.
It follows that

$$x_{A,\gamma} > \frac{m^2-m+2}{2m}$$

Assume for the sake of contradiction that A wins no districts if $x_{A,\gamma} > \frac{m^2-m+2}{2m}$. In the worst case, after all $m-1$ opponents crack A, $x_{A,\gamma} > \frac{1}{m}$. However, the support must have been added to a districting making A win 1 district as in this scenario all $m-1$ would have made A spend $\frac{1}{2}$ support, which is a contradiction. Thus $A(m) \geq 1$. If $2 > \frac{2m}{m^2-m+2}x_{A,\gamma} > 1$ then $1 > \frac{2m}{m^2-m+2}x_{A,\gamma} - 1 > 0$ therefore, $1 > \frac{2m}{m^2-m+2}x_{A,\gamma} - 1$. Thus $A(m) \geq \frac{2m}{m^2-m+2}x_{A,\gamma} - 1$.

Case 2: Assume $1 \geq \frac{2m}{m^2-m+2}x_{A,\gamma} \geq 0$.
It follows that

$$\frac{m^2-m+2}{2m} \geq x_{A,\gamma} \geq 0.$$

Then,

$$\frac{2m}{m^2-m+2}x_{A,\gamma}-1 \leq \frac{2m}{m^2-m+2}\left(\frac{m^2-m+2}{2m}\right)-1=0.$$

Since $A(m) \geq 0$ it follows that for $1 \geq \frac{2m}{m^2-m+2} \geq 0$,

$$A(m) \geq \frac{2m}{m^2-m+2}x_A - 1.$$

Inductive Hypothesis: Suppose that for $n \geq m$, $A(m) \geq \frac{2m}{m^2-m+2}x_{A,n-m}-1$. In the worst case scenario for A, A's $m-1$ opponents crack A meaning A spends $\frac{1}{2}$ support for each opponents turn, thus A spends $\frac{m-1}{2}$, and A will spend $\frac{1}{m}$, from the efficient strategy, to win a district. Therefore to win 1 district per round A spends at most $\frac{1}{m}+\frac{m-1}{2}=\frac{m^2-m+2}{2m}$. Then for n districts it follows that after 1 round, A has 1 win and support of $x_{A,n-m} \geq x_{A,n}-\frac{m^2-m+2}{2m}$.
By inductive hypothesis,

$$\begin{aligned}
A(m) &\geq \frac{2m}{m^2-m+2}x_{A,n-m}-1 \\
&\geq \frac{2m}{m^2-m+2}\left(x_{A,n}-\frac{m^2-m+2}{2m}\right)-1+1 \\
&= \frac{2m}{m^2-m+2}x_{A,n}-2+1 \\
&= \frac{2m}{m^2-m+2}x_{A,n}-1.
\end{aligned}$$

Therefore,by induction

$$A(m) \geq \frac{2m}{m^2-m+2}x_A - 1$$

for all districts $n \geq 0$. □

Proof of ii. Using induction on the number of districts n, we will prove $A(m) \geq \frac{2}{m}x_A - 1$ for a party A using the cracking strategy for n districts.
Base Case: We consider all base cases where $n \leq m-1$. Therefore it follows that the support of $x_{A,\gamma}$ is $0 \leq x_{A,\gamma} \leq m-1$. Thus,

$$\begin{aligned}
x_{A,\gamma} &\leq m-1 \\
&< m \\
&= \frac{2m}{2}.
\end{aligned}$$

Therefore

$$\frac{2}{m}x_{A,\gamma} < 2.$$

There are two cases:
Case 1: $2 > \frac{2}{m}x_{A,\gamma} > 1$.
It follows that

$$x_{A,\gamma} > \frac{m}{2}.$$

Assume for the sake of contradiction that A wins no districts if $x_{A,\gamma} > \frac{m}{2}$. In the worst case, after all $m-1$ opponents crack A, $x_{A,\gamma} \geq \frac{m}{2} - \frac{m-1}{2} = \frac{1}{2}$. However, the support must have been added to a districting making A win 1 district as in this scenario all $m-1$ would have made A spend $\frac{1}{2}$ support, which is a contradiction. Thus $A(m) \geq 1$. If $2 > \frac{2}{m}x_{A,\gamma} > 1$ then $1 > \frac{2}{m}x_{A,\gamma} - 1 > 0$ and therefore, $1 > \frac{2}{m}x_{A,\gamma} - 1$. Thus $A(m) \geq \frac{2}{m}x_{A,\gamma} - 1$.

Case 2: $1 \geq \frac{2}{m}x_{A,\gamma} \geq 0$. It follows that

$$\frac{m}{2} \geq x_{A,\gamma} \geq 0.$$

Therefore,

$$\frac{2}{m}x_{A,\gamma} - 1 \leq \frac{2}{m}\left(\frac{m}{2}\right) - 1 = 0.$$

Since $A(m) \geq 0$ it follows that for $1 \geq \frac{2}{m} \geq 0$,

$$A(m) \geq \frac{2}{m}x_A - 1.$$

Inductive Hypothesis: Suppose that for $n \geq m$, $A(m) \geq \frac{2}{m}x_{A,n-m} - 1$. In the worst case scenario for A, A's $m-1$ opponents crack A making A spend $\frac{1}{2}$ support for each opponents turn, thus A spends $\frac{m-1}{2}$ from opponents, and A will spend $\frac{1}{2}$, from the cracking strategy to win a district. Therefore to win 1 district per round A spends at most $\frac{m}{2}$. Then for n districts it follows that after 1 round, A has 1 win and support of $x_{A,n-m} \geq x_{A,n} - \frac{m}{2}$.
By inductive hypothesis

$$\begin{aligned}
A(m) &\geq \frac{2}{m}x_{A,n-m} - 1 \\
&\geq \frac{2}{m}\left(x_{A,n} - \frac{m}{2}\right) - 1 + 1 \\
&= \frac{2}{m}x_{A,n} - 2 + 1 \\
&= \frac{2}{m}x_{A,n} - 1.
\end{aligned}$$

Therefore, by induction

$$A(m) \geq \frac{2}{m}x_A - 1$$

for all districts $n \geq 0$. □

The efficient strategy provides the maximum lower bound on the number of wins regardless of the strategies played by the party's opponents. However, the strategy that will maximize a party's wins in a given districting relies on knowing their opponents strategies. For example, if all other parties agree to crack 1 other party, the cracking parties will have a higher wins per support spent ratio than if all parties play efficiently. This is true withstanding the support of the cracked party, the ratio is proportional to the support of the cracked party. However, if a party cannot convince the other parties to crack the same target party, some cracking parties will have lower wins per support spent ratios than if all parties played efficiently. Therefore, a party maximizes their wins when they have the knowledge of the other parties strategies and can adapt to what the other parties will do throughout each sub-game.

5 Future Work

While we prove that if there are no geometric constraints, the LRY protocol can return a result as most 2 from the geometric target, we also give an example where if geometric constraints are insisted, this claim is not true. Therefore the question remains of how likely a typical state with geometric constraints is to differ from the geometric target by more than 2. While it seems likely that a contrived state must be created in order to have a *coin flip* deviate from the geometric target by more than 2, a quantification of this likely hood has yet to be explored.

For our m player protocol we provide a lower bound on the number of wins, however there is still a question if one could relax the assumptions of the protocol and still find a lower bound on the number of wins for a party. One way these assumptions could be relaxed is assuming V_{all} is not true and each party has their own voting model that could be different from the accepted voting model V_{all}. Another assumption that could be relaxed is the assumptions that there is an accepted voting model at all, however some constraints would have to be put on the parties so that they cannot exert too much control over the other parties once they no longer have support.

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