Examples of Boundary Layer Associated with Incompressible Newtonian Flows

ε progress on a $\frac{1}{\varepsilon}$ problem

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The Models

- Incompressible Navier-Stokes equations (NSE)

$$
\begin{aligned}
\frac{\partial \mathbf{u}^\nu}{\partial t} + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu - \nu \Delta \mathbf{u}^\nu + \nabla p^\nu &= \mathbf{f}, \\
\nabla \cdot \mathbf{u}^\nu &= 0, \\
\mathbf{u}^\nu \big|_{\partial\Omega} &= \text{given} \\
\mathbf{u}^\nu \big|_{t=0} &= \mathbf{u}_0.
\end{aligned}
$$

- Euler equations for inviscid (dry) fluid

$$
\begin{aligned}
\frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 + \nabla p^0 &= \mathbf{f}, \\
\nabla \cdot \mathbf{u}^0 &= 0, \\
\mathbf{u}^0 \cdot \mathbf{n} \big|_{\partial\Omega} &= \text{given} + \cdots \\
\mathbf{u}^0 \big|_{t=0} &= \mathbf{u}_0.
\end{aligned}
$$

The Main Issue

- Major issue: (model validity)

$$\mathbf{u}^{\nu} \to \mathbf{u}^{0}, \text{ as } \nu \to 0?$$

- Yes for the case without physical boundary. For instance for whole space, periodic box or free-slip: Wolibner, Golovin, Yudovich, Swann, McGrath, Kato, Lions, Xiao&Xin (CPAM),....
- Yes for the case with Navier slip boundary conditions (no leading order boundary layer): Lopez-Filho& Lopez& Planas(SIMA), Kelliher(SIMA), Kim(SIMA), Wang& Wang& Xin (CMS),.....
- Also results on related system: Masmoudi, Grenier, ...
- Case with physical boundary is difficult in general.
  - No convergence in L^{∞} or $H^k, k > \frac{1}{2}$ is possible.
  - Euler can be valid in the interior region (away from the boundary) but not up to the boundary
  - What happens near the boundary is important (for drag ...)
  - d'Alembert's paradox (zero drag for inviscid incompressible potential flow)

Example 1

- Plane parallel channel flow ansatz (characteristic)

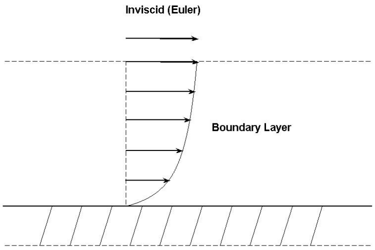
$$\Omega = \mathbb{R}^1 \times (0, \infty), \mathbf{u}^\nu = (u_1^\nu(z,t), 0), p^\nu \equiv 0, u_1^\nu\big|_{z=0} = 0$$

- Reduced NSE: $\frac{\partial u_1^\nu}{\partial t} - \nu \frac{\partial^2 u_1^\nu}{\partial z^2} = f_1, \quad u_1^\nu\big|_{z=0} = \text{given}$
- Reduced Euler: $\frac{\partial u_1^0}{\partial t} = f_1$
- Boundary layer: $u_1^\nu - u_1^0 \approx \theta(\frac{z}{\sqrt{\nu t}})$
- For $u_1^\nu\big|_{t=0} = a, f_1 \equiv b$,
  $u_1^\nu(z,t) = a\,\mathrm{erf}(\frac{z}{2\sqrt{\nu t}}) + b\int_0^t \mathrm{erf}(\frac{z}{2\sqrt{\nu(t-s)}})\,ds, u_1^0(z,t) = a + bt.$
- Generation of vorticity near the boundary.
- Obstruction to convergence in L^∞, H^1, no uniform estimate in $H^1(\Omega)$, uniform only away from the boundary, ...
- Disc version: Matsui, Bona& Wu, Kelliher, ... (see also examples 3 and 4)

Schematic plot for characteristic case

Figure: Schematic plot of boundary layer in the no-slip no penetration case

Figure: horizontal velocity evolution



(boundary layer)

Example 2

- Channel flow with injection/suction (non-characteristic)

$$\Omega = \mathbb{R}^1 \times (0, H), \quad \mathbf{u}^\nu = (u_1^\nu(z,t), -V), \quad p^\nu \equiv 0, \quad u_1^\nu\big|_{z=0,H} = 0$$

- Reduced NSE

$$\frac{\partial u_1^\nu}{\partial t} - V\frac{\partial u_1^\nu}{\partial z} - \nu\frac{\partial^2 u_1^\nu}{\partial z^2} = f_1, \quad u_1^\nu\big|_{z=0,H} = 0$$

- Reduced Euler

$$\frac{\partial u_1^0}{\partial t} - V\frac{\partial u_1^0}{\partial z} = f_1, u_1^0\big|_{z=H} = 0$$

- Steady state case (essentially Friedrichs)

$$u_1^\nu(z) = -\frac{f_1}{V}z + \frac{f_1 H}{V(1 - \exp(-\frac{VH}{\nu}))}(1 - \exp(-\frac{Vz}{\nu}))$$

- Boundary layer (general)

$$u_1^\nu - u_1^0 \approx \theta(\frac{z}{\nu})$$

- Boundary layer in downwind direction only, narrower (more singular).

Schematic plot for the non-characteristic case

Figure: Domain with injection and suction

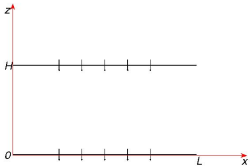
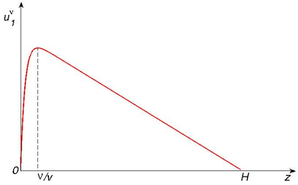


Figure: Horizontal velocity plot



Ludwig Prandtl 1875-1953

Prandtl 1904 theory

- All functions and their horizontal derivatives are bounded
- Viscous and inviscid flows match outside a thin layer (boundary layer) near the boundary.
- The boundary layer is of thickness proportional to $\sqrt{\nu}$.
- Stretched coordinates $Z = \frac{z}{\sqrt{\nu}}$
- Prandtl's 1904 resolution

  The physical processes in the boundary layer between fluid and the solid body can be calculated in a sufficiently satisfactory way if it is assumed that the fluid adheres to the walls, so that the total velocity there is zero - or equal to the velocity of the body. If the viscosity is very small and the path of the fluid along the wall is not too long, the velocity will have again its usual value very near to the wall. In this thin transition layer the sharp changes of velocity, in spite of the small viscosity coefficient, produce noticeable effects.

Prantl equation

Use stretched variable $Z = \frac{z}{\sqrt{\nu}}$ and drop lower order terms.

- Prandtl equation: approximate $\mathbf{u}^\nu = (u_1, \not{u_2}, u_3)$ and $p^\nu = p$ directly

$$\begin{aligned} \partial_t u_1 + u_1 \partial_x u_1 + u_3 \partial_z u_1 - \nu \partial_{zz}^2 u_1 + \partial_x p^0 \big|_{z=0} &= f_1 \big|_{z=0}, \\ \frac{\partial p^\nu}{\partial z} &= 0, \\ \partial_x u_1 + \partial_z u_3 &= 0, \\ u_1\big|_{z=0} = 0, u_1\big|_{z=\infty} = u_1^0(x,t), u_3\big|_{z=0} = 0 & \qquad u_1\big|_{t=0} = u_{10}(x,z) \end{aligned}$$

$$(\partial_t u_1^0 + u_1^0 \partial_x u_1^0 + \partial_x p^0 = f_1, \text{at } z = 0)$$

- Issues with Prandtl theory
  - Well-posedness of the Prandtl equation: Oleinik, Xin&Zhang, E&Engquist, Sammartino&Caflisch, Grenier, Gerard-Varet&Dormy,...
  - Matching of the solutions (essentially open): Nickel, Fife, Serrin, Sammartino&Caflisch, Lombardo&Cannone&Sammartino, Temam&W., ...
  - Expected result: short time

Two open problems from Oleinik's book

1. in Sect. 1.1 it has been shown how on the basis of certain physical assumptions, called the Prandtl hypotheses, the Navier-Stokes equations can be simplified to yield the system of the boundary layer equations. Is it possible to give a strict mathematical justification of this procedure and find the limits of applicability of the Prandtl hypotheses?
2. It follows from the theorems formulated in Sect. 1.2 that under certain assumptions, the solutions of the Prandtl system are close to those of the corresponding Navier-Stokes equations near the boundary of the body. Is it possible to generalize these results and obtain an asymptotic solution of the Navier-Stokes system on the basis of the solutions of the boundary layer equations and the Euler-Bernoulli equations?

from *Mathematical Models in Boundary Layer Theory* by O.A. Oleinik and V.N. Samokhin, Chapman and Hall /CRC, 1999. Open problems 1 & 2 on p. 500.

Olga Arsenievna Oleinik 1925-2001

Prandtl theory: Nickel's result (1963)

- no-slip steady state on $\Omega = (0, L) \times (-H, H)$
- $\Gamma = \{x = 0\} \cup \{|z| = H\}$
- assume
  - (1) $p^\nu \equiv p^0$;
  - (2) $u_{1zz} \leq A, u^\nu_{1zz} \leq A, B(H^2 - z^2) \leq u_1, u^\nu_1 \leq C$,
  - (3) $\lim_{\nu \to 0} \|u^\nu - u\|_{L^\infty(\Gamma)} = 0$;
  - (4) $\lim_{\nu \to 0} \nu \|u_{1xx}\|_{L^\infty(\Omega)} = 0$.
- Then

$$\lim_{\nu \to 0} \|u^\nu - u\|_{L^\infty(\Omega)} = 0$$

Prandtl theory: Fife's result (1965)

- no-slip steady state on $\Omega = (0, L) \times (0, \infty)$
- assume
  - $\|p^\nu\|_{L^\infty(\Omega)} + \|p^0\|_{L^\infty(\Omega)} \le C$;
  - $u_1^\nu \ge Mz, u_1^\nu(0, z) \ge M \min(\frac{z}{\nu}, 1)$;
  - $M^{-1} \min(\frac{z}{\nu}, 1) \le u_1(0, z) \le M \min(\frac{z}{\nu}, 1)$;
  - $u_{1z}(0, z) \le \frac{M}{\nu}$;
  - $|u_1^\nu(0, z) - u_1(0, z)| \le M\nu \max(\sqrt{\frac{z}{\nu}}, \frac{z}{\nu})$,
  - $|p_x(x) - p_x^\nu(x, 0)| \le M\nu$;
  - $|u_{1zz}(0, z) - p_x(0)| \le M(\frac{z}{\nu})^2$ for $0 < \frac{z}{\nu} < z_0(\nu)$.
- Then $\forall \delta \in (0, 3 - 2\sqrt{2})$, $\exists k = k(\delta)$ such that

$$|u_1^\nu(x, z) - u_1(x, z)| \le C\nu^k, z \in (0, \nu^{1-\delta}), \nu \in (0, \nu_1)$$

Prandtl theory: Sammartino & Caflisch analytical data result (1998)

Short time validity of Prandtl theory under the assumptions

- Half space (in 2 or 3D)
- Analytical data (generalized to analytic in tangential variable only in 2D by Lombardo, Cannone and Sammartino)

Prandtl type theory: a small variation

- approximate $\mathbf{u}-\mathbf{u}^0(\approx \boldsymbol{\theta})$ instead of $\mathbf{u}$ (Lyusternik& Vishik, JL Lions, ...)
- Prandtl type equation (use stretched variable $Z=\frac{z}{\sqrt{\nu}}$ or 2 spatial scale expansion):

$$
\begin{aligned}
\partial_t\theta_1+(u_1^0\big|_{z=0}+\theta_1)\partial_x\theta_1+\theta_1\partial_x u_1^0\big|_{z=0}+\theta_3\partial_z\theta_1-\nu\partial_{zz}^2\theta_1 &= 0, && (1)\\
\partial_x\theta_1+\partial_z\theta_3 &= 0, && (2)\\
\theta_1\big|_{z=0}=-u_1^0\big|_{z=0}, \theta_1\big|_{z=\infty}=0, & \quad \theta_3\big|_{z=0}= && (3)
\end{aligned}
$$

- Similar issues as the original approach
- Goal: for $\mathbf{u}^a=\mathbf{u}^0+\boldsymbol{\theta}$

$$\mathbf{u}-\mathbf{u}^a\to 0, \text{as } \nu\to 0$$

Jacques-Louis Lions and Lazar Lyusternik

Figure: Jacques-Louis Lions 1928-2001

Figure: Lazar Lyusternik 1899-1981

Approximate solution, difficulty with convergence

- Assume well-posedness and nice decay. Approximate solution: $\mathbf{u}^a = \mathbf{u}^0 + \boldsymbol{\theta}$ satisfies the NSE with an additional term ($\epsilon = \nu$)

$$\mathbf{f}^e \approx \sqrt{\epsilon}\psi(\frac{z}{\sqrt{\epsilon}})$$

- error equation: $\mathbf{u}^e = \mathbf{u} - \mathbf{u}^a$

$$\begin{aligned} \frac{\partial \mathbf{u}^e}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}^e + (\mathbf{u}^e\cdot\nabla)\mathbf{u}^a - \epsilon\Delta\mathbf{u}^e + \nabla q^e &= \mathbf{f}^e, \\ \nabla\cdot\mathbf{u}^e &= 0, \\ \mathbf{u}^e\big|_{\partial\Omega} &= 0 \\ \mathbf{u}^e\big|_{t=0} &= 0. \end{aligned}$$

- Difficulty in energy approach: advection in the normal direction

$$u_3^e\partial_z\mathbf{u}^a \qquad (\int u_3^e\partial_z\mathbf{u}^a\cdot\mathbf{u}^e)$$

- Difficulty term vanishes if there is no normal (to the boundary) flow.

Spectral Constraint

- Key ingredient in the validity of Prandtl theory: a spectral constraint on the approximate solution $\mathbf{u}^a$: $\exists\Lambda$ *independent of the viscosity* $\nu = \epsilon$ such that

$$\inf_{\mathbf{v}\in\mathcal{A}} \frac{\int_\Omega \left(\nu|\nabla\mathbf{v}|^2 + (\mathbf{v}\cdot\nabla)\mathbf{u}^a\cdot\mathbf{v}\right)}{\|\mathbf{v}\|^2_{L^2}} \geq \Lambda > -\infty \tag{4}$$

  where

$$\mathcal{A} = \left\{\mathbf{v}\in(H^1_0(\Omega))^n, \nabla\cdot\mathbf{v} = 0, \mathbf{v}\neq 0\right\}.$$

- Difficult term: advection in the normal direction

$$\int_\Omega v_3\partial_z\mathbf{u}^a\cdot\mathbf{v}$$

A Meta Theorem on the validity of Prandtl type theory

Theorem (W. CAM2010)

Suppose that the following two conditions are satisfied

1. *The well-posedness of the Prandtl type system (1) together with appropriate decay at infinity;*
2. *The verification of the spectral constraint (4) on the approximate solution* $\mathbf{u}^a$*.*

Then the Prandtl type approximation is valid in the sense that

$$\|\mathbf{u}^e\|_{L^\infty(L^2)} = \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^\infty(L^2)} \leq C\nu^{\frac{3}{4}}, \tag{5}$$

$$\|\mathbf{u}^e\|_{L^2(H^1)} = \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^2(H^1)} \leq C\nu^{\frac{1}{4}}. \tag{6}$$

Example 3: 3D plane parallel flow

- Ansatz (W. IUMJ2001) ($\nu = \epsilon$)

$$\Omega = \mathbb{R}^1 \times \mathbb{R}^1 \times (0,1)$$

$$\mathbf{u} = (u_1(t,z), u_2(t,x,z), 0), \mathbf{u}\big|_{z=j} = (\beta_1^j(t), \beta_2^j(t,x), 0), j = 0, 1,$$
$$\mathbf{u}_0 = (u_{1,0}(z), u_{2,0}(x,z), 0), \mathbf{f} = (f_1(t,z), f_2(t,x,z), 0).$$

- reduced NSE:

$$\begin{aligned} \partial_t u_1 - \epsilon \partial_{zz} u_1 &= f_1, \\ \partial_t u_2 + u_1 \partial_x u_2 - \epsilon \partial_{xx} u_2 - \epsilon \partial_{zz} u_2 &= f_2 \end{aligned}$$

- reduced Euler

$$\begin{aligned} \partial_t u_1^0 &= f_1, \\ \partial_t u_2^0 + u_1^0 \partial_x u_2^0 &= f_2, \end{aligned}$$

- Prandtl type equation (another one at $z = 1$)

$$\begin{aligned} \partial_t \theta_1^0 - \epsilon \partial_{zz} \theta_1^0 &= 0, \\ \partial_t \theta_2^0 - \epsilon \partial_{zz} \theta_2^0 + \theta_1^0 \partial_x \theta_2^0 + u_1^0(t,0) \partial_x \theta_2^0 + \theta_1^0 \partial_x u_2^0(t,x,0) &= 0, \\ (\beta_1^0(t) - u_1^0(t,0), \beta_2^0(t,x) - u_2^0(t,x,0)) &= (\theta_1^0, \theta_2^0)|_{z=0} \end{aligned}$$

Example 3: 3D plane parallel flow (continued)

- Mazzucato& Niu & W. IUMJ2011:

$$\|\mathbf{u}-\mathbf{u}^0-\boldsymbol{\theta}\|_{L^\infty(X)} \leq C\epsilon^{\gamma(X)}, \gamma(L^2)=\frac{3}{4}, \gamma(H^1)=\frac{1}{4}=\gamma(L^\infty)$$

- W. 2001: (via Kato type technique)

$$\|\mathbf{u}-\mathbf{u}^0\|_{L^\infty(L^2)} \to 0$$

- Mazzucato & Taylor 2010: (via parametrix, no Prandtl)

$$L^\infty(L^\infty)$$

Example 4: Parallel pipe flow

- Ansatz in cylindrical coordinates (W. IUMJ2001):

$$\Omega = \{r < 1, \phi \in [0, 2\pi], x \in \mathbb{R}^1\}$$

$$\mathbf{u} = u_\phi(t,r)\mathbf{e}_\phi + u_x(t,r,\phi)\mathbf{e}_x, p = p(t,r), \mathbf{u}\big|_{r=1} = (0, \beta_\phi(t), \beta_x(t,\phi))$$

- reduced NSE

$$\begin{aligned} -(u_\phi)^2 + r\partial_r p &= 0, \\ \partial_t u_\phi &= \frac{\nu}{r}\partial_r(r\partial_r u_\phi) - \frac{\nu}{r^2}u_\phi + f_\phi, \\ \partial_t u_x + \frac{u_\phi}{r}\partial_\phi u_x &= \frac{\nu}{r}\partial_r(r\partial_r u_x) + \frac{\nu}{r^2}\partial_{\phi\phi}u_x + f_x, \end{aligned}$$

- reduced Euler

$$\begin{aligned} -(u_\phi^0)^2 + r\partial_r p^0 &= 0, \\ \partial_t u_\phi^0 &= f_\phi, \\ \partial_t u_x^0 + \frac{u_\phi^0}{r}\partial_\phi u_x^0 &= f_x, \end{aligned}$$

Example 4: Parallel pipe flow (continued)

- Prandtl type

$$\begin{aligned} \partial_t \theta_\phi - \partial^2_{RR} \theta_\phi &= 0, \\ \partial_t \theta_x + \theta_\phi \partial_\phi u^0_x(t,1) + \theta_\phi \partial_\phi \theta_x + u^0_\phi(t,1) \partial_\phi \theta_x &= \partial^2_{RR} \theta_x \end{aligned}$$

- Han & Mazzucato & Niu & W.2011:

$$\|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^\infty(L^2), L^2(H^1), L^\infty(L^\infty)} \to 0$$

- $L^\infty(L^\infty)$ convergence requires anisotropic embedding as well as decomposition of domain into boundary region and interior region.
- Curvature effects reflected in pressure estimates.
- W. 2001: $L^\infty(L^2)$ (via Kato type on more general ansatz $\mathbf{u} = u_\phi(t,r,x)\mathbf{e}_\phi + u_x(t,r,\phi)\mathbf{e}_x$);
- Mazzucato & Taylor 2010: "$L^\infty(H^1)$" (via parametric);

More general parallel pipe flow

- $\mathbf{u} = u_\phi(t, x, r)\mathbf{e}_\phi + u_x(t, r, \phi)\mathbf{e}_x, p(t, x, r)$ reduced NSE

$$\begin{aligned}
-(u_\phi)^2 + r\partial_r p &= 0, \\
\partial_t u_\phi + u_x \partial_x u_\phi &= \frac{\nu}{r}\partial_r(r\partial_r u_\phi) - \frac{\nu}{r^2}u_\phi + \nu\partial_{xx}u_\phi + f_\phi, \\
\partial_t u_x + \frac{u_\phi}{r}\partial_\phi u_x + \partial_x p &= \frac{\nu}{r}\partial_r(r\partial_r u_x) + \frac{\nu}{r^2}\partial_{\phi\phi}u_x + f_x,
\end{aligned}$$

- $\mathbf{u} = u_\phi(t, x, r)\mathbf{e}_\phi + u_x(t, r, \not\phi)\mathbf{e}_x, p(t, x, r)$ reduced NSE (special case of axisymmetric flow)

$$\begin{aligned}
-(u_\phi)^2 + r\partial_r p &= 0, \\
\partial_t u_\phi + u_x \partial_x u_\phi &= \frac{\nu}{r}\partial_r(r\partial_r u_\phi) - \frac{\nu}{r^2}u_\phi + \nu\partial_{xx}u_\phi + f_\phi, \\
\partial_t u_x + \partial_x p &= \frac{\nu}{r}\partial_r(r\partial_r u_x) + f_x,
\end{aligned}$$

Example 5: Non-characteristic case (fully nonlinear)

- $$\Omega = \mathbb{R}^1 \times (0, H), \quad \mathbf{u} = (v_1, -V + v_3), v_j\big|_{z=0,H} = 0, j = 1, 3,$$
- reduced Euler (Antontsev& Kazhikhov 1990: 2D, Petcu 2006: 3D): upwind boundary condition

$$v_1^0\big|_{z=H} = 0,\, v_3^0\big|_{z=0,H} = 0$$

- Prandtl type equation:

$$-V\frac{\partial \theta_1}{\partial z} - \nu\frac{\partial^2 \theta_1}{\partial z^2} = 0, \theta_1\big|_{z=0} = -v_1^0\big|_{z=0}, \theta_1\big|_{z=\infty} = 0, \quad \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_3}{\partial z} = 0$$

- Temam & W.1999, 2001, 2002

$$\|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^\infty(L^2), L^\infty(L^\infty), L^2(H^1)} \to 0$$

Alekseenko1994 ($\|\mathbf{u} - \mathbf{u}^0\|_{L^2} \to 0$)

- Generalization: Hamouda-Temam, Xie, ...

Connection to another Oleinik's open problem

Asymptotic expansion of solutions of the system of boundary layer with strong injection. In this case one considers the Prandtl system with a small parameter ε*:*

$$\begin{aligned} \nu \frac{\partial^2 u}{\partial y^2} - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} &= -\frac{1}{\varepsilon} U(x) \frac{dU(x)}{dx}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

in the domain $D = \{0 < x < X, 0 < y < \infty\}$*, with the conditions*

$$u(x,0) = 0, u(0,y) = 0, v(x,0) = \varepsilon^{-1} v_0(x),$$

$$u(x,y) \to U(x), \ \text{as } y \to \infty,$$

where $U(0) = 0, U(x) > 0$ *for* $0 < x \le X; dU/dx > 0$ *and* $v_0(x) > 0$ *for* $0 \le x \le X, \alpha = const > 0.$ *It is required to find the asymptotic expansion of the solution of this problem as* $\varepsilon \to 0$*.*

Mathematical Models in Boundary Layer Theory by O.A. Oleinik and

Hopf-Kato type approach

- Hopf (1951) and Kato (1984)'s observation: corrector term $\boldsymbol{\theta}$ does not necessarily obey Prandtl type equation or Prandtl scaling
- Kato's result:

$$\|\mathbf{u}^\nu - \mathbf{u}^0\|_{L^2(0,T;L^2)} \to 0 \iff \nu \int_0^T \int_{\Gamma_{c\nu}} |\nabla \mathbf{u}^\nu|^2 \to 0$$

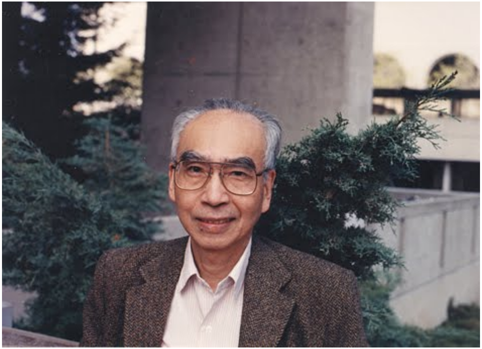
- choose $\boldsymbol{\theta} \approx \boldsymbol{\theta}(\frac{z}{\delta(\nu)}), \quad \text{supp}\boldsymbol{\theta} \in \Gamma_{\delta(\nu)}, \quad \mathbf{u}^a = \mathbf{u}^0 + \boldsymbol{\theta}, \mathbf{u}^e = \mathbf{u} - \mathbf{u}^a,$

$$\begin{aligned}
\nu\|\nabla\boldsymbol{\theta}\cdot\nabla\mathbf{u}^e\|_{L^2} &\leq \frac{\nu}{2}\|\mathbf{u}^e\|_{L^2}^2 + C\frac{\nu}{\delta(\nu)}, \\
\|(\mathbf{u}^0\cdot\nabla)\boldsymbol{\theta}\|_{L^2} &\leq C\delta(\nu)^{\frac{1}{2}}, \\
\|(\boldsymbol{\theta}\cdot\nabla)\mathbf{u}^0\|_{L^2} &\leq C\delta(\nu)^{\frac{1}{2}}, \\
\|(\boldsymbol{\theta}\cdot\nabla)\boldsymbol{\theta}\|_{L^2} &\leq C\delta(\nu)^{\frac{1}{2}}, \\
\|\frac{\partial\boldsymbol{\theta}}{\partial t}\|_{L^2} &\leq C\delta(\nu)^{\frac{1}{2}}
\end{aligned}$$

Hopf and Kato

Figure: Eberhard Hopf 1902-1983

Figure: Tosio Kato 1917-1999



Hopf-Kato type result

- no longer spectral constraint type approach
- Prandtl type assumptions lead to affirmative answer to vanishing viscosity limit (in L^2). Examples under $\nu/\delta(\nu) \to 0$ assumption

$$\nu^{\frac{1}{2}}\|\nabla_\tau u_n^\nu\|_{L^2(0,T;L^2(\Gamma_\delta))} \to 0, \quad (nec.\&suff., W.200$$

$$\nu^{\frac{1}{2}}\|\nabla_\tau \mathbf{u}_\tau^\nu\|_{L^2(0,T;L^2(\Gamma_\delta))} \to 0, \quad (nec.\&suff., W.200$$

$$\nu\|\nabla_\tau p^\nu\|_{L^2(0,T;L^2(\partial\Omega))} \to 0, \quad (2D, Iyer\&Pego\&W.2$$

$$\delta(\nu)^{\frac{1}{2}}\|p^\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} \to 0, \quad (2D, W.2001)$$

$$\delta(\nu)^{\frac{3}{2}}\|\nabla_\tau\nabla_\tau u_n^\nu\|_{L^2(0,T;L^2(\Gamma_\delta))} \to 0, \quad (Temam\&W.1998$$

$$\delta(\nu)^{\frac{3}{2}}\|\nabla_\tau\nabla_\tau \mathbf{u}_\tau^\nu\|_{L^2(0,T;L^2(\Gamma_\delta))} \to 0, \quad (Temam\&W.1998$$

$$\nabla_n u_\tau^\nu \geq 0, \nabla_\tau p^\nu \geq g, \delta(\nu)^{\frac{3}{4}}\|g\|_{L^2(0,T;L^2(\partial\Omega))} \to 0, \quad (2D, Temam\&W.19$$

$$\nabla_n u_\tau^\nu \geq 0, \nabla_\tau p_e \geq g, \nu^{\frac{3}{4}}\|g\|_{L^2(0,T;L^2(\partial\Omega))} \to 0, \quad (2D, Iyer\&Pego\&W.2$$

$$\nu\|(\mathbf{u}^\nu\cdot\nabla)\mathbf{u}^\nu\|^2_{L^2(0,T;L^2)} \to 0, \quad (Iyer\&Pego\&W.20$$

$$\nu^3\|\Delta\mathbf{u}^\nu\|^2_{L^2(0,T;L^2)} \to 0, \quad (Iyer\&Pego\&W.20$$

$$\nu^{-\frac{1}{2}}\|\mathbf{u}^\nu\|_{L^2(0,T;\Gamma_{c\nu})} \to 0, \quad (Kelliher2007)$$

A discrete Hopf-Kato type result

Theorem (Cheng& W. JMP2007)

Assume channel geometry with periodicity in x. $\mathbf{u}^k$ solves the truncated NSE (P_k is projection onto the 1st K_k horizontal modes)

$$\begin{aligned}
\frac{\partial \mathbf{u}^k}{\partial t} + P_k((\mathbf{u}^k\cdot\nabla)\mathbf{u}^k) - \nu_k\Delta\mathbf{u}^k + \nabla p^k &= P_k\mathbf{f},\\
div\,\mathbf{u}^k &= 0,\\
\mathbf{u}^k|_{z=0,h} &= P_k\mathbf{b},\\
\mathbf{u}^k|_{t=0} &= P_k\mathbf{u}_0
\end{aligned}$$

Assume that the following conditions are satisfied

$$\begin{aligned}
K_k &\to \infty \quad (consistency)\\
\nu_k &\to 0 \quad (vanishing\ viscosity)\\
K_k\frac{\nu_k}{LU} &\to 0 \quad (under-resolved\ condition)
\end{aligned}$$

Then

$$\|\mathbf{u}^k-\mathbf{u}^0\|_{L^\infty(0,T;L^2)} \le \kappa((K_k\nu_k)^{\frac{1}{5}} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(}$$

Well-known small scales in fluids

- Prandtl boundary layer thickness

$$\sqrt{\nu T} \tag{8}$$

- Kolmogorov dissipation length (3D)

$$(\frac{\nu^3}{\varepsilon})^{\frac{1}{4}} \sim \nu^{\frac{3}{4}} \tag{9}$$

where ε is the energy dissipation rate per unit volume and is presumably independent of the kinematic viscosity.

- Kraichnan dissipation length (2D)

$$(\frac{\nu^3}{\eta})^{\frac{1}{6}} \sim \nu^{\frac{1}{2}} \tag{10}$$

where η is the enstrophy dissipation rate per unit volume which is presumably independent of the kinematic viscosity.

- Taylor micro length

$$(\frac{\nu U^2}{\varepsilon})^{\frac{1}{2}} \tag{11}$$

Conclusion

- Validity of Prandtl theory for special cases
  - half-space with analytical data
  - case no normal flow/no separation
  - injection plus suction
  - Prandtl type assumptions lead to affirmative answer to vanishing viscosity limit question
- Long way to go ...
  - short time well-posedness of Prandtl (characteristic)
  - short time validity of Prandtl theory (characteristic case)
  - theory after separation
  - vanishing viscosity

Thank You!