

Some remarks on the analyticity of minimizers of free discontinuity problems

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October 15, 2002

Abstract

In this paper we give a partial answer to a conjecture of De Giorgi, namely we prove that in dimension two the regular part of the discontinuity set of a local minimizer of the homogeneous Mumford-Shah functional is analytic with the exception of at most a countable number of isolated points.

1 Introduction

This paper is concerned with the analyticity of the discontinuity set of local minimizers of a class of free discontinuity problems in dimension two. More precisely we consider the functional

$$F(u) := \int_{\Omega} f(\nabla u) \, dx_1 \, dx_2 + \beta \mathcal{H}^1(S(u) \cap \Omega) \quad (1)$$

defined on the space $SBV(\Omega)$ of special functions of bounded variation. Here $\Omega \subset \mathbb{R}^2$ is a bounded open set, $S(u)$ is the jump set of u , and $f : \mathbb{R}^2 \rightarrow [0, \infty)$ a strictly convex, analytic function such that

$$0 \leq f(\xi) \leq C(1 + |\xi|^p), \quad p > 1,$$

for some constant $C > 0$.

The prototype problem is given by the homogeneous Mumford-Shah functional

$$F(u) := \int_{\Omega} |\nabla u|^2 dx_1 dx_2 + \beta \mathcal{H}^1(S(u) \cap \Omega) \quad (2)$$

which was introduced in [36] in connection with a variational approach to Image Segmentation.

The existence of absolute minimizers of the non-homogeneous Mumford-Shah functional

$$F_g(u) := \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\Omega} |u - g|^2 dx + \beta \mathcal{H}^1(S(u) \cap \Omega) \quad (3)$$

was proved by Ambrosio [2, 3] in arbitrary dimensions.

A crucial observation in the study of regularity of minimizers of (3) is that if g is bounded they are quasi-minima for (2). Moreover, as observed in [6], typical blow-up arguments in regularity theory relate the local behavior of minimizers of the non-homogeneous functional (3) to the one of the homogeneous functional (2). Thus it is important to study (2).

The first regularity results are due to De Giorgi, Carriero and Leaci [19] who proved that the discontinuity set of local minimizers of (3) is essentially closed, that is $\mathcal{H}^1\left(\left(\overline{S(u)} \setminus S(u)\right) \cap \Omega\right) = 0$. The same result has been extended by Fonseca and Fusco [21] to energies of the form

$$\int_{\Omega} f(\nabla u) dx + \beta \mathcal{H}^{N-1}(S(u) \cap \Omega)$$

under suitable hypotheses on f and more recently by Fusco, Mingione and Trombetti [23] to integrands $f = f(x, u, \nabla u)$.

Partial regularity for (2) was studied by Ambrosio, Fusco and Pallara (see [8, 5, 6]) who showed that if u is a quasi-minimizer of (2) then there exists an \mathcal{H}^{N-1} -null set $\Sigma \subset \overline{S(u)} \cap \Omega$, relatively closed in Ω such that $\left(\overline{S(u)} \setminus \Sigma\right) \cap \Omega$ is a $C^{1, \frac{1}{4}}$ hypersurface. The same authors in [7] later proved higher regularity, and, in particular, that if $g \in C^\infty(A)$ for some open set $A \subset \Omega$ and $\overline{S(u)} \cap A$ is a $C^{1, \gamma}$ hypersurface, then $\overline{S(u)} \cap A$ is actually a C^∞ hypersurface.

It should also be noted that there is an extensive literature for the regularity of (3) in the two dimensional case $N = 2$, we quote here the results of Bonnet [10], Dal Maso, Morel and Solimini [14], David and Semmes [15], Leger [28], Maddalena and Solimini [30, 31, 33] and refer to [6] for a more detailed bibliography.

Regarding the analyticity of the discontinuity set, the following conjecture was made by De Giorgi:

Conjecture (De Giorgi) *If u is a local minimizer of the functional (2) and $S(u) \cap A$ is a $C^{1, \gamma}$ manifold for some open set A , then $S(u) \cap A$ is analytic.*

We refer to [6] for more details. In this section we give a partial answer to this conjecture, namely we prove the following.

Theorem 1 *Assume that $N = 2$ and let u be a local minimizer of the functional (1) in Ω . Assume that $S(u) \cap A$ is a $C^{1,\gamma}$ curve for some open set $A \subset \Omega$, then $S(u) \cap A$ is analytic with the exception of at most a countable set of isolated points.*

A similar result actually holds for more general functionals of the form (1) (see Section 4 below for more details). The main tool in the proof is the hodograph transform which was first used to study the regularity of the free boundary for elliptic systems by Kinderlehrer, Nirenberg and Spruck [25]. For a detailed exposition of the method we refer to the monograph of Kinderlehrer and Stampacchia [24].

The exceptional set in our result is given by

$$E := \left\{ (x_1, x_2) \in S(u) \cap A : \frac{\partial u^+}{\partial \tau}(x_1, x_2) = \frac{\partial u^-}{\partial \tau}(x_1, x_2) = 0 \right\},$$

where u^+ and u^- are the *approximate upper* and *lower limit* of u (see [6] for more details on *SBV* functions), and τ is the tangent vector to the jump set. To our knowledge there is no boundary analyticity results without some kind of non degeneracy condition.

Our argument works only in the two dimensional case and does not apply when lower order terms appear in the functional. Thus proving De Giorgi conjecture in the higher dimensional case seems to require different methods.

Besides the intrinsic interest of the result, Theorem 1 provides a justification to the analyticity assumption in [34] where it was shown, using the calibration method introduced by Alberti, Bouchitté and Dal Maso in [1], that if u is harmonic outside an analytic curve Γ and satisfies the necessary condition on Γ

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \mathcal{K} \text{ on } \Gamma,$$

where \mathcal{K} is the curvature, then u is a local minimizer of the Mumford-Shah functional in a neighborhood of Γ .

Finally, we refer to [22] and [24] (see also the recent paper [9]) for an extensive bibliography on related regularity results for free boundary problems.

2 Preliminaries

Let V and Y be two topological vector spaces and denote by V^* and Y^* their respective dual spaces. We denote by $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_Y$ the duality pairing. Let $\Lambda : V \rightarrow Y$ be a continuous, linear operator and denote by Λ^* its transpose. Given two convex functionals

$$I : V \rightarrow (-\infty, \infty], \quad H : Y \rightarrow (-\infty, \infty]$$

we consider the minimization problem (\mathcal{P})

$$\inf_{v \in V} [I(v) + H(\Lambda(v))].$$

We define the dual problem (\mathcal{P}^*) as

$$\sup_{z^* \in Y^*} [-I^*(\Lambda^* z^*) - H^*(-z^*)],$$

where I^* and H^* are the polar functions of I and H respectively, that is

$$\begin{aligned} I^*(v^*) &:= \sup_{v \in V} \{\langle v^*, v \rangle_V - I(v)\}, \\ H^*(z^*) &:= \sup_{z \in Y} \{\langle z^*, z \rangle_Y - H(z)\}. \end{aligned}$$

In what follows the operator ∂ denotes the subdifferential. The following result may be found in [20].

Theorem 2 *Let V and V^* , and Y and Y^* be two pairs of topological vector spaces, let $\Lambda : V \rightarrow Y$ be a continuous, linear operator. Consider two convex functionals*

$$I : V \rightarrow (-\infty, \infty], \quad H : Y \rightarrow (-\infty, \infty].$$

Assume that there exists $v_0 \in V$ such that

$$I(v_0) < \infty, \quad H(\Lambda(v_0)) < \infty,$$

H being continuous at $\Lambda(v_0)$. Then

$$\inf_{v \in V} [I(v) + H(\Lambda(v))] = \sup_{z^* \in Y^*} [-I^*(\Lambda^* z^*) - H^*(-z^*)],$$

and (\mathcal{P}^*) admits at least a solution \bar{z}^* .

Moreover, if \bar{v} is a solution of (\mathcal{P}) and \bar{z}^* is a solution of (\mathcal{P}^*) then

$$\Lambda^* \bar{z}^* \in \partial I(\bar{v}), \quad -\bar{z}^* \in \partial H(\Lambda(\bar{v})),$$

or, equivalently,

$$I(\bar{v}) + I^*(\Lambda^* \bar{z}^*) - \langle \Lambda^* \bar{z}^*, \bar{v} \rangle = 0, \quad H(\Lambda(\bar{v})) + H^*(-\bar{z}^*) + \langle \bar{z}^*, \Lambda \bar{v} \rangle = 0.$$

We now present, without proofs, some classical results on the analyticity of solutions of elliptic systems. For more details we refer to the monographs of Kinderlehrer and Stampacchia [24] and of Morrey [35].

In what follows let $\Omega \subset \mathbb{R}^N$ be an open set and set.

$$D = (D_1, \dots, D_N), \quad D_j = \frac{1}{i} \frac{\partial}{\partial y_j} \quad 1 \leq j \leq N.$$

Let $L_{kj}(y, D)$, $1 \leq j, k \leq n$, be linear differential operators with continuous complex valued coefficients. Consider the system of partial differential equations in the dependent variables u^1, \dots, u^n

$$\sum_{j=1}^n L_{kj}(y, D) u^j(y) = f_k(y) \quad \text{in } \Omega, \quad 1 \leq k \leq n. \quad (4)$$

To each equation we assign an integer weight $s_k \leq 0$ and to each dependent variable an integer weight $t_k \geq 0$ such that

$$\begin{aligned} \text{order } L_{kj}(y, D) &\leq s_k + t_j \quad \text{in } \Omega, \quad 1 \leq k \leq n, \\ \max_k s_k &= 0, \end{aligned}$$

where we use the convention that $L_{kj}(y, D) \equiv 0$ if $s_k + t_j < 0$. If we write

$$\overline{L}_{kj}(y, D) = \sum_{|\alpha| \leq s_k + t_j} a_{kj}^\alpha(y) D^\alpha,$$

then the *principal part* of $L_{kj}(y, D)$ is defined by

$$L'_{kj}(y, D) = \sum_{|\alpha| = s_k + t_j} a_{kj}^\alpha(y) D^\alpha.$$

We say that the system (4) is *elliptic* if

$$\text{rank } (L'_{kj}(y, \xi)) = n \text{ for each } \xi \in \mathbb{R}^N \setminus \{0\} \text{ and } y \in \Omega, \quad (5)$$

and for each pair of independent vectors $\xi, \eta \in \mathbb{R}^N$ and $y \in \Omega$ the polynomial

$$p(z) = \det L'_{kj}(y, \xi + z\eta) \quad (6)$$

has exactly $\mu = \frac{1}{2} \deg p$ roots with positive imaginary part and $\mu = \frac{1}{2} \deg p$ roots with negative imaginary part.

A general system of equations

$$F_k(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^\ell \mathbf{u}(y)) = 0 \quad \text{in } \Omega, \quad 1 \leq k \leq n, \quad (7)$$

where $\mathbf{u} = (u_1, \dots, u_n)$ and D^m stands for the set of all partial derivatives of order m , is *elliptic along the solution* \mathbf{u} if the variational equations

$$\sum_{j=1}^n L_{kj}(y, D) \overline{w}^j(y) := \left. \frac{d}{dt} F_k(y, \mathbf{u}(y) + t\overline{\mathbf{u}}, D(\mathbf{u}(y) + t\overline{\mathbf{u}}), \dots, D^\ell(\mathbf{u}(y) + t\overline{\mathbf{u}})) \right|_{t=0} = 0 \quad (8)$$

constitute an elliptic system as defined above.

Let $B_{hj}(y, D)$, $1 \leq h \leq \mu$, $1 \leq j \leq n$, be linear differential operator with continuous coefficients and assume that a portion of the boundary $\partial\Omega$ is contained in the hyperplane $y_N = 0$. We say that the set of boundary conditions

$$\sum_{j=1}^n B_{hj}(y, D) w^j(y) = g_h(y) \text{ on } S \subset \partial\Omega \cap \{y_N = 0\}, \quad 1 \leq h \leq \mu$$

is *coercive* for the system (4) if

(i) the system (4) is elliptic and

$$2\mu = \sum_{j=1}^n (s_j + t_j) \geq 0$$

is even;

(ii) there exist integers r_h , $1 \leq h \leq \mu$, such that order $B_{h_j}(y, D) \leq r_h + t_j$ on S ;

(iii) for every $y_0 \in S$ the homogeneous boundary value problem

$$\begin{aligned} \sum_{j=1}^n L'_{kj}(y_0, D) u^j(y) &= 0 \text{ in } \mathbb{R}_+^N, \quad 1 \leq k \leq n, \\ \sum_{j=1}^n B'_{hj}(y_0, D) u^j(y) &= 0 \text{ on } y_N = 0, \quad 1 \leq h \leq \mu, \end{aligned}$$

where B'_{h_j} is the part of B_{h_j} of order $r_h + t_j$, admits no nontrivial bounded exponential solutions of the form

$$u^j(y) = e^{i\xi' y'} \varphi_j(y_N), \quad 1 \leq j \leq n, \quad \xi' \in \mathbb{R}^{N-1},$$

where as usual $y' = (y_1, \dots, y_{N-1})$.

A set of (nonlinear) boundary conditions

$$\Psi_h(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^s \mathbf{u}(y)) = 0 \quad \text{on } S, \quad 1 \leq h \leq \mu,$$

is coercive for the system (7) along the solution \mathbf{u} if there exist weights r_1, \dots, r_μ such that the set of linearized boundary conditions

$$\sum_{j=1}^n B_{hj}(y, D) \bar{u}^j(y) := \left. \frac{d}{dt} \Psi_k(y, \mathbf{u}(y) + t\bar{\mathbf{u}}, D(\mathbf{u}(y) + t\bar{\mathbf{u}}), \dots, D^s(\mathbf{u}(y) + t\bar{\mathbf{u}})) \right|_{t=0} = 0 \quad (9)$$

in S is coercive for the linearized system (8) on S .

Theorem 3 Let U be a neighborhood of 0 in \mathbb{R}_+^N and $S = \partial U \cap \{y_N = 0\}$. Assume that \mathbf{u} is a solution of the elliptic and coercive system

$$\begin{aligned} F_k(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^\ell \mathbf{u}(y)) &= 0 \quad \text{in } U, \quad 1 \leq k \leq n, \\ \Psi_h(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^s \mathbf{u}(y)) &= 0 \quad \text{on } S, \quad 1 \leq h \leq \mu, \end{aligned}$$

with weights $s_k, t_j, r_h, 1 \leq j, k \leq n, 1 \leq h \leq \mu$.

Suppose also that F_k and Ψ_h are analytic. If $u^j \in \mathbf{C}^{t_h + r_0, \alpha}(U \cup S)$, for some $\alpha > 0$ and where $r_0 = \max_h(0, 1 + r_h)$, then the u^j are analytic in $U \cup S$, $1 \leq j \leq n$.

To test ellipticity and coerciveness of a system it is actually sufficient to verify it at one point. Indeed we have the following

Theorem 4 *Let U be a neighborhood of 0 in \mathbb{R}_+^N and $S = \partial U \cap \{y_N = 0\}$. Assume that 0 is an interior point of S and that \mathbf{u} is a solution of the system*

$$F_k(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^\ell \mathbf{u}(y)) = 0 \quad \text{in } U, \quad 1 \leq k \leq n, \quad (10)$$

$$\Psi_h(y, \mathbf{u}(y), D\mathbf{u}(y), \dots, D^s \mathbf{u}(y)) = 0 \quad \text{on } S, \quad 1 \leq h \leq \mu, \quad (11)$$

with weights $s_k, t_j, r_h, 1 \leq j, k \leq n, 1 \leq h \leq \mu$.

Suppose also that F_k and Ψ_h are analytic. If $u^j \in \mathbf{C}^{t_h+r_0}(U \cup S)$, where $r_0 = \max_h(0, 1 + r_h)$, and if the variational equations (8) and (9) are elliptic and coercive at $y = 0$, then the systems (10) and (11) are respectively elliptic and coercive in a neighborhood $(U \cup S) \cap B(0, \varepsilon)$, for some $\varepsilon > 0$.

3 Duality

Theorem 5 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open bounded set with Lipschitz boundary. Let $f : \mathbb{R}^2 \rightarrow [0, \infty)$ be a strictly convex function of class $C^1(\mathbb{R}^2)$ such that*

$$0 \leq f(\xi) \leq C(1 + |\xi|^p), \quad p > 1.$$

Let $u \in W^{1,p}(\Omega)$ be a weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\nabla f(\nabla u)) = 0 & \text{in } \Omega, \\ \nabla f(\nabla u) \cdot \nu = 0 & \text{on } \Gamma \subset \partial\Omega, \end{cases}$$

where Γ is a connected, relatively closed subset of $\partial\Omega$ with positive length. Then the problem

$$\begin{cases} \operatorname{div}(\nabla g(\nabla w)) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma, \end{cases}$$

where

$$g(\xi) := f^*(\xi^\perp),$$

admits a unique weak solution $w \in W^{1,q}(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\nabla w = (\nabla f(\nabla u))^\perp.$$

Proof. Consider the linear functional

$$\begin{aligned} \Lambda : W^{1,p}(\Omega) &\rightarrow L^p(\Omega; \mathbb{R}^2) \\ v &\mapsto \nabla v \end{aligned}$$

and the convex functionals

$$I : W^{1,p}(\Omega) \rightarrow \{0, \infty\}, \quad H : L^p(\Omega; \mathbb{R}^2) \rightarrow [0, \infty)$$

defined by

$$I(v) := \begin{cases} 0 & \text{if } v = u \text{ on } \partial\Omega \setminus \Gamma, \\ \infty & \text{otherwise,} \end{cases} \quad H(z) := \int_{\Omega} f(z) \, dx.$$

By the convexity assumption of f it is clear that u is a solution of the minimization problem (\mathcal{P})

$$\inf_{\substack{v \in W^{1,p}(\Omega), \\ v = u \text{ on } \partial\Omega \setminus \Gamma}} \int_{\Omega} f(\nabla v) \, dx = \inf_{v \in W^{1,p}(\Omega)} [I(v) + H(\Lambda v)].$$

The dual problem (\mathcal{P}^*) is given by

$$\sup_{z^* \in L^q(\Omega; \mathbb{R}^2)} [-I^*(\Lambda^* z^*) - H^*(-z^*)].$$

It is well-known that

$$H^*(z^*) = \int_{\Omega} f^*(z^*) \, dx,$$

while

$$\begin{aligned} I^*(\Lambda^* z^*) &= \sup_{v \in W^{1,p}(\Omega)} \{\langle \Lambda^* z^*, v \rangle - I(v)\} = \sup_{\substack{v \in W^{1,p}(\Omega), \\ v = u \text{ on } \partial\Omega \setminus \Gamma}} \langle \Lambda^* z^*, v \rangle \\ &= \langle z^*, \Lambda u \rangle + \sup_{\substack{h \in W^{1,p}(\Omega), \\ h = 0 \text{ on } \partial\Omega \setminus \Gamma}} \langle z^*, \Lambda h \rangle. \end{aligned}$$

Since

$$\langle z^*, \Lambda h \rangle = \int_{\Omega} z^* \cdot \nabla h \, dx = - \int_{\Omega} h \operatorname{div} z^* \, dx + \int_{\Gamma} h z^* \cdot \nu \, d\mathcal{H}^1,$$

where we have used the Divergence Theorem and the fact that $h = 0$ on $\partial\Omega \setminus \Gamma$, we have

$$I^*(\Lambda^* z^*) = \begin{cases} \langle z^*, \Lambda u \rangle & \text{if } \operatorname{div} z^* = 0 \text{ in } \Omega \text{ and } z^* \cdot \nu = 0 \text{ on } \Gamma, \\ \infty & \text{otherwise.} \end{cases}.$$

Therefore problem (\mathcal{P}^*) reduces to

$$\sup \left\{ - \int_{\partial\Omega \setminus \Gamma} u z^* \cdot \nu \, d\mathcal{H}^1 - \int_{\Omega} f^*(-z^*) \, dx : z^* \in L^q(\operatorname{div}; \Omega), \quad (12) \right. \\ \left. \operatorname{div} z^* = 0 \text{ in } \Omega \text{ and } z^* \cdot \nu = 0 \text{ on } \Gamma \right\},$$

where

$$L^q(\operatorname{div}; \Omega) := \{z^* \in L^q(\Omega; \mathbb{R}^2) : \operatorname{div} z^* \in L^q(\Omega)\}.$$

Note that the normal trace $z^* \cdot \nu$ is well-defined in $L^q(\operatorname{div}; \Omega)$ (see e.g. [27]). Using the fact that in \mathbb{R}^2 divergence-free vector fields are rotated gradients, namely using the change of variables

$$z^* = -(\nabla w)^\perp, \quad w = 0 \text{ on } \Gamma$$

we may now rewrite problem (\mathcal{P}^*) as

$$\sup \left\{ \int_{\partial\Omega \setminus \Gamma} u \nabla_\tau w \, d\mathcal{H}^1 - \int_\Omega f^*((\nabla w)^\perp) \, dx : w \in W^{1,q}(\Omega), \text{ and } w = 0 \text{ on } \Gamma \right\}, \quad (13)$$

where ∇_τ denotes the tangential gradient.

By Theorem 2, with

$$V := W^{1,p}(\Omega), \quad Y = L^p(\Omega; \mathbb{R}^2),$$

there exists a solution $\bar{z}^* \in L^q(\operatorname{div}; \Omega)$ of the dual problem (12) with

$$H(\Lambda(u)) + H^*(-\bar{z}^*) + \langle \bar{z}^*, \Lambda u \rangle = 0,$$

that is

$$\int_\Omega (f(\nabla u) + f^*(-\bar{z}^*) + \bar{z}^* \cdot \nabla u) \, dx = 0.$$

Since, by Young's inequality

$$f(\nabla u) + f^*(-\bar{z}^*) + \bar{z}^* \cdot \nabla u \geq 0$$

we deduce that

$$f(\nabla u) + f^*(-\bar{z}^*) + \bar{z}^* \cdot \nabla u = 0$$

\mathcal{L}^2 a.e. in Ω , which is equivalent to

$$-\bar{z}^* = \nabla f(\nabla u)$$

If we now consider the function $\bar{w} \in W^{1,q}(\Omega)$ such that $\bar{z}^* = -(\nabla \bar{w})^\perp$ and $\bar{w} = 0$ on Γ , it follows that

$$\nabla \bar{w} = (\nabla f(\nabla u))^\perp,$$

and, since \bar{w} is a solution of (13), we have

$$\begin{cases} \operatorname{div}(\nabla g(\nabla \bar{w})) = 0 & \text{in } \Omega, \\ \bar{w} = 0 & \text{on } \Gamma, \end{cases}$$

where

$$g(\xi) := f^*(\xi^\perp).$$

Note that the function g is still of class C^1 in its domain (see e.g. [38]). This concludes the proof. ■

4 Free discontinuity problems

Consider the functional

$$F(u) := \int_{\Omega} f(\nabla u) \, dx_1 \, dx_2 + \beta \mathcal{H}^1(S(u) \cap \Omega)$$

defined on the space $SBV(\Omega)$ of special functions of bounded variation (see [6] for more details). Here $\Omega \subset \mathbb{R}^2$ is a bounded open set and $f : \mathbb{R}^2 \rightarrow [0, \infty)$.

In this section we prove the following result

Theorem 6 *Assume that $f : \mathbb{R}^2 \rightarrow [0, \infty)$ is a strictly convex, analytic function such that*

$$0 \leq f(\xi) \leq C(1 + |\xi|^p), \quad p > 1,$$

for some constant $C > 0$. Let $u \in SBV(\Omega)$ be a local minimizer of the functional F . Assume that there exists an open set $A \subset \Omega$ such that $S(u) \cap A$ is a C^2 manifold which divides A into two simply connected components and that u is of class C^2 up to the boundary in $A \setminus S(u)$. Suppose also that

$$|\nabla u^+| + |\nabla u^-| \neq 0 \text{ in } S(u) \cap A. \quad (14)$$

Then $S(u) \cap A$ is analytic.

Proof. Fix a point $P \in S(u) \cap A$ such that

$$(|\nabla u^+| + |\nabla u^-|)(P) \neq 0.$$

Without loss of generality we may assume that $P = (0, 0)$,

$$\nu(0, 0) = (0, 1),$$

and

$$\nabla u^+(0, 0) \neq (0, 0).$$

Let

$$\Gamma := S(u) \cap A.$$

From the local minimality and from the regularity assumptions on u and Γ it follows that u is a solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\nabla f(\nabla u)) = 0 & \text{in } A \setminus \Gamma, \\ \nabla f(\nabla u) \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

Moreover a simple variation argument shows that

$$f(\nabla u^+) - f(\nabla u^-) = \mathcal{K} \text{ on } \Gamma, \quad (15)$$

where \mathcal{K} is the curvature.

By Theorem 5 applied on each connected component, the problem

$$\begin{cases} \operatorname{div}(\nabla g(\nabla w)) = 0 & \text{in } A \setminus \Gamma, \\ w = 0 & \text{on } \Gamma, \end{cases} \quad (16)$$

where

$$g(\xi) := f^*(\xi^\perp),$$

admits a unique weak solution $w \in W^{1,q}(A \setminus \Gamma)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\nabla w = (\nabla f(\nabla u))^\perp. \quad (17)$$

Observe that since f is strictly convex and analytic ∇f is invertible and its inverse is still analytic. Hence

$$g(\xi) = f^*(\xi^\perp) = \xi^\perp \cdot (\nabla f)^{-1}(\xi^\perp) - f\left((\nabla f)^{-1}(\xi^\perp)\right)$$

is still analytic and strictly convex. Denote by A^+ and A^- the two connected components of $A \setminus S(u)$ and by w^+ and w^- the restriction of w in A^+ and A^- respectively. Note that $w^\pm \in C^2(\overline{A^\pm})$ by (17), and so (16) and (15) become

$$g_{\xi_1 \xi_1}(\nabla w^+) w_{x_1 x_1}^+ + 2g_{\xi_1 \xi_2}(\nabla w^+) w_{x_1 x_2}^+ + g_{\xi_2 \xi_2}(\nabla w^+) w_{x_2 x_2}^+ = 0 \text{ in } A^+, \quad (18)$$

$$g_{\xi_1 \xi_1}(\nabla w^-) w_{x_1 x_1}^- + 2g_{\xi_1 \xi_2}(\nabla w^-) w_{x_1 x_2}^- + g_{\xi_2 \xi_2}(\nabla w^-) w_{x_2 x_2}^- = 0 \text{ in } A^-, \quad (19)$$

$$w^+ = w^- = 0, \quad h(\nabla w^+) - h(\nabla w^-) = \mathcal{K} \text{ on } \Gamma, \quad (20)$$

where

$$h(\xi) := f\left((\nabla f)^{-1}(\xi^\perp)\right).$$

We consider the transformation

$$\begin{aligned} A^+ &\rightarrow U^+ \\ (x_1, x_2) &\mapsto (y_1, y_2) := (x_1, w^+(x_1, x_2)). \end{aligned}$$

We claim that it is locally invertible in a neighborhood of $(0, 0)$. Indeed

$$\det \begin{pmatrix} 1 & w_{x_1}^+(0, 0) \\ 0 & w_{x_2}^+(0, 0) \end{pmatrix} = w_{x_2}^+(0, 0) \neq 0.$$

Write the inverse function as

$$(y_1, y_2) \mapsto (x_1, x_2) =: (y_1, \psi(y_1, y_2)).$$

Straightforward calculations yield

$$w_{x_1}^+ = -\frac{\psi_{y_1}}{\psi_{y_2}}, \quad w_{x_2}^+ = \frac{1}{\psi_{y_2}}, \quad (21)$$

while

$$\begin{aligned} w_{x_1 x_1}^+ &= -\frac{\psi_{y_1 y_1}}{\psi_{y_2}} + 2\frac{\psi_{y_1}}{\psi_{y_2}^2}\psi_{y_1 y_2} - \frac{\psi_{y_1}^2}{\psi_{y_2}^3}\psi_{y_2 y_2}, \\ w_{x_1 x_2}^+ &= -\frac{\psi_{y_1 y_2}}{\psi_{y_2}^2} + \frac{\psi_{y_1}}{\psi_{y_2}^3}\psi_{y_2 y_2}, \\ w_{x_2 x_2}^+ &= -\frac{\psi_{y_2 y_2}}{\psi_{y_2}^3}. \end{aligned}$$

Note that since $w^+ = w^- = 0$ on Γ we have that the tangential derivative of w^+ and w^- are zero on Γ and since $\nu(0,0) = (0,1)$ it follows that $w_{x_1}^+(0,0) = 0$ and in turn

$$\psi_{y_1}(0,0) = 0. \quad (22)$$

Hence (18) transforms into

$$\begin{aligned} &g_{\xi_1 \xi_1} \left(-\frac{\psi_{y_1}}{\psi_{y_2}}, \frac{1}{\psi_{y_2}} \right) \left(-\frac{\psi_{y_1 y_1}}{\psi_{y_2}} + 2\frac{\psi_{y_1}}{\psi_{y_2}^2}\psi_{y_1 y_2} - \frac{\psi_{y_1}^2}{\psi_{y_2}^3}\psi_{y_2 y_2} \right) \\ &+ 2g_{\xi_1 \xi_2} \left(-\frac{\psi_{y_1}}{\psi_{y_2}}, \frac{1}{\psi_{y_2}} \right) \left(-\frac{\psi_{y_1 y_2}}{\psi_{y_2}^2} + \frac{\psi_{y_1}}{\psi_{y_2}^3}\psi_{y_2 y_2} \right) \\ &+ g_{\xi_2 \xi_2} \left(-\frac{\psi_{y_1}}{\psi_{y_2}}, \frac{1}{\psi_{y_2}} \right) \left(-\frac{\psi_{y_2 y_2}}{\psi_{y_2}^3} \right) = 0 \text{ in } U^+. \end{aligned} \quad (23)$$

Next we consider the change of variables

$$\begin{aligned} U^+ &\rightarrow A^- \\ (y_1, y_2) &\mapsto (x_1, x_2) := (y_1, \psi(y_1, y_2) - Cy_2), \end{aligned}$$

with $C > 0$ to be chosen.

We claim that for C sufficiently large this transformation is locally invertible in a neighborhood of $(0,0)$. Indeed

$$\det \begin{pmatrix} 1 & 0 \\ \psi_{y_1}(0,0) & \psi_{y_2}(0,0) - C \end{pmatrix} = \psi_{y_2}(0,0) - C < 0$$

for $C > \psi_{y_2}(0,0)$. Let

$$\phi(y_1, y_2) := w^-(y_1, \psi(y_1, y_2) - Cy_2) \quad (24)$$

and

$$A(y_1, y_2) := \psi_{y_2}(y_1, y_2) - C.$$

We have

$$w_{x_1}^- = \phi_{y_1} - \frac{\psi_{y_1} \phi_{y_2}}{A}, \quad w_{x_2}^- = \frac{\phi_{y_2}}{A}, \quad (25)$$

while

$$\begin{aligned} w_{x_1 x_1}^- &= -\frac{\phi_{y_2}}{A} \psi_{y_1 y_1} + \frac{2\phi_{y_2} \psi_{y_1}}{A^2} \psi_{y_1 y_2} + \frac{2\phi_{y_2} \psi_{y_1}^2}{A^3} \psi_{y_2 y_2} + \left(1 - 2 \left(\frac{\psi_{y_1}}{A} - \frac{\psi_{y_1}^2}{A^2}\right)\right) \phi_{y_1 y_1} \\ &\quad - 2 \left(\frac{\psi_{y_1}}{A} - \frac{\psi_{y_1}^2}{A^2}\right) \phi_{y_1 y_2} - \frac{\psi_{y_1}^2}{A^2} \phi_{y_2 y_2}, \\ w_{x_1 x_2}^- &= \left(\frac{1}{A} - \frac{\psi_{y_1}}{A^2}\right) \phi_{y_1 y_2} + \frac{\phi_{y_2} \psi_{y_1}}{A^3} \psi_{y_2 y_2} - \frac{\phi_{y_2}}{A^2} \psi_{y_1 y_2}, \\ w_{x_2 x_2}^- &= \frac{\phi_{y_2 y_2}}{A^2} - \frac{\phi_{y_2}}{A^3} \psi_{y_2 y_2}. \end{aligned}$$

Hence (19) reduces to

$$\begin{aligned} &g_{\xi_1 \xi_1} \left(\phi_{y_1} - \frac{\psi_{y_1} \phi_{y_2}}{A}, \frac{\phi_{y_2}}{A} \right) \left[-\frac{\phi_{y_2}}{A} \psi_{y_1 y_1} + \frac{2\phi_{y_2} \psi_{y_1}}{A^2} \psi_{y_1 y_2} + \frac{2\phi_{y_2} \psi_{y_1}^2}{A^3} \psi_{y_2 y_2} \right. \\ &\quad \left. + \left(1 - 2 \left(\frac{\psi_{y_1}}{A} - \frac{\psi_{y_1}^2}{A^2}\right)\right) \phi_{y_1 y_1} - 2 \left(\frac{\psi_{y_1}}{A} - \frac{\psi_{y_1}^2}{A^2}\right) \phi_{y_1 y_2} - \frac{\psi_{y_1}^2}{A^2} \phi_{y_2 y_2} \right] \\ &\quad + 2g_{\xi_1 \xi_2} \left(\phi_{y_1} - \frac{\psi_{y_1} \phi_{y_2}}{A}, \frac{\phi_{y_2}}{A} \right) \left[\left(\frac{1}{A} - \frac{\psi_{y_1}}{A^2}\right) \phi_{y_1 y_2} + \frac{\phi_{y_2} \psi_{y_1}}{A^3} \psi_{y_2 y_2} - \frac{\phi_{y_2}}{A^2} \psi_{y_1 y_2} \right] \\ &\quad + g_{\xi_2 \xi_2} \left(\phi_{y_1} - \frac{\psi_{y_1} \phi_{y_2}}{A}, \frac{\phi_{y_2}}{A} \right) \left[\frac{\phi_{y_2 y_2}}{A^2} - \frac{\phi_{y_2}}{A^3} \psi_{y_2 y_2} \right] = 0. \end{aligned} \quad (26)$$

Set

$$\begin{aligned} \alpha &:= \frac{1}{A(0,0)}, \quad \beta := w_{x_2}^+(0,0), \quad \gamma := w_{x_2}^-(0,0), \\ a &:= g_{\xi_1 \xi_1}(0, \beta), \quad b := g_{\xi_1 \xi_2}(0, \beta), \quad c := g_{\xi_2 \xi_2}(0, \beta) \end{aligned}$$

and note that by the strict convexity condition we have

$$b^2 - ac < 0 \quad (27)$$

To apply Theorem 4 we choose $s_1 = s_2 := 0$ and $t_1 = t_2 := 2$. Then the principal parts of the linearized equations of (23) and (26) at $(0,0)$ are given respectively by

$$-\beta (a\bar{\psi}_{y_1 y_1} + 2b\bar{\psi}_{y_1 y_2} + c\beta^2\bar{\psi}_{y_2 y_2})$$

and

$$a(\bar{\phi}_{y_1 y_1} - \gamma\bar{\psi}_{y_1 y_1}) + 2b\alpha(\bar{\phi}_{y_1 y_2} - \gamma\bar{\psi}_{y_1 y_2}) + c\alpha^2(\bar{\phi}_{y_2 y_2} - \gamma\bar{\psi}_{y_2 y_2}).$$

To check condition (5) for each $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we compute

$$\begin{aligned} & \det(L'_{kj}((0, 0), \xi)) \\ &= \det \begin{pmatrix} -\beta (a\xi_1^2 + 2b\beta\xi_1\xi_2 + c\beta^2\xi_2^2) & 0 \\ -\gamma (a\xi_1^2 + 2b\alpha\xi_1\xi_2 + c\alpha^2\xi_2^2) & a\xi_1^2 + 2b\alpha\xi_1\xi_2 + c\alpha^2\xi_2^2 \end{pmatrix} \\ &= -\beta (a\xi_1^2 + 2b\beta\xi_1\xi_2 + c\beta^2\xi_2^2) (a\xi_1^2 + 2b\alpha\xi_1\xi_2 + c\alpha^2\xi_2^2) \end{aligned}$$

which differs from zero by condition (27).

Next for each pair of independent vectors $\xi, \eta \in \mathbb{R}^2$ the polynomial

$$\begin{aligned} p(z) &= \det L'_{kj}((0, 0), \xi + z\eta) \\ &= -\beta \left[a(\xi_1 + z\eta_1)^2 + 2b\beta(\xi_1 + z\eta_1)(\xi_2 + z\eta_2) + c\beta^2(\xi_2 + z\eta_2)^2 \right] \\ &\quad \times \left[a(\xi_1 + z\eta_1)^2 + 2b\alpha(\xi_1 + z\eta_1)(\xi_2 + z\eta_2) + c\alpha^2(\xi_2 + z\eta_2)^2 \right] \end{aligned}$$

which has roots

$$\begin{aligned} z_{1,2} &:= \frac{-a\xi_1\eta_1 - b\beta\xi_1\eta_2 - b\beta\eta_1\xi_2 - c\beta^2\xi_2\eta_2 \pm \sqrt{(\xi_1\eta_2 - \eta_1\xi_2)^2\beta^2(b^2 - ac)}}{2b\beta\eta_1\eta_2 + a\eta_1^2 + c\beta^2\eta_2^2}, \\ z_{3,4} &:= \frac{-a\eta_1\xi_1 - b\alpha\eta_1\xi_2 - b\alpha\eta_2\xi_1 - c\alpha^2\eta_2\xi_2 \pm \sqrt{(\eta_1\xi_2 - \eta_2\xi_1)^2\alpha^2(b^2 - ac)}}{2b\alpha\eta_1\eta_2 + a\eta_1^2 + c\alpha^2\eta_2^2} \end{aligned}$$

where the denominator does not vanish since $\eta \neq 0$ and by (27). Since $\xi, \eta \in \mathbb{R}^2$ are independent vectors and again by (27) we have that

$$\begin{aligned} (\xi_1\eta_2 - \eta_1\xi_2)^2\beta^2(b^2 - ac) &< 0, \\ (\eta_1\xi_2 - \eta_2\xi_1)^2\alpha^2(b^2 - ac) &< 0, \end{aligned}$$

and hence $p(z)$ has exactly $2 = \frac{1}{2} \deg p$ roots with positive imaginary part and $2 = \frac{1}{2} \deg p$ roots with negative imaginary part.

Thus we have shown that the system is elliptic. We now show the coercivity of the boundary conditions. Since the curve Γ is parametrized by $x_2 = \psi(y_1, 0)$ near $(0, 0)$, it is easy to see that the curvature reduces to

$$\mathcal{K} = \frac{\psi_{y_1 y_1}(y_1, 0)}{(1 + \psi_{y_1}^2(y_1, 0))^{3/2}}.$$

Hence, using (21) and (21), condition (20) reduces to

$$\begin{aligned} & h \left(-\frac{\psi_{y_1}(y_1, 0)}{\psi_{y_2}(y_1, 0)}, \frac{1}{\psi_{y_2}(y_1, 0)} \right) - h \left(\phi_{y_1}(y_1, 0) - \frac{\psi_{y_1}(y_1, 0)\phi_{y_2}(y_1, 0)}{A(y_1, 0)}, \frac{\phi_{y_2}(y_1, 0)}{A(y_1, 0)} \right) \\ & \hspace{15em} (28) \\ &= \frac{\psi_{y_1 y_1}(y_1, 0)}{(1 + \psi_{y_1}^2(y_1, 0))^{3/2}}. \end{aligned}$$

Moreover, since $w^- = 0$ on Γ , we have that $\phi(y_1, 0) = w^-(y_1, \psi(y_1, 0)) = 0$ on $S \subset \{y_2 = 0\}$ and in turn

$$\phi_{y_1 y_1}(y_1, 0) = 0 \text{ on } S. \quad (29)$$

Choosing $r_1 = r_2 := 0$ the principal part of the linearized equations of (28).and (29) at $(0, 0)$ become respectively

$$\bar{\psi}_{y_1 y_1}(y_1, 0)$$

and

$$\bar{\phi}_{y_1 y_1}(y_1, 0).$$

To check coercivity we must show that the only bounded solutions of the homogeneous boundary value problem

$$a(\bar{\phi}_{y_1 y_1} - \gamma \bar{\psi}_{y_1 y_1}) + 2b\alpha(\bar{\phi}_{y_1 y_2} - \gamma \bar{\psi}_{y_1 y_2}) + c\alpha^2(\bar{\phi}_{y_2 y_2} - \gamma \bar{\psi}_{y_2 y_2}) = 0 \quad \text{on } \mathbb{R}_+^2 \quad (30)$$

$$\begin{aligned} \bar{\psi}_{y_1 y_1}(y_1, 0) &= 0 \\ \bar{\phi}_{y_1 y_1}(y_1, 0) &= 0 \end{aligned} \quad \text{on } \mathbb{R} \quad (31)$$

of the form

$$(\bar{\psi}(y_1, y_2), \bar{\phi}(y_1, y_2)) = (e^{i\xi y_1} \varphi_1(y_2), e^{i\xi y_1} \varphi_2(y_2)), \quad \xi \in \mathbb{R}$$

are constant. If $\xi = 0$ then from (30) it follows that $\varphi_1'' = \varphi_2'' = 0$ and so φ_1 and φ_2 must be constant (in order to be bounded). If $\xi \neq 0$ then from (31) we have $\varphi_1(0) = 0$ and from the first equation in (30)

$$-a\xi^2 \varphi_1 + 2b\beta i\xi \varphi_1' + c\beta^2 \varphi_1'' = 0.$$

The general solution is

$$\varphi_1(y_2) = c_1 \exp \left[y_2 \left(\frac{-ib\xi + \xi \sqrt{ac - b^2}}{c\beta} \right) \right] - c_1 \exp \left[y_2 \left(\frac{-ib\xi - \xi \sqrt{ac - b^2}}{c\beta} \right) \right]$$

which is bounded only if $c_1 = 0$. Hence the second equation in (30) reduces to

$$a\bar{\phi}_{y_1 y_1} + 2b\alpha\bar{\phi}_{y_1 y_2} + c\alpha^2\bar{\phi}_{y_2 y_2} = 0$$

and thus the same reasoning shows that φ_2 must be zero.

Thus all the hypotheses of Theorem 4 are satisfied and therefore we may apply Theorem 3 to conclude that ϕ and ψ are analytic. Since the curve Γ is parametrized by $x_2 = \psi(y_1, 0)$ near $(0, 0)$ the proof is concluded. \blacksquare

Remark 7 (i) *The same techniques may also be applied to more general functionals of the form*

$$F(u) := \int_{\Omega} f(x, \nabla u) \, dx_1 \, dx_2 + \int_{S(u) \cap \Omega} \theta(x, \nu) \, d\mathcal{H}^1.$$

We leave the details to the interested reader. However it is not clear how to adapt the proof to include functionals of the type

$$F(u) := \int_{\Omega} f(x, u, \nabla u) \, dx_1 \, dx_2 + \int_{S(u) \cap \Omega} \theta(x, [u], \nu) \, d\mathcal{H}^1.$$

(ii) *If in place of (14) we assume that*

$$|\nabla u^+|, |\nabla u^-| \neq 0 \text{ in } S(u) \cap A,$$

then Theorem 6 continues to hold if we assume that f is analytic in $\mathbb{R}^2 \setminus \{0\}$. This last condition is satisfied in particular by

$$f(\xi) := \frac{1}{p} |\xi|^p, \quad p > 1.$$

As a corollary we may now prove Theorem 1.

Proof of Theorem 1. By the results in [6] we may assume that $S(u) \cap A$ is a C^∞ connected curve. Fix a point $(\bar{x}_1, \bar{x}_2) \in S(u) \cap A$ such that

$$\text{either } \frac{\partial u^+}{\partial \tau}(\bar{x}_1, \bar{x}_2) \neq 0 \text{ or } \frac{\partial u^-}{\partial \tau}(\bar{x}_1, \bar{x}_2) \neq 0. \quad (32)$$

By Theorem 6 it follows that $S(u) \cap A$ is analytic in a neighborhood of (\bar{x}_1, \bar{x}_2) . Thus to conclude the proof it remains to show that the set

$$E := \left\{ (x_1, x_2) \in S(u) \cap A : \frac{\partial u^+}{\partial \tau}(x_1, x_2) = \frac{\partial u^-}{\partial \tau}(x_1, x_2) = 0 \right\}$$

consists of isolated points. Indeed fix $(\bar{x}_1, \bar{x}_2) \in E$ and let $B((\bar{x}_1, \bar{x}_2), \varepsilon) \subset A$ be so small that $S(u) \cap B((\bar{x}_1, \bar{x}_2), \varepsilon)$ divides $B((\bar{x}_1, \bar{x}_2), \varepsilon)$ into two connected open regions $B^+((\bar{x}_1, \bar{x}_2), \varepsilon)$ and $B^-((\bar{x}_1, \bar{x}_2), \varepsilon)$. Let $\Phi : B^+((\bar{x}_1, \bar{x}_2), \varepsilon) \rightarrow D^+ := \Phi(B^+((\bar{x}_1, \bar{x}_2), \varepsilon))$ be an (invertible) conformal mapping of class C^∞ up to the boundary such that

$$D_0 := \Phi(S(u) \cap B((\bar{x}_1, \bar{x}_2), \varepsilon)) \subset \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0\}, \\ D^+ \subset \{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 0\},$$

(see e.g. Theorem 3.6 in [37]). The function $v^+(y_1, y_2) := u(\Phi^{-1}(y_1, y_2))$ is harmonic in D^+ , with $\frac{\partial v^+}{\partial y_2}(y_1, y_2) = 0$ on D_0 . Let $D^- := \{(y_1, y_2) \in \mathbb{R}^2 : (y_1, -y_2) \in D^+\}$ and extend v^+ to D^- by reflection. It is clear that v^+ is harmonic in $D = D^+ \cup D^- \cup D_0$ and so its critical points are isolated. This implies that (\bar{x}_1, \bar{x}_2) is isolated in E and completes the proof. ■

5 Acknowledgment

The authors would like to thank David Kinderlehrer for very useful conversations on the subject of this paper and for pointing out reference [25].

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