CONVERGENCE OF THE DISCONTINUOUS GALERKIN METHOD FOR DISCONTINUOUS SOLUTIONS*

NOEL J. WALKINGTON†

Abstract. We consider linear first order scalar equations of the form $\rho_t + div(\rho v) + a\rho = f$ with appropriate initial and boundary conditions. It is shown that approximate solutions computed using the discontinuous Galerkin method will converge in $L^2[0,T;L^2(\Omega)]$ when the coefficients v and a and data f satisfy the minimal assumptions required to establish existence and uniqueness of solutions. In particular, v need not be Lipschitz, so characteristics of the equation may not be defined, and the solutions being approximated my not have bounded variation.

Key words. Convergence, Discontinuous Galerkin Method, Hyperbolic Equations

AMS subject classifications. 65M60, 65M12

1. Introduction.

1.1. Overview.

We consider approximate solutions $\rho : [0,T) \times \Omega \to \mathbb{R}$ of the equation

$$\rho_t + div(\rho v) + a\rho = f, \quad \text{in } \Omega, \qquad \rho|_{t=0} = \rho_0, \tag{1.1}$$

computed with the discontinuous Galerkin method. Here the coefficient v is vector valued and a is scalar valued. When $a = f = 0$ this equation represents the balance of mass governing the density of a continuum and is present in every fluid simulation. It was this application that motivated the present work.

When modeling the flow of two immiscible fluids the density is discontinuous, and both the density and viscosity, $\mu = \mu(\rho)$, appear as coefficients of the principle terms of the momentum equation. A key ingredient of the existence theory for the momentum equation is the stability solutions the balance of mass under perturbations in v. Specifically, if a sequence of velocity fields $\{v_h\}_{h>0}$ converges weakly in $L^2[0,T;H^1_0(\Omega)]$ then the corresponding densities ρ_h converge strongly in $L^2[0,T;L^2(\Omega)]$ [14]. Strong convergence is needed in order to pass to the limit in the momentum equation to obtain solutions of the coupled system.

Similarly, in order to prove convergence of numerical approximations to the equations modeling immiscible fluids, or flows containing particles, strong convergence of the approximate densities ρ_h will be required when the velocity fields are also computed approximately, and this is what is addressed here. Specifically, we show that approximate solutions of the density equation converge strongly in $L^2[0,T;L^2(\Omega)]$ when the

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†Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213, noelw@cmu.edu.
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coefficients and data satisfy the minimal hypotheses required to obtain existence and uniqueness of the continuous problem.

While many numerical schemes have been proposed for the solution of first order hyperbolic equations, the discontinuous Galerkin method stands out as one of the best schemes in practice. In his introductory text [7] Johnsen states "...the discontinuous Galerkin method performs remarkably well and we know of no (linear) finite difference method that is better". Another advantage is that discontinuous Galerkin approximations of the the density equation couple correctly with natural approximations of the momentum equation so that the discrete scheme inherits estimates on the kinetic energy $\rho(|v|^2/2)$. To recover such energy estimates it is natural to multiply the density equation by $|v|^2/2$. Since stable approximations of the momentum equation typically approximate v with piecewise quadratic functions this suggests that piecewise quartic functions are a natural choice of discrete spaces for the discontinuous Galerkin approximation of the balance of mass. Our results below show that the discontinuous Galerkin scheme will converge when piecewise polynomial approximations of arbitrary degree in the spatial variables are used; however, for technical reasons the degree of the piecewise polynomial temporal variation is restricted to be zero or one.

1.2. Discontinuous Galerkin Method. The discontinuous Galerkin method was introduced to simulate neutron transport, and in this context the coefficients v and a are constant. Most of the analysis of this method concerns rates of convergence [8, 11, 12] and requires the solution to be smooth, so is not applicable to problems involving discontinuous solutions. One exception is the work of Lin and Zhou [13] who consider equations in the form $V.\nabla\rho + a\rho = f$, which is slightly more general than the evolution equation considered here (put $V = (1, v)$ and $\nabla = (\partial_t, \nabla_x)$ to recover the evolution form). Lin and Zhou require $V \in W^{1,1}(\Omega)$ and show that if the solution is in $H^{1/2}(\Omega)$ then piecewise constant solutions of the discontinuous Galerkin method will converge to certain "weak" solutions (the definition of a weak solution in [13] is not standard). Below we exploit the evolution structure of the equation in an essential fashion. This allows us to avoid any smoothness assumptions on v with respect to the time variable, $v \in L^1[0,T; H_0^1(\Omega)]$. Under this assumption ρ will not be of bounded variation and typically will not belong to the fractional Sobolev space $H^{1/2}$.

Since equation (1.1) is a conservation law in divergence form it is natural to consider the substantial literature concerning numerical schemes developed for nonlinear conservation laws of the form $\rho_t + div(F(\rho)) = f$. Essentially all of the methods (including the discontinuous Galerkin method) and theory developed in this context assume that F is independent of x and t; the exception being Kruzkov's original work [10] which allows F to depend upon x and t in a C^1 fashion. In this context solutions of the conservation laws are "regular" in the sense that they have bounded variation and frequently stability of numerical schemes is ensured by "flux limiters" which limit the variation [3, 4, 6, 9]. The best regularity one can expect for the velocity field of immiscible fluids is $v \in L^2[0,T; H_0^1(\Omega)]$ and the variation of the corresponding density will fail to be bounded, so any scheme which limits the variation of the discrete solution could not converge strongly.

The level set method has also been used to solve certain hyperbolic equations [15]. If $a = f = div(v) = 0$ then (1.1) can be written as $\rho_t/|\nabla\rho| = -v.(\nabla\rho/|\nabla\rho|)$ showing that the level sets of ρ move with normal velocity $-v$. Frequently it is possible to find a smooth function ϕ_0 having the same level sets as ρ_0 with $\rho_0 = \beta(\phi_0)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a discontinuous function. Since $\rho = \beta(\phi)$ for all subsequent times this eliminates the need to deal with discontinuous functions. This approach has been applied to problems in fluid mechanics [2].

1.3. Notation. Below $\Omega \subset \mathbb{R}^d$ will be a bounded domain with unit outward normal n. We will consider a regular family of finite element meshes $\{\mathcal{T}_h\}_{h>0}$ each of which we assume triangulates Ω exactly. It is assumed that the finite elements have uniformly bounded aspect ratio, and the parameter $h > 0$ represents the diameter of the largest element in $\mathcal{T}_h$. The discontinuous Galerkin method is constructed using space-time elements of the form $K \times (t^{n-1}, t^n)$ with $K \in \mathcal{T}_h$ where $\{t^n\}_{n=0}^N$ is a partition of $[0, T]$. The space of polynomials of degree k on an element K is denoted $\mathcal{P}_k(K)$. For simplicity we assume that for each $h > 0$ a uniform partition of $[0, T]$ used with $t^n = n\tau$ where $\tau = T/N$ is assumed to converge to zero as h tends to zero. We will denote the approximate solutions by ρ_h; in particular, to the dependence upon τ is implicit. If $a \in \mathbb{R}$ the positive and negative parts are denoted by $a^\pm$ with $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Standard notation is adopted for the Lebesgue spaces, $L^p(\Omega)$, and the Sobolev spaces, $W^{m,p}(\Omega)$ or $H^m(\Omega)$. The dual exponent to p will be denoted by p', $1/p + 1/p' = 1$. Solutions of the evolution equation will be functions from $[0, T]$ into these spaces, and we adopt the usual notion, $L^2[0, T; H^1(\Omega)]$, $C[0, T; H^1(\Omega)]$, etc. to indicate the temporal regularity of such functions. The space of C^∞ test functions having compact support in Ω is denoted by $\mathcal{D}(\Omega)$. For vector valued quantities, such as the velocity v, we write $v \in L^2(\Omega)$, to indicate that each component lies in the specified space. The space $H(div; \Omega)$ is the set of vector valued functions in $L^2[0, T; L^2(\Omega)]$ with divergence in $L^2[0, T; L^2(\Omega)]$. Strong convergence of a sequence will be indicated as $\rho_h \to \rho$, and weak convergence by $\rho_h \rightharpoonup \rho$.

2. Background. In this section we recall the essential results developed by DiPerna and Lions [5] for equation (1.1) and recall the discontinuous Galerkin method for approximating solutions of (1.1). Our proof of convergence is essentially a verification of the old adage "a stable consistent scheme is convergent". To make this rigorous we use Theorem 2.2, taken from [14], in a crucial fashion. This theorem essentially states that certain elementary relations that hold for classical solutions of (1.1) continue to hold for weak solutions. As stated above, our convergence results will require the approximate solutions to be either piecewise constant or piecewise linear in time. This can be directly attributed to a lack of stability; in general, there are no estimates for the time derivative of the discrete solution. Lacking bounds on the time derivative we can't show that natural piecewise constant approximations converge weakly to the same limit as higher degree piecewise polynomial approximations; see Corollary 3.2 below.

2.1. DiPerna Lions Theory. For technical reasons DiPerna and Lions [5] considered velocity fields v which vanished on the boundary, and for this reason we will always require $v(t) \in H^1_0(\Omega)$. In this situation no boundary conditions are required for ρ; otherwise, ρ would be specified on the "inflow boundary" where $v.n < 0$. The following definition of a weak solution of (1.1) is standard and allows us to admit the possibility of discontinuous solutions.

DEFINITION 2.1. *Let $v|_{\partial\Omega} = 0$, then $\rho : [0,T) \times \Omega \to \mathbb{R}$ is a* weak solution of (1.1) *if*

$$-\int_0^T \int_\Omega \rho(\psi_t + v.\nabla\psi + a\psi) = \int_\Omega \rho_0 \psi(0) + \int_0^T \int_\Omega f\psi \tag{2.1}$$

for all $\psi \in \mathcal{D}([0,T) \times \Omega)$.

If $\beta : \mathbb{R} \to \mathbb{R}$ is differentiable, then multiplying equation (1.1) by $\beta'(\rho)$ and formally rearranging the derivatives shows that $\beta(\rho)$ satisfies

$$\beta(\rho)_t + div(\beta(\rho)v) + \big(\rho\beta'(\rho) - \beta(\rho)\big)\, div(v) + a\rho\beta'(\rho) = f\beta'(\rho). \tag{2.2}$$

The following theorem by DiPerna and Lions [5] states that weak solutions of (1.1) will also be weak solutions of (2.2) provided each term is integrable.

THEOREM 2.2 (DiPerna & Lions). *Let $1 \le p \le \infty$ and suppose that*

$$v \in L^1[0,T, W_0^{1,p'}(\Omega)], \qquad a, div(v) \in L^1[0,T; L^\infty(\Omega)], \qquad f \in L^1[0,T; L^p(\Omega)].$$

Then for each $\rho_0 \in L^p(\Omega)$ then there exists a unique weak solution $\rho \in L^\infty[0,T, L^p(\Omega)]$ of (1.1). This solution satisfies

- *$\rho \in C[0,T; L^p(\Omega)]$ if $p < \infty$,*
- *If $\beta \in C^1(\mathbb{R})$ satisfies $\beta'(t) \le C(1+|t|^r)$ for $C > 0$, and $r = p-1$ if $p < d/(d-1)$, $r < p-1$ if $p = d/(d-1)$, and $r = p/d$ if $p > d/(d-1)$ (r arbitrary if $p = \infty$), then equation (2.2) is satisfied weakly.*
- *If $\beta \in C^1(\mathbb{R})$ satisfies $\beta'(t) \le C(1+|t|^r)$ for $C > 0$ and $r \le p-1$ (β arbitrary if $p = \infty$) then*

$$\frac{d}{dt}\int_\Omega \beta(\rho) + \int_\Omega div(v)(\rho\beta'(\rho) - \beta(\rho)) + a\rho\beta'(\rho) = \int_\Omega f\beta'(\rho). \tag{2.3}$$

Remark: The restrictions on r in the second statement of the theorem and the Sobolev embedding theorem guarantee that the term $\beta(\rho)v.\nabla\psi$ is integrable. Similarly, the restriction $r \le p-1$ in the third statement guarantees that each term is integrable.

2.2. Discontinuous Galerkin Method. We will allow for the possibility that the coefficients are only computed approximately on each mesh; $v \simeq v_h$ and $a \simeq a_h$. Since $div(v_h)$ may not be bounded in $L^1[0,T; L^\infty(\Omega)]$ care is required to construct a stable approximation scheme. Let

$$\mathcal{R}_h = \{\rho \in L^2[0,T; L^2(\Omega)] \mid \rho|_{K\times(t^{n-1},t^n)} \in \mathcal{P}_k(K)\otimes\mathcal{P}_\ell(t^{n-1},t^n),\ K \in \mathcal{T}_h,\ n = 1,2,\dots\}$$

The discontinuous Galerkin method requires $\rho \in \mathcal{R}_h$ to satisfy

$$\int_K \rho^n \psi(t^n_-) - \rho^{n-1}\psi(t^{n-1}_+) - \int_{t^{n-1}}^{t^n}\int_K \rho\left(\psi_t + v_h.\nabla\psi + (1/2)(div(v_h) - div(v))\psi - a_h\psi\right)$$

$$(2.4) \qquad + \int_{t^{n-1}}^{t^n}\int_{\partial K} \rho_{in}(v_h.n)\psi = \int_{t^{n-1}}^{t^n}\int_K f\psi,$$

for each $K \in \mathcal{T}_h$, $n = 1, 2, \dots$ and $\psi \in \mathcal{R}_h$. Since functions in $\mathcal{R}_h$ are discontinuous at the boundary of each space-time element, $K \times (t^{n-1}, t^n)$, we specify how the traces are to be evaluated. In all instances traces of ρ are taken from the "upwind" direction, and traces of ψ are taken from within $K \times (t^{n-1}, t^n)$. That is, ρ^n and ρ^{n-1} are the traces taken from below, $\rho^n = \lim_{s\nearrow t^n}\rho(s)$, ρ_{in} is the inflow trace, $\rho_{in}(x) = \lim_{\epsilon\searrow 0}\rho(x - \epsilon v_h)$, and the subscripts $t_\pm$ are used to indicate the traces of ψ at each end of the time interval.

If $(v.n) = (v.n)^+ + (v.n)^-$ is the decomposition of $v.n$ into positive and negative parts and $e = K \cap K_-$ is an edge (face in 3d) common to K and K_- then the upwind term can be written as

$$\rho_{in}(v.n) = \rho_-(v.n)^- + \rho(v.n)^+,$$

where ρ_- is the density on the element K_-. If an orientation of e is determined by selecting one of its normals N, then the weak statement on each element can be summed to give

$$\int_\Omega \rho^n\psi(t^n_-) - \sum_{k=0}^{n-1}\int_\Omega \rho^k[\psi^k] - \int_0^{t^n}\int_\Omega \rho\left(\psi_t + v_h.\nabla\psi + (1/2)(div(v_h) - div(v))\psi - a_h\psi\right)$$

$$(2.5) \qquad -\sum_e \int_0^{t^n}\int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi] = \int_\Omega \rho_0\psi(t^0_-) + \int_0^{t^n}\int_\Omega f\psi.$$

The jump in ψ on the element boundaries is denoted by $[\psi] = \psi_+ - \psi_-$ with $\psi_\pm$ determined by the orientation N on an edge and the positive time direction at a temporal interface. We have abused the notation by writing $\sum_K \int_K(.) = \int_\Omega(.)$ for terms involving gradients and similarly for temporal integrals. When ψ is continuous the above expression reduces to a standard weak statement of equation (1.1).

The assumptions $v \in L^1[0,T;H^1_0(\Omega)]$ and $div(v) \in L^1[0,T;L^\infty(\Omega)]$ are required for uniqueness of the solution of equation (1.1); however, the approximation v_h of v used in the computations need not converge to v in these spaces. In order for the scheme to be well defined traces of $v_h.n$ must exist on element boundaries and the traces from each side must agree. Our proof of convergence will require v_h and $div(v_h)$ to converge in $L^1[0,T;L^2(\Omega)]$. For these reasons we will require v_h to lie in the space

$$V_h = \{v \in L^1[0,T;H(div;\Omega)] \mid v(t)|_K \in \mathcal{P}_n(K)\}$$

for some fixed integer $n \geq 0$.

3. Stability. The following stability result is standard.

THEOREM 3.1 (Stability). *Let $\rho \in \mathcal{R}_h$ be the approximate solution of equation (1.1) obtained with the discontinuous scheme (2.4) and suppose that $\rho^0 \in L^2(\Omega)$, $v_h \in V_h$,*

$$v \in L^1[0,T; H_0^1(\Omega)], \qquad a_h,\ div(v) \in L^1[0,T; L^\infty(\Omega)], \qquad f \in L^1[0,T; L^2(\Omega)],$$

then

$$\begin{aligned} &(1/2)\|\rho^n\|^2_{L^2(\Omega)} + (1/2)\sum_{k=0}^{n-1} \|[\rho^k]\|^2_{L^2(\Omega)} + (1/2)\sum_e \int_0^{t^n}\int_e |v_h.n|[\rho]^2 \\ &\qquad + \int_0^{t^n}\int_\Omega (div(v)/2 + a_h)\rho^2 = (1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^{t^n}\int_\Omega f\rho. \end{aligned} \tag{3.1}$$

(1) If $(div(v)/2 + a_h) \geq c > 0$ and $f \in L^2[0,T; L^2(\Omega)]$ then

$$\begin{aligned} \|\rho^n\|^2_{L^2(\Omega)} + \sum_{k=0}^{n-1} \|[\rho^k]\|^2_{L^2(\Omega)} + \sum_e \int_0^{t^n}\int_e |v_h.n|[\rho]^2 + c\int_0^{t^n} \|\rho(s)\|^2_{L^2(\Omega)}\, ds \\ \leq \|\rho^0\|^2_{L^2(\Omega)} + (1/c)\int_0^{t^n} \|f(s)\|^2_{L^2(\Omega)}\, ds. \end{aligned}$$

(2) If ρ_h is piecewise constant in time or piecewise linear in time, ($\ell = 0$ or 1 in the definition of $\mathcal{R}_h$), and τ is sufficiently small then

$$\|\rho^n\|^2_{L^2(\Omega)} + \sum_{k=0}^{n-1} \|[\rho^k]\|^2_{L^2(\Omega)} + \sum_e \int_0^{t^n}\int_e |v_h.n|[\rho]^2 \leq C_1\|\rho^0\|^2_{L^2(\Omega)} \exp(C_2 t^n)$$

where C_1 and C_2 depend upon the coefficients v and a_h and the data f.

Proof. Selecting $\psi = \rho$ in equation (2.4) gives

$$\begin{aligned} &(1/2)\int_K (\rho^n)^2 + [\rho^{n-1}]^2 - (\rho^{n-1})^2 + \int_{t^{n-1}}^{t^n}\int_K (div(v)/2 + a_h)\rho^2 \\ &+(1/2)\int_{t^{n-1}}^{t^n}\int_{\partial K} \rho^2 (v_h.n)^+ + \rho_-^2 (v_h.n)^- - [\rho]^2 (v_h.n)^- = \int_{t^{n-1}}^{t^n}\int_K f\rho. \end{aligned}$$

Summing this expression and collecting terms establishes (3.1), and if $(div(v)/2 + a_h) \geq c > 0$ the statement (1) follows immediately.

If ρ_h is piecewise constant in time then

$$\int_0^{t^n}\int_\Omega f\rho - (div(v)/2 + a_h)\rho^2 \leq F^n \max_{1\leq k\leq n} \|\rho^k\|_{L^2(\Omega)} + \sum_{k=1}^{n} \beta_k \|\rho^k\|^2_{L^2(\Omega)}$$

where

$$\beta_k = \int_{t^{k-1}}^{t^k} \left((1/2)\|div(v(s))\|_{L^\infty(\Omega)} + \|a_h(s)\|_{L^\infty(\Omega)}\right) ds, \qquad F^n = \int_0^{t^n} \|f(s)\|_{L^2(\Omega)}\, ds.$$

If ρ_h is piecewise linear in time, then for $s \in (t^{n-1}, t^n)$

$$\begin{aligned}
\|\rho_h(s)\|_{L^2(\Omega)} &= \|\rho^n(s-t^{n-1})/\tau + \rho_+^{n-1}(t^n - s)/\tau\|_{L^2(\Omega)} \\
&= \|\rho^n(s-t^{n-1})/\tau + ([\rho^{n-1}] - \rho^{n-1})(t^n - s)/\tau\|_{L^2(\Omega)} \\
&\le \|\rho^n\|_{L^2(\Omega)}(s-t^{n-1})/\tau + \|[\rho^{n-1}] - \rho^{n-1}\|_{L^2(\Omega)}(t^n - s)/\tau
\end{aligned}$$

Then

$$\|\rho_h\|_{L^\infty[0,t^n;L^2(\Omega)]} \le \max_{0\le k\le n} \|\rho^k\|_{L^2(\Omega)} + \max_{0\le k\le n-1} \|[\rho^k]\|_{L^2(\Omega)}$$

and

$$\|\rho_h(s)\|^2_{L^2(\Omega)} \le \|\rho^n\|^2_{L^2(\Omega)} + 2\|\rho^{n-1}\|^2_{L^2(\Omega)} + 2\|[\rho^{n-1}]\|^2_{L^2(\Omega)}.$$

It follows that

$$\begin{aligned}
&\int_0^{t^n}\int_\Omega f\rho - (div(v)/2 + a)\rho^2 \le F^n \left(\max_{0\le k\le n} \|\rho^k\|_{L^2(\Omega)} + \max_{0\le k\le n-1} \|[\rho^k]\|_{L^2(\Omega)} \right) \\
&+ \left(\beta_n\|\rho^n\|^2_{L^2(\Omega)} + 2\beta_1\|\rho_0\|^2_{L^2(\Omega)}\right) + \sum_{k=1}^{n-1} \left((\beta_k + 2\beta_{k+1})\|\rho^k\|^2_{L^2(\Omega)} + 2\beta_k\|[\rho^{k-1}]\|^2_{L^2(\Omega)}\right)
\end{aligned}$$

In each instance an estimate of the form

$$\begin{aligned}
&\|\rho^n\|^2_{L^2(\Omega)} + (1-\gamma)\sum_{k=0}^{n-1} \|[\rho^k]\|^2_{L^2(\Omega)} + \sum_e \int_0^{t^n}\int_e |v_h.n|[\rho]^2 \\
&\qquad\qquad \le (1+\gamma_0)\|\rho_0\|^2_{L^2(\Omega)} + (C/c)(F^n)^2 + \sum_{k=1}^{n} \left(\gamma_k\|\rho^k\|^2_{L^2(\Omega)}\right) \\
&\qquad\qquad + c\left(\max_{0\le k\le n} \|\rho^k\|^2_{L^2(\Omega)} + \max_{0\le k\le n-1} \|[\rho^k]\|^2_{L^2(\Omega)} \right)
\end{aligned}$$

holds, where $c > 0$ is arbitrary and γ and γ_k are bounded by integrals of functions in $L^1[0,T]$ over intervals of length τ. If τ is sufficiently small, all of these constants will be less than $1/2$, and statement (2) follows from the discrete Gronwall inequality. □

When $div(v)/2 + a_h = 0$ the stability estimate only bounds ρ at the discrete times $\{t^n\}_{n=0}^N$. If $\bar\rho_h(t) = \rho^n$ on $(t^{n-1}, t^n]$ the lemma shows that $\bar\rho_h$ can be bounded in $L^\infty[0,T;L^2(\Omega)]$. Clearly $\bar\rho_h = \rho_h$ when $\ell = 0$; however, when $\ell > 1$ it may happen that $\bar\rho_h$ and ρ_h have different weak limits. The following corollary shows that this will not happen when $\ell = 1$.

Corollary 3.2. *Let $\rho_h \in \mathcal{R}_h$ be piecewise linear in time ($\ell = 1$) and $\bar\rho_h \in \mathcal{R}_h$ be the function piecewise constant in time equal to ρ^n on $(t^{n-1}, t^n]$.*

- *If $\psi \in L^2[0,T;L^2(\Omega)]$ then*

$$\begin{aligned}
&\left|\int_0^T\int_\Omega (\bar\rho_h - \rho_h)\psi\right| \le \|\bar\rho_h\|_{L^2[0,T;L^2(\Omega)]}\|\psi - \psi(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]} \\
&+\sqrt{(\tau/2)}\left(\|\rho^N\|_{L^2(\Omega)} + \|\rho^0\|_{L^2(\Omega)} + \left(\sum_{n=0}^{N-1}\|[\rho^n]\|^2_{L^2(\Omega)}\right)^{1/2}\right)\|\psi\|_{L^2[0,T;L^2(\Omega)]}
\end{aligned}$$

- $\|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]} \le \|\bar\rho_h\|_{L^\infty[0,T;L^2(\Omega)]} + \max_{0\le n\le N-1} \|[\rho_h^n]\|_{L^2(\Omega)}$ *and*

$$\|\bar\rho_h - \rho_h\|^2_{L^2[0,T;L^2(\Omega)]} \le (2/3)\left(\|\bar\rho_h - \bar\rho(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]} + \tau \sum_{n=0}^{N-1} \|[\rho^n]\|^2_{L^2(\Omega)} \right)$$

Proof. On the interval $(t^{n-1}, t^n]$ we have $\bar\rho_h - \rho_h = (\rho^n - \rho_+^{n-1})(t^n - t)/\tau$ (recall that $\rho^n = \rho_-^n$). Then

$$\begin{aligned}
\int_0^T \int_\Omega (\bar\rho_h - \rho_h)\psi &= \int_\Omega \sum_{n=1}^N \int_0^\tau (\rho^n - \rho_+^{n-1})(1 - s/\tau)\psi(t^{n-1}+s)\, ds \\
&= \int_\Omega \int_0^\tau \Bigg(\rho^N \psi(t^{N-1}+s) - \rho_+^0 \psi(s) \\
&\quad + \sum_{n=1}^{N-1} (\rho^n \psi(t^{n-1}+s) - \rho_+^n \psi(t^n+s)) \Bigg) (1 - s/\tau)\, ds \\
&= \int_\Omega \int_0^\tau \Bigg(\rho^N \psi(t^{N-1}+s) - \rho_h^0 \psi(s) \\
&\quad + \sum_{n=1}^{N-1} \rho^n (\psi(t^{n-1}+s) - \psi(t^n+s)) \\
&\quad + \sum_{n=0}^{N-1} (\rho^n - \rho_+^n)\psi(t^n+s) \Bigg) (1 - s/\tau)\, ds.
\end{aligned}$$

The first statement then follows upon observing that the last term can be bounded as

$$\int_\Omega \int_0^\tau \sum_{n=0}^{N-1} (\rho^n - \rho_+^n)\psi(t^n+s)(1 - s\tau) \le \sqrt{(\tau/2)} \left(\sum_{n=0}^{N-1} \|[\rho^n]\|^2_{L^2(\Omega)} \right)^{1/2} \|\psi\|_{L^2[0,T;L^2(\Omega)]}.$$

To establish the second statement we compute

$$\begin{aligned}
\|\bar\rho_h - \rho_h\|^2_{L^2[0,T;L^2(\Omega)]} &= \sum_{n=1}^N \int_0^\tau \|\rho^n - \rho_+^{n-1}\|^2 (1 - s/\tau)^2\, ds \\
&\le \sum_{n=1}^N (\tau/3)\|\rho^n - \rho^{n-1} + [\rho^{n-1}]\|^2 \\
&\le (2/3)\left(\|\bar\rho_h - \bar\rho(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]} + \tau \sum_{n=0}^{N-1} \|[\rho^n]\|^2_{L^2(\Omega)} \right)
\end{aligned}$$

□

Remark: When $div(v)$ and a are bounded it is possible to introduce a change of variables to guarantee that $div(v)/2 + a \ge c > 0$. Specifically, if $\rho = re^{\alpha t}$ then r satisfies

$$r_t + div(rv) + (\alpha + a)r = e^{-\alpha t} f.$$

4. Consistency. The bounds established above show that subsequences of the approximate solutions converge weakly-star in $L^\infty[0,T;L^2(\Omega)]$. In this section we show that the limits of these sequences are weak solutions of equation (1.1). This is easy to show when the space $\mathcal{R}_h$ contains the continuous finite element spaces; however, when $k=0$ the only continuous functions are constants which complicates the proof. We begin with a technical lemma.

LEMMA 4.1. *Let $K\subset\mathbb{R}^d$, $v\in H^1(K)^d$ and $\psi\in W^{1,p}(K)$ with $p\geq 4d/(d-3)$. Then there exists a constant C depending only upon d and the aspect ratio of K such that*

$$\int_{\partial K}|v.n||\psi-\bar{\psi}|^2\leq C|K|^{(1/2-2/p)}h_K\|v\|_{H^1(K)}\|\psi\|^2_{W^{1,p}(K)},$$

where $\bar{\psi}=(1/|K|)\int_K\psi$ is the average of ψ on K and h_K is the diameter of K.

If each component of v is a polynomial of degree n then

$$\int_{\partial K}|v.n||\psi-\bar{\psi}|^2\leq C|K|^{(1/2-2/p)}h_K\|v\|_{L^2(K)}\|\psi\|^2_{W^{1,p}(K)},$$

where C may now depend additionally upon n.

Proof. Let $\hat{K}$ be the usual reference simplex and $\mathcal{F}(\xi)=x_0+B\xi$ be an affine mapping of $\hat{K}$ to K. We use a hat to denote the natural correspondence between functions defined on K and $\hat{K}$, $\hat{\psi}=\psi\circ\mathcal{F}$. Writing the integral over the boundary as the sum over the faces $F\subset\partial K$ and using the trace theorem we obtain

$$\begin{aligned}\int_{\partial K}|v.n||\psi-\bar{\psi}|^2&=\sum_{F\subset\partial K}\int_F|v.n||\psi-\bar{\psi}|^2\\&=\sum_{\hat{F}\subset\partial\hat{K}}\frac{|F|}{|\hat{F}|}\int_{\hat{F}}|\hat{v}.n||\hat{\psi}-\bar{\psi}|^2\\&\leq C\sum_{\hat{F}\subset\partial\hat{K}}|F|\|\hat{v}\|_{H^1(\hat{K})}\|\hat{\psi}-\bar{\psi}\|^2_{W^{1,p}(\hat{K})}\\&\leq C\sum_{\hat{F}\subset\partial\hat{K}}|F|\|\hat{v}\|_{H^1(\hat{K})}|\hat{\psi}|^2_{W^{1,p}(\hat{K})}.\end{aligned}$$

We used the fact that the average of ψ is the average of $\hat{\psi}$ and the Poincare inequality to pass to the $W^{1,p}$ semi-norm in the last line. Notice that if $v\in\mathcal{P}_n(K)^d$ then $\|\hat{v}\|_{H^1(\hat{K})}$ is equivalent to $\|\hat{v}\|_{L^2(\hat{K})}$ since $\mathcal{P}_n(\hat{K})$ is finite dimensional.

Recalling that $\|\hat{v}\|_{L^2(\hat{K})}=(|\hat{K}|/|K|)^{1/2}\|\hat{v}\|_{L^2(K)}$,

$$|\hat{\psi}|_{W^{1,p}(\hat{K})}\leq(|\hat{K}|/|K|)^{1/p}|B||\psi|_{W^{1,p}(K)}\leq C(|\hat{K}|/|K|)^{1/p}h_K|\psi|_{W^{1,p}(K)}$$

and $|\nabla\hat{v}|_{L^2(\hat{K})}=C(|\hat{K}|/|K|)^{1/2}h_K\|\hat{v}\|_{L^2(K)}$ we deduce that

$$\int_{\partial K}|v.n||\psi-\bar{\psi}|^2\leq C\sum_{F\subset\partial K}\frac{|F|}{|K|^{(1/2+2/p)}}h_K^2\|v\|_{H^1(K)}|\psi|^2_{W^{1,p}(K)},$$

(with $\|v\|_{L^2(K)}$ replacing $\|v\|_{H^1(K)}$ if $v \in \mathcal{P}_n(K)^d$). Since

$$|K| = (1/d)|F| \times (\text{perpendicular height}) \geq c|F|h_K,$$

where c depends upon the aspect ratio of K, it follows that $|F|h_K \leq C|K|$ and the proof follows. $\Box$

The following lemma provides sufficient conditions on the coefficients v_h and a_h that suffice to establish consistency of the discontinuous Galerkin method.

LEMMA 4.2. *Let $\{\rho_h\}$ be a (sub) sequence of solutions of the discontinuous Galerkin scheme (2.4) computed on a sequence of quasi-regular meshes and suppose that $\rho_h \rightharpoonup^* \rho$ in $L^\infty[0,T;L^2(\Omega)]$. Assume that $f \in L^1[0,T;L^2(\Omega)]$, $v \in L^1[0,T;H^1_0(\Omega)]$, $v_h \in V_h$, $\rho_h^0 \rightharpoonup \rho_0$ in $L^2(\Omega)$,*

$$v_h \to v, \quad \textit{and} \quad a_h \to a, \quad \textit{in } L^1[0,T;L^2(\Omega)],$$

and either (1) $div(v_h) \to div(v)$ in $L^1[0,T;L^2(\Omega)]$, or (2) $(div(v_h) - div(v))|_K \perp \mathcal{P}_k(K)$ in $L^2(K)$ for each $K \in \mathcal{T}_h$. Then ρ is a weak solution of equation (1.1).

Proof. When $k,\ \ell > 0$ $\mathcal{R}_h$ contains the usual continuous finite element spaces, so if $\psi \in \mathcal{D}([0,T) \times \Omega)$ then the interpolant $\psi_h \in \mathcal{R}_h \cap C([0,T] \times \bar{\Omega})$ converges to ψ in $W^{1,\infty}((0,T) \times \Omega)$. When ψ_h is substituted into equation (2.5) all of the jump terms vanish to give

$$-\int_0^T \int_\Omega \rho_h \left(\psi_{ht} + v_h.\nabla\psi_h + (1/2)(div(v_h) - div(v))\psi_h - a_h\psi_h\right) = \int_\Omega \rho_h^0 \psi(0) + \int_0^T \int_\Omega f\psi_h.$$

If $div(v_h) \to div(v)$ in $L^1[0,T;L^2(\Omega)]$ the hypotheses on the suffice to pass to the limit term by term in the above equation to show that ρ is a weak solution of equation (1.1). If $(div(v_h) - div(v)|_K \perp \mathcal{P}_k(K)$ for $K \in \mathcal{T}_h$ that the term involving $div(v_h) - div(v)$ still vanishes since

$$\begin{aligned}
\int_0^T \int_\Omega \rho_h(div(v_h) - div(v))\psi_h &= \int_0^T \int_\Omega \rho_h(div(v_h) - div(v))(\psi_h - \bar{\psi}) \\
&\leq \|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]} \|div(v_h) + div(v)\|_{L^1[0,T;L^2(\Omega)]} Ch \\
&\to 0,
\end{aligned}$$

where $\bar{\psi}(t,x)$ is the average of $\psi_h(t,.)$ over the element K containing x.

When $\ell = 0$ or $k = 0$ functions in $\mathcal{R}_h$ are piecewise constant in time or space respectively. In this situation the terms involving ψ_{ht} or $\nabla\psi_h$ vanish and it is necessary to show that the corresponding jump terms in (2.5) approximate the missing terms: $-\sum_{k=0}^{N-1} \int_\Omega \rho^k [\psi^k] \sim -\int_0^T \int_\Omega \rho\psi_t$ and

$$-\sum_e \int_0^T \int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h] \sim \int_0^T \int_\Omega \rho v.\nabla\psi.$$

If $\ell = 0$ and $\psi \in \mathcal{D}([0,T) \times \Omega)$ let ψ^k be a interpolant of $\psi(t^k)$ onto the (spatial) finite element space and let $\hat{\psi}_h$ denote the piecewise linear interpolant of $\{\psi^k\}$ in time. Then

$\hat{\psi}_h$ converges to ψ in $W^{1,\infty}[0,T;L^\infty(\Omega)]$ and temporal jump terms become

$$-\sum_{k=0}^{N-1}\int_\Omega \rho^k[\psi^k] = -\sum_{k=0}^{N-1}\int_\Omega \rho^k(\psi^{k+1}-\psi^k) = -\int_0^T\int_\Omega \rho_h\hat{\psi}_{ht} \to -\int_0^T\int_\Omega \rho\psi_t$$

as required.

Finally consider the piecewise constant scheme, $k=0$, and consider ψ piecewise polynomial of degree ℓ in time with values in $\mathcal{D}(\Omega)$. Selecting ψ_h to be the spatial average of ψ over each element the spatial jump terms in equation (2.5) become

$$\begin{aligned}
&-\sum_e\int_0^T\int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h] \\
&= -\sum_e\int_0^T\int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h-\psi] \\
&= -\int_0^T\sum_K\int_{\partial K}\left(\rho_{K_-}(v_h.n)^- + \rho_K(v_h.n)^+\right)(\psi_K-\psi)
\end{aligned}$$

where ψ_K is the average value of ψ on K, ρ_K is the value of ρ_h on K and ρ_{K_-} the value of ρ_h on the upwind element K_-. Then

$$\begin{aligned}
&-\sum_e\int_0^T\int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h] \\
&= -\int_0^T\sum_K\int_{\partial K}\left((\rho_{K_-}-\rho_K)(v_h.n)^- + \rho_K(v_h.n)\right)(\psi_K-\psi) \\
&= -\int_0^T\sum_K\int_K div(\rho_K v_h(\psi_K-\psi)) + \int_{\partial K}[\rho_h](v_h.n)^-(\psi_K-\psi) \\
&= -\int_0^T\int_\Omega \left(\rho_h v_h.\nabla\psi + \rho_h\, div(v_h)(\psi_h-\psi)\right) + \int_0^T\sum_K\int_{\partial K}[\rho_h](v_h.n)^-(\psi_K-\psi)
\end{aligned}$$

Clearly the first term converges to $\int\int \rho v.\nabla\psi$; we need to show that the second term vanishes in the limit. From Lemma 4.1 (with $p=4$) we obtain

$$\begin{aligned}
&\int_0^T\sum_K\int_{\partial K}[\rho_h](v_h.n)^-(\psi_K-\psi) \\
&\le \left(\int_0^T\sum_K\int_{\partial K}|v_h.n|[\rho_h]^2\right)^{1/2}\left(\int_0^T\sum_K\int_{\partial K}|v_h.n|(\psi_K-\psi)^2\right)^{1/2} \\
&\le \left(\int_0^T 2\sum_e\int_e|v_h.N|[\rho_h]^2\right)^{1/2}\left(\sum_K Ch_K\int_0^T\|v_h\|_{L^2(K)}\|\psi\|^2_{W^{1,4}(K)}\right)^{1/2} \\
&\le \left(\int_0^T\sum_e\int_e|v_h.N|[\rho_h]^2\right)^{1/2} C\sqrt{h}\,\|v_h\|^{1/2}_{L^1[0,T;L^2(\Omega)]}\|\psi\|_{L^\infty[0,T;W^{1,4}(\Omega)]}.
\end{aligned}$$

Theorem 3.1 shows that the first term is bounded so this term vanishes as $h \to 0$. $\square$

Remark: The variant of the discontinuous Galerkin scheme given in Section 2 was formulated to be convergent when the velocity field v was only known approximately but the divergence of the v was known precisely. The canonical example of this would be when v_h is an approximation of the velocity of an incompressible fluid for which $div(v) = 0$. Typically $v_h \to v$ in $L^2[0,T;L^2(\Omega)]$, but $div(v_h) \neq 0$, since it is difficult to construct divergence free subspaces of $H^1_0(\Omega)$; in particular, $div(v_h) \not\to 0$ in $L^1[0,T;L^2(\Omega)]$.

The second hypotheses on the divergence in Lemma 4.2 is useful in this canonical situation. Let $\tilde{v}_h \in L^1[0,T;H^1_0(\Omega)]$ be a piecewise polynomial approximation of v and suppose $\tilde{v}_h \to v$ in $L^1[0,T;L^2(\Omega)]$. We can then construct $v_h \in V_h$ satisfying the hypothesis of the Lemma as follows. At each time t let $v_h(t)$ be the $L^2(\Omega)$ projection of $\tilde{v}_h(t)$ onto the space

$$\bar{V}_h(v) = \{v_h \in RT^k_h(\Omega) \mid \int_K div(v_h)p(x) = \int_K div(v)p(x),\ p \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h\}.$$

Here $RT^k_h(\Omega)$ is the Raviart-Thomas subspace of $H(div;\Omega)$ constructed using piecewise polynomials of degree k on $\mathcal{T}_h$ [1].

By construction $div(v_h) - div(v)$ is orthogonal to $\mathcal{P}_k(K)$ for each element K. To see that v_h also converges to v first observe that

$$(\tilde{v}_h - v_h, w_h)_{L^2(\Omega)} = 0 \qquad \forall\ w_h \in \bar{V}_h \equiv \bar{V}_h(0).$$

Then

$$\begin{aligned}\|\tilde{v}_h - v_h\|^2_{L^2(\Omega)} &= (\tilde{v}_h - v_h, \tilde{v}_h - I_h v + I_h v - v_h)_{L^2(\Omega)} \\ &= (\tilde{v}_h - v_h, \tilde{v}_h - I_h v)_{L^2(\Omega)}\end{aligned}$$

where $I_h v$ is the "interpolant" of v onto the Raviart-Thomas space [1]. The degrees of freedom of I_h are constructed to guarantee that $I_h v - v_h \in \bar{V}_h$. Then

$$\begin{aligned}\|\tilde{v}_h - v_h\|_{L^2(\Omega)} &\le \|\tilde{v}_h - v\|_{L^2(\Omega)} + \|v - I_h v\|_{L^2(\Omega)} \\ &\le \|\tilde{v}_h - v\|_{L^2(\Omega)} + Ch\|v\|_{H^1_0(\Omega)}\end{aligned}$$

so $v_h \to v$ if $\tilde{v}_h \to v$ and v is bounded in $L^1[0,T;H^1_0(\Omega)]$.

5. Convergence. When ρ_h is piecewise constant or linear in time ($\ell = 0$ or 1) the stability estimate guarantees that it is possible to pass to a subsequence for which

$$\bar{\rho}_h \rightharpoonup^* \bar{\rho} \text{ and } \rho_h \rightharpoonup^* \rho \qquad \text{in } L^\infty[0,T;L^2(\Omega)],$$

and Corollary 3.2 shows that the weak limits coincide, $\rho = \bar{\rho}$. (Recall that $\bar{\rho}_h$ is piecewise constant in time and assumes the values ρ^n in $(t^{n-1}, t^n]$.) If additionally $v \in L^1[0,T;H^1_0(\Omega)]$ then Theorem 2.2 shows that ρ is unique. In this situation the whole sequence $\{\rho_h\}$ converges weakly and in the following theorem we establish strong convergence in $L^2[0,T;L^2(\Omega)]$.

THEOREM 5.1. *Let $\{\rho_h\}_{h>0}$ be the sequence of solutions of the discontinuous Galerkin scheme (2.4) with either $\ell = 0$ or $\ell = 1$ (ρ_h piecewise constant or linear in time) computed on a sequence of quasi-regular meshes. Assume $\rho_0 \in L^2(\Omega)$,*

$$v \in L^1[0,T; H_0^1(\Omega)], \qquad a,\ div(v) \in L^1[0,T; L^\infty(\Omega)], \qquad f \in L^1[0,T; L^2(\Omega)],$$

and $div(v)/2+a \geq 0$. If $\rho_h^0 \to \rho_0$ in $L^2(\Omega)$, $a_h \to a$ in $L^1[0,T; L^\infty(\Omega)]$, and $v_h \in V_h$ is an approximation of v for which $v_h \to v$ in $L^2[0,T; L^2(\Omega)]$ and either (1) $div(v_h) \to div(v)$ in $L^1[0,T; L^2(\Omega)]$, or (2) $(div(v_h) - div(v))|_K \perp \mathcal{P}_k(K)$ in $L^2(K)$ for each $K \in \mathcal{T}_h$, then ρ_h converges in $L^2[0,T; L^2(\Omega)]$ to ρ, the weak solution of equation (1.1). Moreover, the jump term

$$J_h^N = (1/2) \sum_{k=0}^{N-1} \|[\rho^k]\|_{L^2(\Omega)}^2 + (1/2) \sum_e \int_0^T \int_e |v_h.n| [\rho]^2$$

converge to zero.

Proof. The idea of the proof is to show that $\liminf_h \|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]} \leq \|\rho\|_{L^2[0,T;L^2(\Omega)]}$. If $\ell = 0$ then $\bar{\rho}_h = \rho_h$ and strong convergence of ρ_h follows. When $\ell = 1$ the first estimate in Corollary 3.2 shows that $\bar{\rho}_h$ and ρ_h have the same weak limit so $\bar{\rho}_h$ converges strongly to ρ. The second estimate in Corollary 3.2 shows that strong convergence of $\bar{\rho}_h$ implies strong convergence of ρ_h.

The key step is to observe that equation (2.3) in Theorem 2.2 an *equation* instead of the usual *inequality*. The hypotheses on v allow us to select $\beta(s) = (1/2)s^2$ in Theorem 2.2 to obtain

$$\begin{aligned} (1/2)\|\rho(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega ((1/2)div(v) + a)\rho^2 &= (1/2)\|\rho_0\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega f\rho \\ &= \liminf_{h\to 0} \left((1/2)\|\rho_h^0\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega f\rho_h \right). \end{aligned} \tag{5.1}$$

Integrating both sides with respect to t and using the dominated convergence theorem to interchange the limit and integral gives

$$(1/2)\|\rho\|_{L^2[0,T;L^2(\Omega)]}^2 + \int_0^T \beta(t;\rho)\, dt = \liminf_{h\to 0} \int_0^T \left((1/2)\|\rho_h^0\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega f\rho_h \right) dt,$$

where

$$\beta(t;\rho) = \int_0^t \int_\Omega ((1/2)div(v) + a)\rho^2.$$

Below we will use equation (3.1) to show that

$$\begin{aligned} &\liminf_{h\to 0} \int_0^T \left((1/2)\|\rho_h^0\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega f\rho_h \right) dt \\ &\geq \liminf_{h\to 0} \left((1/2)\|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]}^2 + \int_0^T \beta(t;\rho_h)\, dt \right), \end{aligned} \tag{5.2}$$

so that

$$(1/2)\|\rho\|_{L^2[0,T;L^2(\Omega)]}^2 + \int_0^T \beta(t;\rho)\, dt \geq \liminf_{h\to 0} \left((1/2)\|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]}^2 + \int_0^T \beta(t;\rho_h)\, dt \right).$$

Since $0 \leq (div(v)/2 + a) \in L^1[0,T;L^\infty(\Omega)]$ it follows that $\beta(t,.)$ is non-negative and lower semi-continuous with respect to weak star convergence in $L^\infty[0,T;L^2(\Omega)]$. Then applying Fatou's lemma to the right hand side of the above expression shows that $\|\rho\|_{L^2[0,T;L^2(\Omega)]} \geq \liminf_h \|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]}$ as required.

To establish equation (5.2), multiply equation (3.1) by τ and sum to obtain

$$\begin{aligned}
\int_0^T \left((1/2)\|\rho_h^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho_h \right) dt = (1/2)\|\bar{\rho}_h\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \beta(t,\rho_h) + \sum_{n=1}^N \tau J_h^n \\
+ \int_0^T \int_0^t \int_\Omega f\rho_h - \sum_{n=1}^N \tau \int_0^{t^n} \int_\Omega f\rho_h \\
+ \sum_{n=1}^N \tau\beta(t^n;\rho_h) - \int_0^T \beta(t,\rho_h)\, dt \\
+ \sum_{n=1}^N \tau \int_0^{t^n} \int_\Omega (a-a_h)\rho_h^2.
\end{aligned}$$

To complete the convergence proof we show that the terms in the last three lines vanish as $h \to 0$.

- The terms involving f can be combined as

$$\begin{aligned}
\int_0^T \int_0^t \int_\Omega f\rho_h - \sum_{n=1}^N \tau \int_0^{t^n} \int_\Omega f\rho_h &= \sum_{n=1}^N \int_\Omega \left(\int_{t^{n-1}}^{t^n} \int_0^t f\rho_h - \tau \int_0^{t^n} f\rho_h \right) \\
&= \sum_{n=1}^N \int_\Omega \left(\int_{t^{n-1}}^{t^n} \int_{t^{n-1}}^t f\rho_h - \tau \int_{t^{n-1}}^{t^n} f\rho_h \right) \\
&= \sum_{n=1}^N \int_\Omega \int_{t^{n-1}}^{t^n} (t^n - s - \tau) f(s)\rho_h(s)\, ds
\end{aligned}$$

  where the last line follows upon interchanging the order of integration. It follows that

$$\left| \int_0^T \int_0^t \int_\Omega f\rho_h - \sum_{n=1}^N \tau \int_0^{t^n} \int_\Omega f\rho_h \right| \leq 2\tau \|f\|_{L^1[0,T;L^2(\Omega)]} \|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]}$$

- An similar calculation is used to estimate the terms involving $\beta(.;\rho_h)$,

$$\sum_{n=1}^N \tau\beta(t^n;\rho_h) - \int_0^T \beta(t,\rho_h)\, dt = \sum_{n=1}^N \int_\Omega \int_{t^{n-1}}^{t^n} (t^n - s - \tau) \left(div(v)/2 + a \right) \rho_h^2\, ds$$

  so that

$$\begin{aligned}
&\left| \int_0^T \int_0^t \beta(s;\rho_h)\, ds\, dt - \sum_{n=0}^N \tau \int_0^{t^n} \beta(s;\rho_h)\, ds \right| \\
&\qquad \leq 2\tau \|(1/2)div(v) + a\|_{L^1[0,T;L^\infty(\Omega)]} \|\rho_h\|^2_{L^\infty[0,T;L^2(\Omega)]}
\end{aligned}$$

- The last term is bounded by $T\|a - a_h\|_{L^1[0,T;L^\infty(\Omega)]}\|\rho_h\|^2_{L^\infty[0,T;L^2(\Omega)]}$

To show that the jump terms converge to zero we combine equations (5.1) and (3.1) to get

$$\|\rho(t)\|^2_{L^2(\Omega)} + \beta(t,\rho) = \lim_{h\to 0}\left(\|\rho^n\|_{L^2(\Omega)} + \beta(t^n,\rho_h) + J_h^n\right)$$

where $n = n(h)$ is chosen so that $t \in (t^{n-1}, t^n]$. With this choice $\rho^n = \bar\rho_h(t)$, and since $\|\bar\rho_h(t)\|_{L^2(\Omega)}$ converges to $\|\rho(t)\|_{L^2(\Omega)}$ in $L^2[0,T]$, selecting t to be a Lebesgue point and noting that $\beta(.,.)$ is continuous on $\mathbb{R}\times L^2[0,T;L^2(\Omega)]$ shows that $\lim_{h\to 0} J_h^n = 0$. Since $J_h^n \ge J_h^N$ for $n \ge N$, and observing that the final time can be chosen arbitrarily, we can chose a Lebesgue point $t \ge T$ to conclude that $J_h^N \to 0$. □

6. Monotonicity of the Piecewise Constant Scheme. As stated above, the form of the discontinuous Galerkin method proposed in Section 2 was chosen to guarantee convergence when the velocity field was only known approximately. In particular, the factor of 1/2 in the term $(1/2)(div(v_h) - div(v))$ guarantees stability in $L^2(\Omega)$; however, $L^p(\Omega)$ estimates would require a different weight. In particular, the piecewise constant scheme, $k = \ell = 0$ may fail to be monotone.

When $k = \ell = 0$ the discontinuous Galerkin scheme (2.4) can be written as

$$|K|(\rho^n - \rho^{n-1}) + \int_{t^{n-1}}^{t^n}\int_K \left(a_h + (1/2)(div(v) + div(v_h))\right)\rho^n + \int_{t^{n-1}}^{t^n}\int_{\partial K}(\rho_-^n - \rho^n)(v_h.n)^- = \int_{t^{n-1}}^{t^n}\int_K f.$$

Notice that if K is selected so that $\rho^n|_K$ is maximal/minimal then the boundary term in non-negative/positive so

$$\max(\rho^n) \le \frac{\max(\rho^{n-1}) + F^n}{1 - C^n} \qquad \text{and} \qquad \min(\rho^n) \ge \frac{\min(\rho^{n-1})}{1 + C^n} - \frac{F^n}{1 - C^n},$$

where

$$F^n = \int_{t^{n-1}}^{t^n} \|f\|_{L^\infty(\Omega)}, \quad \text{and} \quad C^n = \int_{t^{n-1}}^{t^n} \|a_h + (1/2)(div(v) + div(v_h))\|_{L^\infty(\Omega)},$$

and we have assumed that $\min(\rho^{n-1}) \ge 0$ and τ is sufficiently small to guarantee that $C^n < 1$. Defining

$$\gamma(t) = \int_0^t \|a + (1/2)(div(v) + div(v_h))\|_{L^\infty(\Omega)}$$

then, assuming $\max(\rho^0) \ge 0$, we compute

$$\max(\rho^n) \le e^{\gamma(t^n)}\max(\rho^0) + \int_0^{t^n} e^{\gamma(t^n)-\gamma(s)}\|f(s)\|_{L^\infty(\Omega)}\,ds,$$

and if $\min(\rho^0) \ge 0$

$$\min(\rho^n) \ge e^{-\gamma(t^n)}\min(\rho^0) - \int_0^{t^n} e^{\gamma(t^n)-\gamma(s)}\|f(s)\|_{L^\infty(\Omega)}\,ds$$

whenever the right hand side is non-negative.

Since it is unlikely that $div(v_h)$ is bounded in $L^1[0,T;L^\infty(\Omega)]$ this estimate is not particularly useful as it stands. For example, typically it will not be possible to chose τ sufficiently small to guarantee that $C^n < 1$. However, if the construction at the end of Section 4 is used to guarantee that $div(v_h)$ and $div(v)$ have the same average on each element, then the piecewise constant discontinuous Galerkin scheme becomes

$$|K|(\rho^n - \rho^{n-1}) + \int_{t^{n-1}}^{t^n}\int_K (a_h + div(v))\rho^n + \int_{t^{n-1}}^{t^n}\int_{\partial K}(\rho^n_- - \rho^n)(v_h.n)^- = \int_{t^{n-1}}^{t^n}\int_K f.$$

In this situation the scheme will be monotone and convergent. The following theorem summarizes these observations.

THEOREM 6.1. *Let $\{\rho_h\}_{h>0}$ be the sequence of solutions of the piecewise constant ($k = \ell = 0$) discontinuous Galerkin scheme (2.4). Assume that $v \in L^1[0,T;H^1_0(\Omega)]$, $v_h \in V_h$ and that the averages of $div(v_h)$ and $div(v)$ agree on each element $K \in \mathcal{T}_h$. If a_h, $div(v)$, and f are bounded in $L^1[0,T;L^\infty(\Omega)]$, $0 \le \alpha \le \rho^0 \le \beta$, and τ is sufficiently small then*

$$\max(\rho^n) \le e^{\gamma(t^n)}\max(\rho^0) + \int_0^{t^n} e^{\gamma(t^n)-\gamma(s)}\|f(s)\|_{L^\infty(\Omega)}\,ds,$$

and

$$\min(\rho^n) \ge e^{-\gamma(t^n)}\min(\rho^0) - \int_0^{t^n} e^{\gamma(t^n)-\gamma(s)}\|f(s)\|_{L^\infty(\Omega)}\,ds.$$

provided the right hand side is non-negative. Here

$$\gamma(t) = \int_0^t \|a_h + div(v)\|_{L^\infty(\Omega)},$$

and τ "sufficiently small" is interpreted to mean $\gamma(t^n+\tau) - \gamma(t^n) < 1$ for $n = 0, 1, \ldots$.

Remark: The monotonicity estimates above can be improved slightly if one sided bounds are used instead of absolute values. For example, in the upper bound $\|a_h + div(v))\|_{L^\infty(\Omega)}$ can be replaced by $\|(a_h + div(v))^-\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$ by $\|f^+\|_{L^\infty(\Omega)}$.

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