Equi-integrability results for 3D-2D dimension reduction problems

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Abstract

3D-2D asymptotic analysis for thin structures rests on the mastery of scaled gradients $\left(\nabla_\alpha u_\varepsilon \middle| \frac{1}{\varepsilon}\nabla_3 u_\varepsilon\right)$ bounded in $L^p(\Omega;\mathbb{R}^9)$, $1 < p < +\infty$. Here it is shown that, up to a subsequence, u_ε may be decomposed as $w_\varepsilon + z_\varepsilon$, where z_ε carries all the concentration effects, i.e. $\left\{\left|\left(\nabla_\alpha w_\varepsilon \middle| \frac{1}{\varepsilon}\nabla_3 w_\varepsilon\right)\right|^p\right\}$ is equi-integrable, and w_ε captures the oscillatory behavior, i.e. $z_\varepsilon \to 0$ in measure. In addition, if $\{u_\varepsilon\}$ is a recovering sequence then $z_\varepsilon = z_\varepsilon(x_\alpha)$ nearby $\partial\Omega$.

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1 Introduction

The treatment of nonlinear 3D-2D dimensional reduction has been undertaken successfully in recent years via asymptotic analysis in the Calculus of Variations (e.g. see [3], [7], [8], [9], [11], [12], [15], [18], [24], [25], [32]). Consider a homogeneous thin 3D domain with reference configuration $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where ω is a bounded domain in $\mathbb{R}^2$, and whose elastic energy density is a continuous function $W : \mathbb{M}^{3\times 3} \to \mathbb{R}$ satisfying the p - growth and coercivity condition

$$\frac{1}{C}\|A\|^p - C \le W(A) \le C\left(1 + \|A\|^p\right) \tag{1.1}$$

for some $1 < p < +\infty$, where $C > 0$ is a real constant and $\mathbb{M}^{3\times 3}$ denotes the space of real 3×3 matrices endowed with the usual Euclidean norm $\|A\| := \sqrt{\mathrm{tr}\,(A^T A)}$. The total energy of the film under a deformation $u : \Omega_\varepsilon \to \mathbb{R}^3$ is given by

$$\tilde{E}_\varepsilon(u) := \int\limits_{\Omega_\varepsilon} W(\nabla u(y))\, dy - \int\limits_{\Omega_\varepsilon} f^\varepsilon(y) \cdot u(y)\, dy,$$

where $f^\varepsilon \in L^{p'}(\Omega_\varepsilon, \mathbb{R}^3)$ stands for an appropriate dead loading body force density, and p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

We assume that the (quasi-static) equilibrium states of the film correspond to minimizers of $\tilde{E}_\varepsilon$ over the space of admissible deformations. To study the effective behavior of a very thin film, we consider a sequence $\{\varepsilon_n\}$ of positive real numbers (thickness) converging to zero and we rescale the

problem by an $1/\varepsilon_n$- dilatation in the transverse direction x_3 in order to recast energy functionals over varying domains Ω_{ε_n} into functionals with a fixed domain of integration. Precisely, performing the change of variables

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = \frac{1}{\varepsilon_n} y_3, \tag{1.2}$$

and with $v(x) := u(y(x))$, we obtain

$$\frac{1}{\varepsilon_n} \tilde{E}_{\varepsilon_n}(u) = E_n(v),$$

where

$$E_n(v) := \int_\Omega W\left(\nabla_\alpha v(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v(x)\right) dx - \frac{1}{\varepsilon_n} \int_\Omega f^{\varepsilon_n}(x_\alpha, \varepsilon_n x_3) \cdot v(x)\ dx,$$

$\Omega := \omega \times (-1,1)$, $\nabla_\alpha v$ is the 3×2 matrix of partial derivatives $\frac{\partial u_i}{\partial x_\alpha}$, $i \in \{1,2,3\}, \alpha \in \{1,2\}$, and $(A|a)$ denotes a 3×3 matrix whose first two columns are those of the 3×2 matrix A and the last column is the vector $a \in \mathbb{R}^3$.

As explained in [18], the simplest assumption on the rescaled body force density $\frac{1}{\varepsilon_n} f^{\varepsilon_n}(x_\alpha, \varepsilon_n x_3)$ so as to be of order $O(1)$ and, together with the total bulk energy, to entail a nonlinear membrane behavior in the zero thickness limit is that $\frac{1}{\varepsilon_n} f^{\varepsilon_n}(x_\alpha, \varepsilon_n x_3)$ be independent of n. Thus, the study of the effective energy of the limiting system rests on the limit of the energies $I_{\varepsilon_n}(v_n) := \int_\Omega W\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right) dx$, where v_n is a minimizer of E_n. In view of (1.1), for energy bounded sequences we have the apriori bound

$$\sup_{n \in \mathbb{N}} \int_\Omega \left| \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right)\right|^p (x_\alpha, x_3) dx_\alpha dx_3 < +\infty,$$

and, up to a subsequence, Poincaré-Friedrichs inequality entails the weak convergence in $W^{1,p}(\Omega;\mathbb{R}^3)$ of $\left\{ v_n - \int_\Omega v_n(y) dy \right\}$ to a deformation $v \in W^{1,p}(\Omega;\mathbb{R}^3)$, where $v = v(x_\alpha)$. Due to the possible presence of oscillations in the sequence of scaled gradients $\left\{ \left(\nabla_\alpha v_n \big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right)\right\}$, it may happen that

$$\lim_{n \to \infty} \int_\Omega W\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right) dx \neq \int_\Omega W\left(\nabla_\alpha v | 0\right) dx.$$

The characterization of (oscillatory) limits of nonlinear quantities in the Calculus of Variations has been analyzed in several contexts by means of Young measures. Young measures were introduced in Optimal Control Theory by L.C. Young in connection to nonconvex problems, thus providing the appropriate framework for the description of generalized minimizers in the Calculus of Variations (see [33], [34]). Later L. Tartar developed the use of Young measures in the PDE framework (see [29], [30], [31]). For a detailed study of Young measures, we refer the reader to [4], [5], [6], [17], [20], [21], [22], [23], [27], among others. It turns out that the ability to determine $\lim_{n \to +\infty} \int_\Omega W\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right) dx$ in terms of parametrized probability measures is restricted to the

case where it is apriori known that the sequence $\left\{W(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n)\right\}$ is equi-integrable, or, in light of (1.1), that $\left\{\left|\left(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p\right\}$ is equi-integrable. While this cannot be guaranteed in general for the original sequence $\{v_n\}$, we are able to show that, up to a subsequence (not relabelled), it is possible to decompose $\left\{\left(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ as a sum of a sequence $\left\{\left(\nabla_\alpha w_n|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ whose p-th power is equi-integrable and a remainder that converges to zero in measure. We may say that $\left\{\left(\nabla_\alpha w_n|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ carries the oscillations, while the remainder accounts for the concentration effects. The main objective of this paper is to prove this decomposition result, controlling the uniform smallness of $\left\{\left(\nabla_\alpha w_n|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ on small sets by means of the initial sequence only. An alternative argument using De La Vallée - Poussin criterion has been recently communicated to the authors by A. Braides (see [10]).

Theorem 1.1 *Let $\Omega := \omega \times (-1,1)$, where $\omega \subset \mathbb{R}^2$ is an open, bounded Lipschitz domain, let $\{\varepsilon_n\}$ be a sequence of positive real numbers converging to zero, and let $\{v_n\}$ be a bounded sequence in $W^{1,p}(\Omega,\mathbb{R}^3)$ $(1 < p < +\infty)$ satisfying*

$$\sup_{n\in\mathbb{N}} \int_\Omega \left|\left(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p (x_\alpha, x_3) dx_\alpha dx_3 < +\infty.$$

Suppose further that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega,\mathbb{R}^3)$ and $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup b$ in $L^p(\Omega,\mathbb{R}^3)$. Then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a sequence $\{w_k\} \subset W^{1,\infty}(\Omega,\mathbb{R}^3)$ such that

(i) $\lim_{k\to\infty} \mathcal{L}^3\left(\{x \in \Omega : w_k(x) \neq v_{n_k}(x)\}\right) = 0$;

(ii) $\left\{\left(\nabla_\alpha w_k|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\}$ *is p-equi-integrable;*

(iii) $w_k \rightharpoonup v$ *weakly in* $W^{1,p}(\Omega,\mathbb{R}^3)$;

(iv) $\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k \rightharpoonup b$ *weakly in* $L^p(\Omega,\mathbb{R}^3)$.

The corresponding decomposition result for the case of unscaled gradients has received much attention in recent years, due to the central role that it plays in a number of key results in the Calculus of Variations, among them the celebrated lower semicontinuity result of E. Acerbi and N. Fusco [1], the characterization of $W^{1,p}$ Young measures by D. Kinderlehrer and P. Pedregal [20], [21], as well as numerous relaxation results for nonconvex integrands. While the technique needed to establish the decomposition property was essentially used in [1], the result was first isolated only ten years later in [22]. The proof in [22] is based on estimates for perturbed Hodge decomposition [19], and more recently I. Fonseca, S. Müller and P. Pedregal [17] proposed another proof using L^p estimates for maximal functions, Lipschitz extensions of $W^{1,p}$ functions off small sets, and Young measures. We follow a similar argument, although the degeneracy of the coercivity condition in the x_3 - direction leads us to repeat v_n periodically on an infinite strip of copies of Ω in the x_3 - direction, thus obtaining non-degenerate uniform bounds for the resulting vertically periodized sequence. De Giorgi's slicing method will now come into play to help us selecting, via an averaging process, one of these layers with small energy concentration. Up to a subsequence, the restriction of v_n to this copy of Ω will eventually become the new w_n.

As a first consequence of Theorem 1.1, we show that the Dirichlet problem on an arbitrarily large cylinder for fixed affine lateral boundary conditions

$$\min\left\{\int_{\omega\times(-L,L)} W(\nabla v)dx : v \in W^{1,p}(\omega\times(-L,L);\mathbb{R}^3), v = v_0 \text{ on } \partial\omega\times(-L,L)\right\},$$

admits p-equi-integrable minimizing sequences energetically prefering thinner and thinner reference domains. Precisely, after changing variables as in (1.2),

Corollary 1.2 *Let $\Omega := \omega \times (-1,1)$, where $\omega \subset \mathbb{R}^2$ is an open, bounded Lipschitz domain. Let $u_0 \in W^{1,p}(\omega;\mathbb{R}^3)$ be an affine mapping $(1 < p < +\infty)$, and let $W : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying (1.1). Define*

$$Q^*_\omega W(u_0) := \inf_{(u,L)} \left\{ \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W(\nabla_\alpha u | L\nabla_3 u) dx : L > 0, u \in W^{1,p}(\Omega;\mathbb{R}^3), u_{|\partial\omega\times(-1,1)} = u_0 \right\}.$$

Then

$$Q^*_\omega W(u_0) = Q\overline{W}(\xi_0), \tag{1.3}$$

where $\overline{W}(\xi) := \min_{z\in\mathbb{R}^3} W(\xi|z)$, for all $\xi \in \mathbb{M}^{3\times 2}$, and $u_0(x_\alpha) = \xi_0 x_\alpha$ for some $\xi_0 \in \mathbb{M}^{3\times 2}$. Moreover, given any sequence $\{\varepsilon_n\}$ of positive real numbers converging to zero, there exist a subsequence (not relabelled) of $\{\varepsilon_n\}$, and a sequence $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that

(i) $\lim_{n\to\infty} \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W\left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right) dx = Q\overline{W}(\xi_0)$;

(ii) $\lim_{n\to\infty} \|u_n - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0$;

(iii) $u_{n_{|\partial\omega\times(-1,1)}} = u_0$;

(iv) $\left\{\left(\nabla_\alpha u_n \big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ *is p-equi-integrable.*

Note that in view of (2.5) below, the inequality

$$Q^*_\omega W(u_0) \le \frac{1}{\mathcal{L}^2(\omega)} \int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha)) dx_\alpha,$$

is valid for any $u_0 \in W^{1,p}(\omega;\mathbb{R}^3)$ not necessarily affine. However, in general the opposite inequality may fail. Indeed, as an example consider $W(\xi) := |\xi|^2$, let φ_0 be any function in $W^{1,2}(\omega)$ and set $u_0(x_\alpha) := (\varphi_0(x_\alpha), 0, 0)$. Here $Q\overline{W}(\nabla_\alpha u_0) = |\nabla_\alpha \varphi_0|^2$, and if

$$Q^*_\omega W(u_0) = \frac{1}{\mathcal{L}^2(\omega)} \int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha)) dx_\alpha$$

was satisfied then φ_0 would need to be harmonic in ω.

Finally, in the last part of the paper we prove that energy recovering sequences $\{(v_n, \varepsilon_n)\}$ (see Definition 2.3) may be modified on a small set (up to a subsequence) so as to obtain a new recovering sequence which depends only on x_α nearby $\partial\Omega$. We view this result as a first step towards matching recovering sequences to their limits u_0 nearby $\partial\Omega$. This strategy (two-step matching) was previously introduced by S. Conti, I. Fonseca and G. Leoni in the context of second-order phase transitions (see [13]). The ability to match recovering sequences $\{(v_n, \varepsilon_n)\}$ to their limiting configurations u_0, would enable us to periodize $(v_n - u_0)$ on a cylindrical cell and, as a consequence, the additional symmetry hypothesis $W(F_\alpha|F_3) = W(F_\alpha| - F_3)$ considered in, say [8], would no longer be needed.

Corollary 1.3 *Let $\Omega := \omega \times (-1,1)$, where $\omega \subset \mathbb{R}^2$ is an open, bounded Lipschitz domain. Let $u_0 \in W^{1,p}(\omega;\mathbb{R}^3)(1 < p < +\infty)$ be given and let $W : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying (1.1). Consider $\{(v_n, \varepsilon_n)\}$ to be a recovering sequence in the sense of Definition 2.3. Then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a sequence $\{w_k\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that*

(i) $\lim_{n\to\infty} \mathcal{L}^3(\{x \in \Omega : w_k(x) \neq v_{n_k}(x)\}) = 0$;

(ii) $\left\{\left(\nabla_\alpha w_k | \frac{1}{\varepsilon_{n_k}} \nabla_3 w_k\right)\right\}$ *is p-equi-integrable;*

(iii) $w_k = w_k(x_\alpha)$ *nearby* $x_3 = \pm 1$;

(iv) $\{(w_k, \varepsilon_{n_k})\}$ *is still a recovering sequence.*

2 Preliminaries

We first recall some well-known facts about maximal functions (see [28] for details). Given a Borel measurable function $u : \mathbb{R}^N \to \mathbb{R}^d$, the *maximal function* of u is defined by

$$M(u)(x) := \sup_{r>0} \frac{1}{\mathcal{L}^N(B(x,R))} \int\limits_{B(x,R)} |u(y)|dy.$$

If $u \in L^p(\mathbb{R}^N;\mathbb{R}^m)$ with $p > 1$, then $|u(x)| \leq M(u)(x)$ for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$, and

$$\|M(u)\|_{L^p(\mathbb{R}^N)} \leq C(N,p)\|u\|_{L^p(\mathbb{R}^N;\mathbb{R}^m)}. \tag{2.1}$$

We will make use of the following decomposition result:

Proposition 2.1 [17, Lemma 1.2] *Let $\Omega \in \mathbb{R}^N$ be an open and bounded set and let $\{w_n\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. There exists a subsequence, $\{w_{n_k}\}$ of $\{w_n\}$, and a sequence $\{z_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\lim_{k\to+\infty} \mathcal{L}^N\left(\{x \in \Omega : z_k(x) \neq w_{n_k}(x)\}\right) = 0,$$

and $\{|\nabla z_k|^p\}$ is equi-integrable. If Ω is Lipschitz (or, more generally, an extension domain), then each z_j may be chosen to be a Lipschitz function.

The following Poincaré-type inequality is well known (see, e.g., [35]). We include a proof for the convenience of the reader.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, Lipschitz domain. Then for every $\alpha > 0$ there exists $C(\alpha) > 0$ such that*

$$\int\limits_\Omega |u(x)|^p dx \leq C(\alpha) \int\limits_\Omega |\nabla u(x)|^p dx,$$

for all $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ satisfying

$$\mathcal{L}^N(\{x \in \Omega : u(x) = 0\}) \geq \alpha.$$

Proof. Without loss of generality, we may assume that $\mathcal{L}^N(\Omega) = 1$. If the result was false, then there would exist $\alpha > 0$ and a sequence $\{u_n\} \subset W^{1,p}(\Omega,\mathbb{R}^d)$ such that $\mathcal{L}^N(\{x \in \Omega : u_n(x) = 0\}) \geq \alpha$ and

$$\int\limits_\Omega |u_n(x)|^p dx > n \int\limits_\Omega |\nabla u_n(x)|^p dx. \tag{2.2}$$

Define

$$w_n := \frac{u_n}{\|u_n\|_{L^p(\Omega,\mathbb{R}^d)}},$$

and note that $\|w_n\|_{L^p(\Omega,\mathbb{R}^d)} = 1,\ \mathcal{L}^N(\{x \in \Omega : w_n(x) = 0\}) \geq \alpha$. By (2.2) we obtain that

$$\int_\Omega |\nabla w_n(x)|^p dx \to 0 \ \text{ as } n \to \infty,$$

and thus the sequence $\{w_n\}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^d)$. Up to a subsequence (not relabelled) we have $w_n \rightharpoonup w$ weakly in $W^{1,p}(\Omega,\mathbb{R}^d)$ and $w_n \to w$ strongly in $L^p(\Omega,\mathbb{R}^d)$, for some $w \in W^{1,p}(\Omega,\mathbb{R}^d)$. We deduce that $\nabla w = 0\ \mathcal{L}^N$ a.e. in Ω, and since $\|w\|_{L^p(\Omega,\mathbb{R}^d)} = 1$ we have that $w = c\ \mathcal{L}^N$ a.e. in Ω, where $|c| = 1$. Since $w_n \to w$ in measure, we have

$$\lim_{n\to\infty} \mathcal{L}^N\left(\left\{x \in \Omega : |w_n(x) - c| > \frac{1}{2}\right\}\right) = 0,$$

and thus

$$0 < \alpha \leq \liminf_{n\to\infty} \mathcal{L}^N(\{x \in \Omega : w_n(x) = 0\}) \leq \lim_{n\to\infty} \mathcal{L}^N\left(\left\{x \in \Omega : |w_n(x) - c| > \frac{1}{2}\right\}\right) = 0.$$

We have reached a contradiction. □

Let $W : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying (1.1). We define the two-dimensional energy density $\overline{W} : \mathbb{M}^{3\times 2} \to \mathbb{R}$, by

$$\overline{W}(\xi) := \min_{z\in\mathbb{R}^3} W(\xi|z),\ \xi \in \mathbb{M}^{3\times 2}. \tag{2.3}$$

The function $\overline{W}$ is continuous and satisfies growth and coercivity estimates similar to (1.1) (see [24]). We will denote by $Q\overline{W}$ the quasiconvex envelope of $\overline{W}$, precisely

$$Q\overline{W}(\eta) := \inf\left\{\frac{1}{\mathcal{L}^2(\omega)}\int_\omega \overline{W}(\eta + \nabla_\alpha\varphi(x_\alpha))dx_\alpha : \varphi \in W_0^{1,\infty}(\omega;\mathbb{R}^3)\right\},\ \eta \in \mathbb{M}^{3\times 2}, \tag{2.4}$$

where we have used the well known fact that quasiconvex envelope of $\overline{W}$ is independent of the domain of integration in (2.4) (see [14]).

Using Γ - convergence techniques, it has been shown by Le Dret and Raoult [24] (see also [8], [9], [11], [12], [15]) that $Q\overline{W}$ is the effective two-dimensional asymptotic thin film energy for the family $\{I_\varepsilon\}$. Indeed, if $u_0 \in W^{1,p}(\omega;\mathbb{R}^3),\ u_0 = u_0(x_\alpha)$, then

$$\int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha = \inf\left\{\liminf_{n\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)dx :\right.$$

$$\left.\varepsilon_n \to 0^+, u_n \in W^{1,p}(\Omega;\mathbb{R}^3),\ \lim_{n\to\infty}\|u_n - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0, u_{n|\partial\omega\times(-1,1)} = u_0\right\}. \tag{2.5}$$

Definition 2.3 *A sequence $\{(v_n,\varepsilon_n)\} \subset W^{1,p}(\Omega;\mathbb{R}^3)\times(0,+\infty)$ is called a recovering sequence if $\varepsilon_n \to 0^+,\ v_{n|\partial\omega\times(-1,1)} = u_0,\ \lim_{n\to\infty}\|v_n - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0$, and if*

$$\lim_{n\to\infty}\int_\Omega W\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)dx = 2\int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha.$$

Remark 2.4 Using the definition of the infimum in (2.5) and a standard diagonalization argument, it can be shown that a recovering sequence in the sense of the Definition 2.3 always exists.

3 Proof of Theorem 1.1

In what follows, and without loss of generality, we assume that $\varepsilon_n \in (0, 1/12)$. Also, $C > 0$ denotes a generic constant which may vary from line to line and expression to expression within the same formula.

Step 1: For every $n \in \mathbb{N}$, set $S_i^{(n)} := \omega \times ((2i-1)\varepsilon_n, (2i+1)\varepsilon_n)$, with $i \in [-L(n), L(n)] \cap \mathbb{Z}$, where the number $L(n) \in \mathbb{N}$ of layers is given by

$$L(n) := \begin{cases} \frac{1}{2\varepsilon_n}, & \text{if } \frac{1}{2\varepsilon_n} \in \mathbb{N}, \\ \left[\frac{1}{2\varepsilon_n}\right] + 1, & \text{otherwise.} \end{cases}$$

Note that

$$\frac{1}{\varepsilon_n} \leq 2L(n) \leq 2 + \frac{1}{\varepsilon_n} \tag{3.1}$$

Define $u_n : S_0^{(n)} \to \mathbb{R}^3$ as $u_n(x_\alpha, x_3) := v_n\left(x_\alpha, \frac{1}{\varepsilon_n} x_3\right)$. We have, changing variables,

$$\frac{1}{\varepsilon_n} \int\limits_{S_0^{(n)}} |\nabla u_n(x)|^p dx = \int\limits_{\Omega} \left| \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right)(y) \right|^p dy,$$

and thus,

$$\sup_{n \in \mathbb{N}} \frac{1}{\varepsilon_n} \int\limits_{S_0^{(n)}} |\nabla u_n(x)|^p dx < +\infty. \tag{3.2}$$

Define $F_i^{(n)} : S_i^{(n)} \to S_0^{(n)}$ by

$$F_i^{(n)}(x_\alpha, x_3) := (x_\alpha, (-1)^i x_3 + 2(-1)^{i+1} i \varepsilon_n),$$

set $\Omega_n := \omega \times (-(2L(n)+1)\varepsilon_n, (2L(n)+1)\varepsilon_n)$, and let $F_n : \Omega_n \to S_0^{(n)}$ be given by

$$F_n(x) := \sum_{i=-L(n)}^{L(n)} \chi_{S_i^{(n)}}(x) F_i^{(n)}(x) \text{ if } x \in \bigcup_{i=-L(n)}^{L(n)} S_i^{(n)},$$

where F_n is extended continuously to Ω_n. We consider now the vertical periodization of u_n, precisely $\bar{u}_n : \Omega_n \to \mathbb{R}^3$ defined as

$$\bar{u}_n(x) := u_n(F_n(x)).$$

Using the fact that for every $n \in \mathbb{N}$ and $i \in [-L(n), L(n)] \cap \mathbb{Z}$,

$$\nabla F_i^{(n)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^i \end{pmatrix} \text{ in } S_i^{(n)}, \tag{3.3}$$

by (3.1) we can estimate

$$\begin{aligned} \int\limits_{\Omega} |\nabla \bar{u}_n(x)|^p dx &= \sum_{i=-L(n)}^{L(n)} \int\limits_{S_i^{(n)} \cap \Omega} |\nabla u_n(F_i^{(n)}(x))|^p dx \\ &\leq \sum_{i=-L(n)}^{L(n)} \int\limits_{S_0^{(n)}} |\nabla u_n(y)|^p dy \leq \left(3 + \frac{1}{\varepsilon_n}\right) \int\limits_{S_0^{(n)}} |\nabla u_n(y)|^p dy. \end{aligned}$$

By virtue of (3.2),

$$\sup_{n\in\mathbb{N}}\int_\Omega |\nabla \bar{u}_n(x)|^p dx < +\infty.$$

A similar computation gives

$$\sup_{n\in\mathbb{N}}\int_\Omega |\bar{u}_n(x)|^p dx \le \frac{5}{4}\sup_{n\in\mathbb{N}}\int_\Omega |v_n(x)|^p dx < +\infty,$$

and thus,

$$\sup_{n\in\mathbb{N}} \|\bar{u}_n\|_{W^{1,p}(\Omega,\mathbb{R}^3)} < +\infty. \tag{3.4}$$

In view of Lemma 2.1, there exists a subsequence $\{\bar{u}_{n_k}\}$ of $\{\bar{u}_n\}$, and a sequence $\{z_k\} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that the sequence $\{|\nabla z_k|^p\}$ is equi-integrable and

$$\lim_{k\to+\infty} \mathcal{L}^3(E_k) = 0,$$

where

$$E_k := \{x \in \Omega : z_k(x) \neq \bar{u}_{n_k}(x)\}.$$

Define $\Lambda : (0,t) \to (0,+\infty)$ by

$$\Lambda(t) := \sup\left\{\int_A |\nabla z_m(x)|^p dx : m \in \mathbb{N}, A \in \mathcal{B}(\Omega), \mathcal{L}^3(A) \le t\right\},$$

where $\mathcal{B}(\Omega)$ denotes the class of Borel subsets of Ω. We note that the equi-integrability property of $\{|\nabla z_k|^p\}$ ensures that

$$\lim_{t\to 0^+} \Lambda(t) = 0. \tag{3.5}$$

Step 2: Set $I := [-L(n_k), L(n_k)] \cap \mathbb{Z}$ and $I_\star := [-L(n_k)+2, L(n_k)-2] \cap \mathbb{Z}$. Note that if $i \in I_\star$ then $S_i^{(n_k)} \subset \Omega$. By (3.1) we have

$$\frac{1}{\varepsilon_{n_k}} - 3 \le \ \mathrm{card} I_\star \le \frac{1}{\varepsilon_{n_k}} - 1. \tag{3.6}$$

If for an infinite number of indices k (not relabelled), we have that $\mathcal{L}^3(E_k) = 0$, then given any Borel subset A of Ω and setting $A_k := \{y \in \Omega : (y_\alpha, y_3) = (x_\alpha, \varepsilon_{n_k} x_3),\ x \in A\} \subset S_0^{(n_k)}$ we can estimate

$$\begin{aligned}
\int_A \left|\left(\nabla_\alpha v_{n_k} \Big| \frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}\right)\right|^p \ dx &= \frac{1}{\varepsilon_{n_k}}\int_{A_k} |\nabla u_{n_k}|^p \ dx \le 2\sum_{i\in I_\star} \int_{\left(F_i^{(n_k)}\right)^{-1}(A_k)} |\nabla \bar{u}_{n_k}|^p \ dx \\
&= 2 \int_{\bigcup_{i\in I_\star}\left(F_i^{(n_k)}\right)^{-1}(A_k)} |\nabla z_k|^p \ dx \\
&\le 2\Lambda(\mathcal{L}^3(A)),
\end{aligned}$$

where we have used (3.3), (3.6), and the fact that $\mathcal{L}^3\left(\bigcup_{i\in I_\star}\left(F_i^{(n_k)}\right)^{-1}(A_k)\right) \le \mathcal{L}^3(A)$. In this case we conclude that $w_k := v_{n_k}$ satisfies (i)-(ii).

Consider now the case where $\mathcal{L}^3(E_k) > 0$ except maybe for a finite number of indices k. In what follows we will assume, without loss of generality, that $\mathcal{L}^3(E_k) > 0$ for all $k \in \mathbb{N}$. Define

$$I_k := \left\{ i \in I_\star : \mathcal{L}^3\left(E_k \cap S_i^{(n_k)}\right) \leq 2\varepsilon_{n_k}\mathcal{L}^3(E_k) \right\}.$$

We have

$$\mathcal{L}^3(E_k) \geq \mathcal{L}^3\left(\bigcup_{i\in I_\star}\left(E_k \cap S_i^{(n_k)}\right)\right) \geq \sum_{i\in I_\star\setminus I_k} \mathcal{L}^3\left(E_k \cap S_i^{(n_k)}\right) \geq 2\varepsilon_{n_k}\mathcal{L}^3(E_k)\,(2L(n_k) - 3 - \text{ card } I_k),$$

and since $\mathcal{L}^3(E_k) > 0$, by (3.1) we find

$$\text{card } I_k \geq \frac{1}{2\varepsilon_{n_k}} - 3.$$

Without loss of generality, we may assume that all indices $i \in I_k$ are even and we replace now the above lower bound for card I_k by

$$\text{card } I_k \geq \frac{1}{2}\left(\frac{1}{2\varepsilon_{n_k}} - 3\right). \tag{3.7}$$

For each $i \in I_k$ define $G^{(i,k)} : \Omega_{n_k} \to S_i^{(n_k)}$ by

$$G^{(i,k)}(x) := \sum_{j\in I} \chi_{S_j^{(n_k)}}(x) G_j^{(i,k)}(x) \text{ if } x \in \bigcup_{j\in I} S_j^{(n_k)},$$

where $G_j^{(i,k)} : S_j^{(n_k)} \to S_i^{(n_k)}$ is given by

$$G_j^{(i,k)}(x) := \left(\left(F_i^{(n_k)}\right)^{-1} \circ F_j^{(n_k)}\right)(x)$$

and $G^{(i,k)}$ is extended to the whole Ω_{n_k} by continuity.

Also, for each $i \in I_k$ set

$$X^{(i,k)} := \left\{ x \in \Omega : z_k(x) \neq z_k\left(G^{(i,k)}(x)\right) \right\}.$$

Claim 1: *For each $k \in \mathbb{N}$, there exists an index $i_0 \in I_k$ and a nonnegative real number a_k such that*

$$\sum_{j\in I_k} \int_{G_j^{(i_0,k)}\left(X^{(i_0,k)}\cap S_j^{(n_k)}\right)} |\nabla z_k(x)|^p dx \leq a_k, \text{ and } \lim_{k\to+\infty} a_k = 0. \tag{3.8}$$

We postpone the proof of Claim 1 until after Step 3 is completed.

Step 3: Assuming that the Claim 1 holds, define $w_k : \Omega \to \mathbb{R}^3$ by

$$w_k(x) := z_k\left(\left(F_{i_0}^{(n_k)}\right)^{-1}(x_\alpha, \varepsilon_{n_k} x_3)\right),$$

for each $k \in \mathbb{N}$, where the index $i_0 \in I_k$ has been selected according to (3.8). We will first show that this sequence satisfies (i) and (ii).

For every $x, y \in \Omega$ we have

$$\begin{aligned} |w_k(x) - w_k(y)| &= \left| z_k \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (x_\alpha, \varepsilon_{n_k} x_3) \right) - z_k \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (y_\alpha, \varepsilon_{n_k} y_3) \right) \right| \\ &\leq \mathrm{Lip}(z_k) \left| \left(F_{i_0}^{(n_k)} \right)^{-1} (x_\alpha, \varepsilon_{n_k} x_3) - \left(F_{i_0}^{(n_k)} \right)^{-1} (y_\alpha, \varepsilon_{n_k} y_3) \right| \\ &\leq \mathrm{Lip}(z_k) |x - y|, \end{aligned}$$

and thus w_k is still a Lipschitz function and we have

$$\mathrm{Lip}(w_k) \leq \mathrm{Lip}(z_k). \tag{3.9}$$

To prove (i), define

$$M_k := \{x \in \Omega : w_k(x) \neq v_{n_k}(x)\}. \tag{3.10}$$

We have

$$\begin{aligned} \mathcal{L}^3(M_k) &= \mathcal{L}^3(\{x \in \Omega : w_k(x_\alpha, x_3) \neq u_{n_k}(x_\alpha, \varepsilon_{n_k} x_3)\}) \\ &= \mathcal{L}^3 \left(\left\{ x \in \Omega : z_k \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (x_\alpha, \varepsilon_{n_k} x_3) \right) \neq \bar{u}_{n_k} \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (x_\alpha, \varepsilon_{n_k} x_3) \right) \right\} \right) \\ &= \frac{1}{\varepsilon_{n_k}} \mathcal{L}^3 \left(\left\{ y \in S_0^{(n_k)} : z_k \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (y) \right) \neq \bar{u}_{n_k} \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (y) \right) \right\} \right) \\ &= \frac{1}{\varepsilon_{n_k}} \mathcal{L}^3 \left(\left\{ y \in S_0^{(n_k)} : \left(F_{i_0}^{(n_k)} \right)^{-1} (y) \in E_k \right\} \right) = \frac{1}{\varepsilon_{n_k}} \mathcal{L}^3 \left(\left(F_{i_0}^{(n_k)} \right)^{-1} (S_0^{(n_k)}) \cap E_k \right) \\ &= \frac{1}{\varepsilon_{n_k}} \mathcal{L}^3 \left(S_{i_0}^{(n_k)} \cap E_k \right) \leq 2\mathcal{L}^3(E_k) \to 0 \text{ as } k \to \infty, \end{aligned} \tag{3.11}$$

where we have used the fact that $i_0 \in I_k \subset I_\star$.

In order to prove (ii), i.e. the equi-integrability of $\left\{ \left| \left(\nabla_\alpha w_k \middle| \frac{1}{\varepsilon_{n_k}} \nabla_3 w_k \right) \right|^p \right\}$, let A be any Borel subset of Ω, and set

$$A_k := \left\{ y \in S_0^{(n_k)} : y_\alpha = x_\alpha, y_3 = \varepsilon_{n_k} x_3, x \in A \right\}$$

and

$$\bar{A}_k := \left(F_{i_0}^{(n_k)} \right)^{-1} (A_k).$$

Changing variables, by (3.3) and (3.7) we have

$$\begin{aligned} \int_A \left| \left(\nabla_\alpha w_k \middle| \frac{1}{\varepsilon_{n_k}} \nabla_3 w_k \right) (x_\alpha, x_3) \right|^p dx &= \frac{1}{\varepsilon_{n_k}} \int_{\bar{A}_k} |\nabla z_k(x)|^p \, dx \leq 8 \sum_{j \in I_k} \int_{\bar{A}_k} |\nabla z_k(x)|^p dx \\ &= 8 \sum_{j \in I_k} \int_{\left(G_j^{(i_0,k)} \right)^{-1} (\bar{A}_k)} \left| \nabla z_k \left(G_j^{(i_0,k)}(y) \right) \right|^p dy. \end{aligned}$$

Splitting the last integral into a sum of two integrals over $\left(G_j^{(i_0,k)} \right)^{-1} (\bar{A}_k) \cap \left(\Omega \setminus (X^{(i_0,k)} \cap S_j^{(n_k)}) \right)$ and $\left(G_j^{(i_0,k)} \right)^{-1} (\bar{A}_k) \cap \left(X^{(i_0,k)} \cap S_j^{(n_k)} \right)$ respectively, using the definition of $X^{(i_0,k)}$ and the fact

that

$$\mathcal{L}^3\left(\bigcup_{j\in I_k}\left(G_j^{(i_0,k)}\right)^{-1}(\bar{A}_k)\right)\leq(\text{card }\ I_k)\varepsilon_{n_k}\mathcal{L}^3(A)\leq\mathcal{L}^3(A)$$

we have

$$\begin{aligned}\sum_{j\in I_k}\int_{\left(G_j^{(i_0,k)}\right)^{-1}(\bar{A}_k)\cap\left(\Omega\setminus(X^{(i_0,k)}\cap S_j^{(n_k)})\right)}\left|\nabla z_k\left(G_j^{(i_0,k)}(y)\right)\right|^p dy &\leq \int_{\bigcup_{j\in I_k}\left(G_j^{(i_0,k)}\right)^{-1}(\bar{A}_k)}|\nabla z_k(y)|^p dy\\ &\leq \Lambda(\mathcal{L}^3(A)),\end{aligned}$$

and in view of (3.8) we obtain

$$\int_A\left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)(x_\alpha,x_3)\right|^p dx\leq 8\left(\Lambda(\mathcal{L}^3(A))+a_k\right).$$

The equi-integrability of $\left\{\left|\left(\nabla_\alpha w_k\big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p\right\}$ now follows from (3.5).

Step 4 (Proof of Claim 1). The selection of $i_0\in I_k$ will rest on De Giorgi's slicing argument to identify a layer of small energy concentration via an averaging process. We start by proving that for each $k\in\mathbb{N}$ and $i,j\in I_k$,

$$X^{(i,k)}\cap S_j^{(n_k)}\subset\left(E_k\cap S_j^{(n_k)}\right)\cup\left(\left(G_j^{(i,k)}\right)^{-1}\left(E_k\cap S_i^{(n_k)}\right)\right).\tag{3.12}$$

Indeed, if $x\in\left(X^{(i,k)}\cap S_j^{(n_k)}\right)\setminus\left(E_k\cap S_j^{(n_k)}\right)$, then

$$\bar{u}_{n_k}(x)=z_k(x)\neq z_k(G_j^{(i,k)}(x)),$$

and

$$\bar{u}_{n_k}(x)=\bar{u}_{n_k}(G_j^{(i,k)}(x)),$$

where we have used the vertical periodicity of $\bar{u}_{n_k}$ and the fact that $i,j\in I_k$ are even. Thus

$$\bar{u}_{n_k}(G_j^{(i,k)}(x))\neq z_k(G_j^{(i,k)}(x)),\ \text{i.e. } G_j^{(i,k)}(x)\in E_k\cap S_i^{(n_k)},$$

and we conclude that

$$\left(X^{(i,k)}\cap S_j^{(n_k)}\right)\setminus\left(E_k\cap S_j^{(n_k)}\right)\subset\left(G_j^{(i,k)}\right)^{-1}\left(E_k\cap S_i^{(n_k)}\right),$$

thus proving (3.12). By the definition of I_k, and in view of (3.3), we have

$$\mathcal{L}^3\left(\bigcup_{i\in I_k}G_j^{(i,k)}\left(E_k\cap S_j^{(n_k)}\right)\right)\leq\sum_{i\in I_k}\mathcal{L}^3\left(E_k\cap S_j^{(n_k)}\right)\leq 2\mathcal{L}^3(E_k)\to 0\text{ as }k\to+\infty,$$

and

$$\mathcal{L}^3\left(\bigcup_{i\in I_k}\left(E_k\cap S_i^{(n_k)}\right)\right)\leq 2\mathcal{L}^3(E_k)\to 0\text{ as }k\to+\infty.$$

We obtain

$$\begin{aligned}
&\sum_{i\in I_k}\sum_{j\in I_k}\int\limits_{G_j^{(i,k)}\left(X^{(i,k)}\cap S_j^{(n_k)}\right)}|\nabla z_k(x)|^p dx\\
&\leq \sum_{i\in I_k}\sum_{j\in I_k}\int\limits_{G_j^{(i,k)}\left(E_k\cap S_j^{(n_k)}\right)\cup\left(E_k\cap S_i^{(n_k)}\right)}|\nabla z_k(x)|^p dx\\
&\leq \sum_{j\in I_k}\left(\sum_{i\in I_k}\int\limits_{G_j^{(i,k)}\left(E_k\cap S_j^{(n_k)}\right)}|\nabla z_k(x)|^p dx\right)+\sum_{j\in I_k}\sum_{i\in I_k}\int\limits_{E_k\cap S_i^{(n_k)}}|\nabla z_k(x)|^p dx\\
&\leq \sum_{j\in I_k}\int\limits_{\bigcup\limits_{i\in I_k}G_j^{(i,k)}\left(E_k\cap S_j^{(n_k)}\right)}|\nabla z_k(x)|^p dx+\frac{1}{\varepsilon_{n_k}}\int\limits_{\bigcup\limits_{i\in I_k}\left(E_k\cap S_i^{(n_k)}\right)}|\nabla z_k(x)|^p dx\\
&\leq \frac{2}{\varepsilon_{n_k}}\Lambda(2\mathcal{L}^3(E_k)) \qquad (3.13)
\end{aligned}$$

Define $a_k := 16\Lambda(2\mathcal{L}^3(E_k))$. In view of (3.13), we deduce the existence of an index $i_0 \in I_k$ that satisfies (3.8). Note that by (3.5) and since $\mathcal{L}^3(E_k) \to 0$ as $k \to +\infty$, we have that $a_k \to 0$ as $k \to +\infty$, thus proving our Claim 1.

Step 5: The remaining of the proof is dedicated to establishing (iii) and (iv). By (ii) it follows that

$$\{\nabla w_k\} \text{ is bounded in } L^p(\Omega,\mathbb{R}^9), \qquad (3.14)$$

so in order to ensure the weak convergence of (a subsequence of) $\{w_k\}$ in $W^{1,p}(\Omega,\mathbb{R}^3)$ we first need to prove that

$$\sup_{k\in\mathbb{N}}\int\limits_{\Omega}|w_k(x)|^p dx < +\infty. \qquad (3.15)$$

Since $\mathcal{L}^3(M_k) \to 0$ as $k \to \infty$ (see (3.11)), there exists $k_1 \in \mathbb{N}$ such that

$$\mathcal{L}^3(\{x\in\Omega : (w_k - v_{n_k})(x) = 0\}) \geq \frac{\mathcal{L}^3(\Omega)}{2},$$

for all $k \geq k_1$. By virtue of Lemma 2.2 there exists a constant $C(\Omega) > 0$ such that

$$\int\limits_{\Omega}|w_k(x) - v_{n_k}(x)|^p dx \leq C(\Omega)\int\limits_{\Omega}|\nabla w_k(x) - \nabla v_{n_k}(x)|^p dx \text{ for all } \ k \geq k_1,$$

and thus, using (3.14) and the fact that $\{v_n\}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^3)$ we obtain (3.15).

Since the sequence $\{w_k\}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$, up to a subsequence (not relabelled) it converges weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ to some function $w \in W^{1,p}(\Omega;\mathbb{R}^3)$, and (i) entails (iii), i.e. $w = v$ $\mathcal{L}^3$ a.e. $x \in \Omega$ (note also that the full sequence $\{w_k\}$ must converge weakly to v).

It remains to prove (iv). In what follows we will denote by $[a]_\beta,\ \beta \in \{1,2,3\}$ the components of $a \in \mathbb{R}^3$. Let $\varphi \in L^{p'}(\Omega)$ be given.

We have

$$\begin{aligned}\int_\Omega \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x)\right]_\beta \varphi(x)dx &= \int_{\Omega\setminus M_k} \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta \varphi(x)dx \\ &\quad + \int_{M_k} \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x)\right]_\beta \varphi(x)dx \\ &= \int_\Omega \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta \varphi(x)dx \\ &\quad + \int_{M_k} \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x) - \frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta \varphi(x)dx.\end{aligned}$$

Since $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup b$ weakly in $L^p(\Omega,\mathbb{R}^3)$, we have

$$\int_\Omega \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta \varphi(x)dx \to \int_\Omega [b(x)]_\beta \varphi(x)dx \ \text{ as } j\to\infty.$$

On the other hand, by (ii) the sequence $\left\{\left|\left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right]_\beta\right|^p\right\}$ is equi-integrable, hence bounded in $L^1(\Omega)$, and by Hölder's inequality we obtain

$$\begin{aligned}&\left|\int_{M_k} \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x) - \frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta \varphi(x)dx\right| \\ &\le \left(\int_{M_k} \left|\left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x) - \frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta\right|^p (x)dx\right)^{1/p} \left(\int_{M_k} |\varphi(x)|^{p'}dx\right)^{1/p'} \\ &\le \left\|\left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x) - \frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}(x)\right]_\beta\right\|_{L^p(\Omega)} \left(\int_{M_k} |\varphi(x)|^{p'}dx\right)^{1/p'} \to 0 \text{ as } j\to\infty,\end{aligned}$$

where we have used (3.11). We conclude that

$$\int_\Omega \left[\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k(x)\right]_\beta \varphi(x)dx \to \int_\Omega [b(x)]_\beta \varphi(x)dx \ \text{ as } j\to\infty$$

for every $\varphi\in L^{p'}(\Omega)$ and each $\beta\in\{1,2,3\}$, and the proof of (iv) is complete. □

It is possible to prove the relative weak compactness of the sequence $\{w_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ without making use of Lemma 2.2 (see Step 5 above). We will use a different approach, following closely the construction of the sequence $\{z_k\}$ in Step 1. To this end, we recall a proof of Proposition 2.1 hinged on the two propositions below.

Proposition 3.1 [16, Corollary 3.3.3] *Let $\Omega\in\mathbb{R}^N$ be an open and bounded set and let $\{u_n\}$ be a sequence of functions uniformly bounded in $L^1(\Omega)$. For $\lambda>0$ consider the truncation $\tau_\lambda:\mathbb{R}\to\mathbb{R}$ defined by*

$$\tau_\lambda(z) := \begin{cases} z & \text{if } |z|\le\lambda, \\ \frac{z}{|z|}\lambda & \text{if } |z|>\lambda. \end{cases}$$

Then there exists a subsequence of $\{u_n\}$ (not relabelled) and an increasing sequence of numbers $\lambda_n \to +\infty$ such that the truncated sequence $\{\tau_{\lambda_n} \circ u_n\}$ is equi-integrable and

$$\lim_{n\to\infty} \mathcal{L}^N \left(\{x \in \Omega : u_n(x) \neq (\tau_{\lambda_n} \circ u_n)(x)\}\right) = 0.$$

Proposition 3.2 (see [1], [2], [26]) *Let $p > 1$ and let $w \in W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$. Given $\lambda > 0$ there exists a Lipschitz function z in $\mathbb{R}^N$ such that $w = z$ $\mathcal{L}^N$ a.e. on $\{x \in \mathbb{R}^N : M(\nabla w)(x) < \lambda\}$ and the Lipschitz constant of z is bounded by $C(N)\lambda$, where $C(N)$ is a constant depending only upon dimension.*

We now turn to the construction of the sequence $\{z_k\}$. As Ω is a bounded, Lipschitz domain we can extend the functions $\bar{u}_n$ (see Step 1) to $\mathbb{R}^3 \setminus \Omega$ in such a way that the extensions $U_n \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ satisfy

$$\|U_n\|_{W^{1,p}(\mathbb{R}^3,\mathbb{R}^3)} \leq C\|\bar{u}_n\|_{W^{1,p}(\Omega,\mathbb{R}^3)},$$

for some $C > 0$. Therefore, by (3.4) and (2.1) we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^3} |M(\nabla U_n)|^p dx < +\infty. \tag{3.16}$$

Since the sequence $\{|M(\nabla U_n)|^p\}$ is uniformly bounded in $L^1(\Omega)$, Proposition 3.1 now guarantees the existence of a subsequence $\{|M(\nabla U_{n_k})|^p\}$ of $\{|M(\nabla U_n)|^p\}$ and of an increasing sequence $\{\lambda_k\}$, with $\lambda_k \nearrow +\infty$, such that the truncated sequence $\{\tau_{\lambda_k} \circ |M(\nabla U_{n_k})|^p\}$ is equi-integrable and

$$\lim_{k\to\infty} \mathcal{L}^3 \left(\{x \in \Omega : |M(\nabla U_{n_k})|^p(x) \neq (\tau_{\lambda_k} \circ |M(\nabla U_{n_k})|^p)(x)\}\right) = 0.$$

By Proposition 3.2, for each $k \in \mathbb{N}$ there exists $z_k \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^3)$ such that $z_k = U_{n_k}$ $\mathcal{L}^3$ a.e. on $\mathbb{R}^3 \setminus R_k$ and $\mathrm{Lip}(z_k) \leq C\lambda_k^{\frac{1}{p}}$ (and thus $|\nabla z_k(x)| \leq C\lambda_k^{\frac{1}{p}}$ for $\mathcal{L}^3$ a.e. $x \in \mathbb{R}^3$), where

$$R_k := \{x \in \mathbb{R}^3 : M(\nabla U_{n_k})(x) \geq \lambda_k^{\frac{1}{p}}\},$$

and $C > 0$ is a constant. Note that

$$E_k = \{x \in \Omega : z_k(x) \neq U_{n_k}(x)\},$$

where the set E_k has been defined in Step 1. The definition of R_k and (3.16) imply that

$$\mathcal{L}^3(E_k) \leq \mathcal{L}^3(R_k \cap \Omega) \leq \frac{1}{\lambda_k} \int_\Omega |M(\nabla U_{n_k})(x)|^p dx \to 0 \ \text{ as } k \to \infty. \tag{3.17}$$

To prove that the sequence $\{|\nabla z_k|^p\}$ is equi-integrable, note that for $\mathcal{L}^3$ a.e. $x \in \Omega \setminus R_k$ we have

$$\begin{aligned} |\nabla z_k(x)|^p &= |\nabla U_{n_k}(x)|^p \leq |M\left(\nabla U_{n_k}\right)(x)|^p \\ &= \left(\tau_{\lambda_k} \circ |M\left(\nabla U_{n_k}\right)|^p\right)(x), \end{aligned}$$

while for $\mathcal{L}^3$ a.e. $x \in \Omega \cap R_k$,

$$|\nabla z_k(x)|^p \leq C^p \lambda_k = C^p \left(\tau_{\lambda_k} \circ |M\left(\nabla U_{n_k}\right)|^p\right)(x).$$

Thus,

$$|\nabla z_k(x)|^p \leq \max\{1, C^p\} \left(\tau_{\lambda_k} \circ |M\left(\nabla U_{n_k}\right)|^p\right)(x) \text{ for } \mathcal{L}^3 \text{ a.e. } x \in \Omega,$$

and the equi-integrability of $\{|\nabla z_k|^p\}$ follows from the equi-integrability of $\{\tau_{\lambda_k} \circ |M(\nabla U_{n_k})|^p\}$. For each $k \in \mathbb{N}$, define the Lipschitz function w_k and the set M_k as in Step 3.

To assert the relative weak compactness of $\{w_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ we first need to prove that

$$\sup_{j\in\mathbb{N}} \int_\Omega |w_{k_j}(x)|^p dx < +\infty. \tag{3.18}$$

for some subsequence $\{w_{k_j}\}$ of $\{w_k\}$. To this end, we claim that there exists a subsequence $\{M_{k_j}\}$ of $\{M_k\}$ such that

$$\mathcal{L}^3\left(\bigcap_{j\in\mathbb{N}} (\Omega \setminus M_{k_j})\right) > 0, \tag{3.19}$$

where the sets M_k have been introduced in (3.10).

Since $\mathcal{L}^3(M_k) \to 0$ as $k \to \infty$ (see (3.11) and (3.17)), there exists $k_1 \in \mathbb{N}$ such that $\mathcal{L}^3(\Omega \setminus M_k) > \frac{\mathcal{L}^3(\Omega)}{2}$ for all $k \geq k_1$, and thus

$$\mathcal{L}^3(\Omega \setminus M_{k_1}) - \mathcal{L}^3(M_{k_1}) > 0. \tag{3.20}$$

Let $k_2 > k_1$ be such that $\mathcal{L}^3(M_{k_2}) < \frac{\mathcal{L}^3(M_{k_1})}{2}$. Since $\Omega \setminus M_{k_1} \subset M_{k_2} \cup ((\Omega \setminus M_{k_1}) \cap (\Omega \setminus M_{k_2}))$, we have

$$\mathcal{L}^3((\Omega \setminus M_{k_1}) \cap (\Omega \setminus M_{k_2})) \geq \mathcal{L}^3(\Omega \setminus M_{k_1}) - \frac{\mathcal{L}^3(M_{k_1})}{2}.$$

Inductively, we construct a sequence $k_j \to \infty$ such that

$$\mathcal{L}^3\left(\bigcap_{i=1}^{j} (\Omega \setminus M_{k_i})\right) \geq \mathcal{L}^3(\Omega \setminus M_{k_1}) - \mathcal{L}^3(M_{k_1}) \sum_{i=1}^{j} \frac{1}{2^i},$$

and letting $j \to \infty$ we obtain

$$\mathcal{L}^3\left(\bigcap_{i=1}^{\infty} (\Omega \setminus M_{k_i})\right) \geq \mathcal{L}^3(\Omega \setminus M_{k_1}) - \mathcal{L}^3(M_{k_1}),$$

which, together with (3.20), establishes (3.19).

Since $v_{n_{k_j}} \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, there exists a subsequence (not relabelled) such that $v_{n_{k_j}}(x) \to v(x)$ $\mathcal{L}^3$ a.e. $x \in \Omega$. Set $N := \{x \in \Omega : v_{n_{k_j}}(x) \nrightarrow v(x)\}$. By virtue of (3.19) and since $\mathcal{L}^3(N) = 0$ we deduce that

$$\left(\bigcap_{j=1}^{\infty} (\Omega \setminus M_{k_j})\right) \cap (\Omega \setminus N) \neq \emptyset,$$

and we choose an element of this intersection, say x_0. Note that

$$w_{k_j}(x_0) = v_{n_{k_j}}(x_0) \to v(x_0) \text{ as } j \to \infty.$$

Since w_{k_j} is a Lipschitz function and $\text{Lip}\,(w_{k_j}) \leq C\lambda_{k_j}^{1/p}$ (see (3.9)), we have

$$|w_{k_j}(x)| \leq |w_{k_j}(x_0)| + C\lambda_{k_j}^{1/p}|x - x_0| \text{ for all } x \in \Omega,$$

and thus

$$\begin{aligned}\int\limits_{M_{k_j}} |w_{k_j}(x)|^p dx &\leq 2^{p-1}|w_{k_j}(x_0)|^p \mathcal{L}^3\left(M_{k_j}\right) + 2^{p-1}C^p\lambda_{k_j}\int\limits_{M_{k_j}} |x-x_0|^p dx \\ &\leq 2^{p-1}|w_{k_j}(x_0)|^p \mathcal{L}^3(M_{k_j}) + C\lambda_{k_j}\mathcal{L}^3\left(M_{k_j}\right) \\ &\leq 2^p|w_{k_j}(x_0)|^p \mathcal{L}^3(E_{k_j}) + C\int\limits_{\Omega}\left|M\left(\nabla U_{n_{k_j}}\right)(x)\right|^p dx, \end{aligned} \tag{3.21}$$

where we have used (3.11) and (3.17). By the definition of M_{k_j}, we have

$$\int\limits_{\Omega} |w_{k_j}(x)|^p dx = \int\limits_{\Omega\setminus M_{k_j}} |v_{n_{k_j}}(x)|^p dx + \int\limits_{M_{k_j}} |w_{k_j}(x)|^p dx \leq \|v_{n_{k_j}}\|_{L^p(\Omega,\mathbb{R}^3)} + \int\limits_{M_{k_j}} |w_{k_j}(x)|^p dx,$$

which, together with (3.16) and (3.21), asserts (3.18). Thus, taking into account (3.14), the sequence $\{w_{k_j}\}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$ and therefore $\{w_k\}$ is relatively weakly compact in $W^{1,p}(\Omega;\mathbb{R}^3)$.

4 Proof of Corollary 1.2

To show that $Q^*_\omega W(u_0) = Q\overline{W}(\xi_0)$, let (u, L) be admissible for $Q^*_\omega W(u_0)$. Using a density argument and in view of (1.1), without loss of generality we may assume that $u \in W^{1,p}(\Omega;\mathbb{R}^3) \cap C^\infty(\overline{\Omega};\mathbb{R}^3)$. The very definition of $\overline{W}$ (see (2.3)), (2.4), and Fubini's Theorem yield

$$\frac{1}{\mathcal{L}^3(\Omega)}\int\limits_{\Omega} W(\nabla_\alpha u|L\nabla_3 u)dx \geq Q\overline{W}(\xi_0),$$

and thus,

$$Q^*_\omega W(u_0) \geq Q\overline{W}(\xi_0). \tag{4.1}$$

Conversely, let us fix $\delta > 0$. In view of (2.5), there exist $\varepsilon^\delta_n \to 0^+$ and $\{u^\delta_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$, with $u^\delta_{n|\partial\omega\times(-1,1)} = u_0$ and $u^\delta_n \to u_0$ strongly in $L^p(\Omega;\mathbb{R}^3)$, such that

$$\begin{aligned} Q\overline{W}(\xi_0) + \delta &\geq \liminf_{n\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int\limits_{\Omega} W\left(\nabla_\alpha u^\delta_n|\frac{1}{\varepsilon^\delta_n}\nabla_3 u^\delta_n\right)dx \\ &\geq Q^*_\omega W(u_0), \end{aligned}$$

and letting $\delta \to 0^+$ we obtain

$$Q\overline{W}(\xi_0) \geq Q^*_\omega W(u_0).$$

This, together with (4.1), asserts (1.3).

Step 1: We will first treat the particular case where $\omega = Q'_c := (-c/2, c/2)^2$, with $c > 0$ fixed. Let $\{(v_n, L_n)\} \subset W^{1,p}(\Omega;\mathbb{R}^3)\times(0,+\infty)$ be an infimizing sequence for $Q^*_\omega W(u_0)$, and extend $v_n - u_0$ by periodicity to $\mathbb{R}^2 \times (-1,1)$. Now let $\{\varepsilon_n\}$ be any sequence of positive numbers converging to zero. Define $v_{n,k} : \Omega \to \mathbb{R}^3$ by

$$v_{n,k}(x) := \varepsilon_k L_n(v_n - u_0)\left((\varepsilon_k L_n)^{-1}x_\alpha, x_3\right).$$

By the Riemann-Lebesgue lemma, we have that

$$\lim_{n\to\infty}\lim_{k\to\infty}\|v_{n,k}\|_{L^p(\Omega;\mathbb{R}^3)}=0,$$

and

$$\lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha v_{n,k}\Big|\frac{1}{\varepsilon_k}\nabla_3 v_{n,k}\right)dx=Q^*_\omega W(u_0).$$

Using a diagonalization argument, we can construct an increasing sequence $\{k(n)\}\subseteqq\mathbb{N}$, with $k(n)\nearrow+\infty$ as $n\to+\infty$, such that

$$\lim_{n\to\infty}\|\widetilde{v}_n\|_{L^p(\Omega;\mathbb{R}^3)}=0, \tag{4.2}$$

and

$$\lim_{n\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha \widetilde{v}_n\Big|\frac{1}{\widetilde{\varepsilon}_n}\nabla_3\widetilde{v}_n\right)dx=Q^*_\omega W(u_0), \tag{4.3}$$

where $\widetilde{v}_n:=v_{n,k(n)}$ and $\widetilde{\varepsilon}_n:=\varepsilon_{k(n)}$. Thus, in view of (1.1),

$$\sup_{n\in\mathbb{N}}\|\widetilde{v}_n\|_{W^{1,p}(\Omega,\mathbb{R}^3)}<+\infty,$$

and up to a subsequence (not relabelled), by (4.2) we have that $\widetilde{v}_n\rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$. By Theorem 1.1 we deduce that there exist a subsequence $\{\widetilde{v}_{n_k}\}$ of $\{\widetilde{v}_n\}$ and a sequence $\{w_k\}\subset W^{1,\infty}(\Omega;\mathbb{R}^3)$ such that

$$\lim_{k\to\infty}\mathcal{L}^3(\widetilde{M}_k)=0, \tag{4.4}$$

with $\widetilde{M}_k:=\{x\in\Omega:\widetilde{v}_{n_k}(x)\neq w_k(x)\}$,

$$\lim_{k\to\infty}\|w_k\|_{L^p(\Omega;\mathbb{R}^3)}=0, \tag{4.5}$$

and

$$\left\{\left(\nabla_\alpha w_k\Big|\frac{1}{\widetilde{\varepsilon}_{n_k}}\nabla_3 w_k\right)\right\}\quad\text{is } p-\text{equi}-\text{integrable}. \tag{4.6}$$

By (1.1) we have that

$$\begin{aligned}\int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha w_k\Big|\frac{1}{\widetilde{\varepsilon}_{n_k}}\nabla_3 w_k\right)dx\quad\leq\quad&\int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha \widetilde{v}_{n_k}\Big|\frac{1}{\widetilde{\varepsilon}_{n_k}}\nabla_3\widetilde{v}_{n_k}\right)dx\\&+\int_{\widetilde{M}_k}W\left(\nabla_\alpha u_0+\nabla_\alpha w_k\Big|\frac{1}{\widetilde{\varepsilon}_{n_k}}\nabla_3 w_k\right)dx+C\mathcal{L}^3(\widetilde{M}_k),\end{aligned}$$

and thus, in view of (4.3), (4.4) and (4.6),

$$\limsup_{k\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha w_k\Big|\frac{1}{\widetilde{\varepsilon}_{n_k}}\nabla_3 w_k\right)dx\leq Q^*_\omega W(u_0). \tag{4.7}$$

For each $j\in\mathbb{N}$ define $\omega_j:=\left\{x\in\omega:\operatorname{dist}(x,\partial\omega)<\frac{1}{j}\right\}$, and consider cut-off functions $\theta_j\in C^\infty_c(\omega;[0,1])$ such that θ_j vanishes in a neighborhood of $\partial\omega$, $\theta_j\equiv 1$ in $\omega\setminus\omega_j$, and $\|\nabla\theta_j\|_{L^\infty(\omega;\mathbb{R}^2)}\leq Cj$, for some constant $C>0$. Define $w_{k,j}:\Omega\to\mathbb{R}^3$ by

$$w_{k,j}(x_\alpha,x_3):=u_0(x_\alpha)+\theta_j(x_\alpha)w_k(x_\alpha,x_3).$$

Since by (4.5) we have that

$$\lim_{j\to\infty}\lim_{k\to\infty}\|w_{k,j}-u_0\|_{L^p(\Omega;\mathbb{R}^3)}=0,$$

a diagonalization argument provides an increasing sequence $k(j)\nearrow+\infty$ as $j\to+\infty$ such that

$$\lim_{j\to\infty}\|u_j-u_0\|_{L^p(\Omega;\mathbb{R}^3)}=0,$$

and

$$\|w_{k(j)}\|^{\frac{1}{2}}_{L^p(\Omega;\mathbb{R}^3)}\leq\frac{1}{j},\tag{4.8}$$

where we have denoted $w_{k(j),j}$ by u_j. Set $\bar{\varepsilon}_j:=\widetilde{\varepsilon}_{n_{k(j)}}$. Note that we have

$$\nabla_\alpha u_j(x_\alpha,x_3)=\xi_0+\nabla\theta_j(x_\alpha)\otimes w_{k(j)}(x_\alpha,x_3)+\theta_j(x_\alpha)\nabla_\alpha w_{k(j)}(x_\alpha,x_3),$$

and

$$\nabla_3 u_j(x_\alpha,x_3)=\theta_j(x_\alpha)\nabla_3 w_{k(j)}(x_\alpha,x_3).$$

Thus,

$$\int_A\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 u_j\right)\right|^p dx\leq C\left(|\xi_0|^p\mathcal{L}^3(A)+j^p\int_A|w_{k(j)}|^p dx+\int_A\left|\left(\nabla_\alpha w_{k(j)}\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 w_{k(j)}\right)\right|^p dx\right),$$

for all Borel subsets A of Ω, and by (4.6) and (4.8) we deduce that (iv) holds (after relabelling). It remains to prove that (i) is satisfied. To this end, let us define

$$T_j:=\left\{x\in\Omega:u_j(x)\neq w_{k(j)}(x)+u_0(x_\alpha)\right\},$$

and note that

$$\mathcal{L}^3(T_j)\leq\mathcal{L}^3(\omega_j)\to 0\text{ as }j\to\infty.\tag{4.9}$$

We have,

$$\begin{aligned}\int_\Omega W\left(\nabla_\alpha u_j\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 u_j\right)dx &= \int_\Omega W\left(\nabla_\alpha u_0+\nabla_\alpha w_{k(j)}\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 w_{k(j)}\right)dx+\int_{T_j}W\left(\nabla_\alpha u_j\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 u_j\right)dx\\ &\quad-\int_{T_j}W\left(\nabla_\alpha u_0+\nabla_\alpha w_{k(j)}\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 w_{k(j)}\right)dx,\end{aligned}$$

and since the last two integrands are equi-integrable (use (1.1)), (4.7) and (4.9) imply that

$$\limsup_{j\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_j\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 u_j\right)dx\leq Q^*_\omega W(u_0).$$

As $\left(u_j,\frac{1}{\bar{\varepsilon}_j}\right)$ is admissible for $Q^*_\omega W(u_0)$, we conclude that

$$\lim_{j\to\infty}\frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega W\left(\nabla_\alpha u_j\Big|\frac{1}{\bar{\varepsilon}_j}\nabla_3 u_j\right)dx=Q^*_\omega W(u_0),$$

thus asserting (i).

Step 2: In this part of the proof we remove the restriction of ω being a cube. Set $Q' := (-1/2,1/2)^2$, and let

$$\omega^{(m)} := \bigcup_{i=1}^{N(m)} (a_i + L_i Q') \subset\subset \omega$$

be a union of pairwise disjoint cubes such the $\mathcal{L}^2(\omega \setminus \omega^{(m)}) \leq \frac{1}{m}$, where $m \in \mathbb{N}$, and $L_i > 0$ $(i = 1, \cdots, N(m))$. Since the definition of $Q^*_\omega W(u_0)$ is invariant under translations of the projection ω of the domain of integration, given a sequence $\{\varepsilon_n\}$ of positive real numbers converging to zero we can repeatedly apply Step 1 to each cube $a_i + L_i Q'$, thus finding a subsequence $\{\varepsilon_{n,m}\}$ of $\{\varepsilon_n\}$ and sequences $\{u_n^{(i)}\} \subset W^{1,p}((a_i + L_i Q') \times (-1,1); \mathbb{R}^3)$ $(i = 1, \cdots, N(m))$, such that (i)-(iii) hold (with ω, u_n and ε_n replaced by $a_i + L_i Q'$, $u_n^{(i)}$ and $\varepsilon_{n,m}$, respectively). Define $u_{n,m} \in W^{1,p}(\Omega; \mathbb{R}^3)$ by

$$u_{n,m}(x_\alpha, x_3) := \sum_{i=1}^{N(m)} u_n^{(i)}(x_\alpha, x_3) \chi_{(a_i + L_i Q') \times (-1,1)}(x_\alpha, x_3) + u_0(x_\alpha) \chi_{(\omega \setminus \omega^{(m)}) \times (-1,1)}(x_\alpha, x_3).$$

Note that $u_{n,m|\partial\omega \times (-1,1)} = u_0$, and that we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \|u_{n,m} - u_0\|^p_{L^p(\Omega;\mathbb{R}^3)} = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{N(m)} \|u_n^{(i)} - u_0\|^p_{L^p((a_i + L_i Q') \times (-1,1); \mathbb{R}^3)} = 0.$$

In view of (1.1) we obtain

$$\begin{aligned}
&\frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W\left(\nabla_\alpha u_{n,m} \Big| \frac{1}{\varepsilon_{n,m}} \nabla_3 u_{n,m}\right) dx \\
&= \frac{1}{\mathcal{L}^3(\Omega)} \left(\sum_{i=1}^{N(m)} \int_{(a_i + L_i Q') \times (-1,1)} W\left(\nabla_\alpha u_n^{(i)} \Big| \frac{1}{\varepsilon_{n,m}} \nabla_3 u_n^{(i)}\right) dx + \int_{(\omega \setminus \omega^{(m)}) \times (-1,1)} W(\xi_0|0) dx \right) \\
&\leq \frac{1}{\mathcal{L}^3(\Omega)} \left(\sum_{i=1}^{N(m)} \int_{(a_i + L_i Q') \times (-1,1)} W\left(\nabla_\alpha u_n^{(i)} \Big| \frac{1}{\varepsilon_{n,m}} \nabla_3 u_n^{(i)}\right) dx + \frac{2C(1 + |\xi_0|^p)}{m} \right),
\end{aligned}$$

and thus,

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W\left(\nabla_\alpha u_{n,m} \Big| \frac{1}{\varepsilon_{n,m}} \nabla_3 u_{n,m}\right) dx \leq Q\overline{W}(\xi_0).$$

By the usual diagonalization argument we may find $n(m) \nearrow +\infty$ as $m \to +\infty$, such that

$$\lim_{m\to\infty} \|u_{n(m),m} - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0,$$

and

$$\limsup_{m\to\infty} \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W\left(\nabla_\alpha u_{n(m),m} \Big| \frac{1}{\varepsilon_{n(m),m}} \nabla_3 u_{n(m),m}\right) dx \leq Q\overline{W}(\xi_0).$$

Since (1.3) holds and as $\left(u_{n(m),m}, \frac{1}{\varepsilon_{n(m),m}}\right)$ is admissible for $Q^*_\omega W(u_0)$, we obtain that

$$\lim_{m\to\infty} \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega W\left(\nabla_\alpha u_{n(m),m} \Big| \frac{1}{\varepsilon_{n(m),m}} \nabla_3 u_{n(m),m}\right) dx = Q\overline{W}(\xi_0).$$

It remains to prove the p -equi-integrability required in (iv). While at this stage this is not necessarily satisfied, we can nevertheless mimic the arguments used in Step 1 (first apply Theorem 1.1 to get a p -equi-integrable sequence, then modify the new sequence near the lateral boundary $\partial\omega\times(-1,1)$) to construct a sequence $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ and a subsequence of $\{\varepsilon_n\}$ such that (i)-(iv) hold simultaneously. The details of this construction are identical to those of the construction performed in Step 1, and we will not repeat them here.

5 Proof of Corollary 1.3

The proof is strongly motivated by that of Proposition 6.3 (Step 1) in [13]. Theorem 1.1 implies that there exist a subsequence $\{v_{n(k)}\}$ of $\{v_n\}$ and a sequence $\{\widetilde{w}_k\} \subset W^{1,\infty}(\Omega;\mathbb{R}^3)$ such that $\lim_{n\to\infty} \mathcal{L}^3(\{x \in \Omega : \widetilde{w}_k(x) \neq v_{n(k)}(x)\}) = 0$, $\lim_{k\to\infty} \|\widetilde{w}_k - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0$ and the sequence $\left\{\left(\nabla_\alpha \widetilde{w}_k \Big| \frac{1}{\varepsilon_{n(k)}} \nabla_3 \widetilde{w}_k\right)\right\}$ is p - equi-integrable. Thus, taking into account (1.1) and using the fact that $\{(v_{n(k)}, \varepsilon_{n(k)})\}$ is a recovering sequence, we obtain that

$$\limsup_{k\to+\infty} \int_\Omega W\left(\nabla_\alpha \widetilde{w}_k \Big| \frac{1}{\varepsilon_{n(k)}} \nabla_3 \widetilde{w}_k\right) dx \leq 2 \int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha)) dx_\alpha. \tag{5.1}$$

Repeating the arguments used in the proof of Corollary 1.2, we may modify $\widetilde{w}_k$ appropriately near the lateral boundary $\partial\omega \times (-1,1)$ so that we obtain subsequences $\{\overline{\varepsilon}_j\}$ and $\{\overline{v}_j\}$ of $\{\varepsilon_{n(k)}\}$ and $\{v_{n(k)}\}$ respectively, and a sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that

$$\lim_{j\to\infty} \mathcal{L}^3(\{x \in \Omega : u_j(x) \neq \overline{v}_j(x)\}) = 0, \tag{5.2}$$

$$\lim_{j\to\infty} \|u_j - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0, \tag{5.3}$$

$$\left\{\left(\nabla_\alpha u_j \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j\right)\right\} \quad \text{is } p-\text{equi}-\text{integrable}, \tag{5.4}$$

and

$$u_{j|\partial\omega\times(-1,1)} = u_0, \tag{5.5}$$

hold. Moreover, in view of (2.5), (5.1) and (5.4), we have that

$$\lim_{j\to\infty} \int_\Omega W\left(\nabla_\alpha u_j \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j\right) dx = 2 \int_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha)) dx_\alpha. \tag{5.6}$$

Let us define $\Omega_k^+ := \omega\times\left(1-\frac{1}{k},1\right)$, $\Omega_k^- := \omega\times\left(-1,-1+\frac{1}{k}\right)$, and put $\Omega_k := \Omega_k^+ \cup \Omega_k^-$. Changing variables, we obtain that

$$\int_{\Omega\setminus\Omega_k} W\left(\nabla_\alpha u_j \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j\right) dx = \left(1-\frac{1}{k}\right) \int_\Omega W\left(\nabla_\alpha v_{j,k} \Big| \frac{1}{\overline{\varepsilon}_j \left(1-\frac{1}{k}\right)} \nabla_3 v_{j,k}\right) dx, \tag{5.7}$$

where $v_{j,k} : \Omega \to \mathbb{R}^3$ is defined by $v_{j,k}(x_\alpha, x_3) := u_j\left(x_\alpha, \left(1-\frac{1}{k}\right)x_3\right)$. Note that by (5.5) we have $v_{j,k|\partial\omega\times(-1,1)} = u_0$, and also

$$\begin{aligned}\|v_{j,k}-u_0\|^p_{L^p(\Omega;\mathbb{R}^3)} &= \left(1-\frac{1}{k}\right)^{-1}\int\limits_{\Omega\setminus\Omega_k}|u_j(y_\alpha,y_3)-u_0(y_\alpha)|^p dy_\alpha dy_3\\ &\leq \left(1-\frac{1}{k}\right)^{-1}\|u_j-u_0\|^p_{L^p(\Omega;\mathbb{R}^3)}.\end{aligned}$$

Thus,

$$\lim_{j\to\infty}\|v_{j,k}-u_0\|_{L^p(\Omega;\mathbb{R}^3)}=0 \text{ for all } k\geq 2,$$

and we deduce that for each integer $k\geq 2$ fixed the sequence $\left\{\left(v_{j,k},\left(1-\frac{1}{k}\right)\overline{\varepsilon}_j\right)\right\}$ is admissible for the infimum in (2.5). This implies that we have

$$\liminf_{j\to\infty}\int\limits_\Omega W\left(\nabla_\alpha v_{j,k}\Big|\frac{1}{\left(1-\frac{1}{k}\right)\overline{\varepsilon}_j}\nabla_3 v_{j,k}\right)dx\geq 2\int\limits_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha,$$

and in view of (5.6) and (5.7), we obtain that

$$\begin{aligned}&\limsup_{j\to\infty}\int\limits_{\Omega_k}W\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)dx\\ &=\limsup_{j\to\infty}\left(\int\limits_\Omega W\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)dx-\left(1-\frac{1}{k}\right)\int\limits_\Omega W\left(\nabla_\alpha v_{j,k}\Big|\frac{1}{\left(1-\frac{1}{k}\right)\overline{\varepsilon}_j}\nabla_3 v_{j,k}\right)dx\right)\\ &\leq 2\int\limits_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha-\liminf_{j\to\infty}\left(1-\frac{1}{k}\right)\int\limits_\Omega W\left(\nabla_\alpha v_{j,k}\Big|\frac{1}{\left(1-\frac{1}{k}\right)\overline{\varepsilon}_j}\nabla_3 v_{j,k}\right)dx\\ &\leq\frac{2}{k}\int\limits_\omega Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha.\end{aligned}$$

Taking into account (1.1), we deduce that

$$\limsup_{j\to\infty}\int\limits_{\Omega_k}\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)\right|^p dx\leq\frac{C}{k}\text{ for all } k\geq 2,\tag{5.8}$$

for some constant $C>0$.

Partition Ω_k^- and Ω_k^+ into k horizontal layers of height $\frac{1}{k^2}$, denoted by $L^-_{k,i}$ and $L^+_{k,i}$ respectively (indexed in a symmetric way with respect to the plane $x_3=0; i=1,\cdots,k$). In view of (5.8), for each integer $k\geq 2$ there exists $J(k)\in\mathbb{N}$ such that

$$\sum_{i=1}^k\int\limits_{L^-_{k,i}\cup L^+_{k,i}}\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)\right|^p dx\leq\frac{2C}{k},\text{ for all } j\geq J(k).$$

Thus, for each $k\geq 2$ and $j\geq J(k)$, there exists an index $i(k,j)\in\{1,\cdots,k\}$ such that

$$\int\limits_{L^-_{k,i(k,j)}\cup L^+_{k,i(k,j)}}\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)\right|^p dx\leq\frac{2C}{k^2}.\tag{5.9}$$

Put $L^-_{k,i(k,j)} := \omega \times (-c_{k,j}, -b_{k,j})$ and $L^+_{k,i(k,j)} := \omega \times (b_{k,j}, c_{k,j})$. In particular, (5.9) yields

$$\int\limits_{-c_{k,j}}^{-\frac{b_{k,j}+c_{k,j}}{2}} \int\limits_{\omega} \left| \left(\nabla_\alpha u_j \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j \right) \right|^p dx \leq \frac{2C}{k^2},$$

and

$$\int\limits_{\frac{b_{k,j}+c_{k,j}}{2}}^{c_{k,j}} \int\limits_{\omega} \left| \left(\nabla_\alpha u_j \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j \right) \right|^p dx \leq \frac{2C}{k^2},$$

and thus, there exist $a^-_{k,j} \in \left(-c_{k,j}, -\frac{b_{k,j}+c_{k,j}}{2}\right)$ and $a^+_{k,j} \in \left(\frac{b_{k,j}+c_{k,j}}{2}, c_{k,j}\right)$ such that we have

$$\int\limits_{\omega} \left| \left(\nabla_\alpha u_j(x_\alpha, a^-_{k,j}) \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j(x_\alpha, a^-_{k,j}) \right) \right|^p dx \leq 4C, \tag{5.10}$$

and

$$\int\limits_{\omega} \left| \left(\nabla_\alpha u_j(x_\alpha, a^+_{k,j}) \Big| \frac{1}{\overline{\varepsilon}_j} \nabla_3 u_j(x_\alpha, a^+_{k,j}) \right) \right|^p dx \leq 4C. \tag{5.11}$$

Consider smooth cut-off functions $\varphi_{k,j} = \varphi_{k,j}(x_3)$ such that

$$\varphi_{k,j} \equiv 1 \text{ on } \left(a^-_{k,j} + \frac{1}{k^2}, a^+_{k,j} + \frac{1}{k^2} \right),$$

$$\varphi_{k,j} \equiv 0 \text{ on } \left(-\infty, -\frac{b_{k,j}+c_{k,j}}{2} \right) \cup \left(\frac{b_{k,j}+c_{k,j}}{2}, +\infty \right),$$

$$\|\varphi'_{k,j}\|_\infty \leq ck^2,$$

and define $w_{k,j} \in W^{1,p}(\Omega; \mathbb{R}^3)$ by

$$\begin{aligned} w_{k,j}(x_\alpha, x_3) \quad := \quad & \varphi_{k,j}(x_3) u_j(x_\alpha, x_3) + (1 - \varphi_{k,j}(x_3)) u_j(x_\alpha, a^-_{k,j}) \chi_{\Omega^-_k}(x_\alpha, x_3) \\ & + (1 - \varphi_{k,j}(x_3)) u_j(x_\alpha, a^+_{k,j}) \chi_{\Omega^+_k}(x_\alpha, x_3). \end{aligned}$$

It can be easily seen that

$$w_{k,j|\partial\omega\times(-1,1)} = u_0, \tag{5.12}$$

and

$$\lim_{k\to\infty} \limsup_{j\to\infty} \mathcal{L}^3(\{x \in \Omega : w_{k,j}(x) \neq u_j(x)\}) = 0. \tag{5.13}$$

Claim: *The following identities hold:*

$$\limsup_{j\to\infty} \|w_{k,j} - u_0\|_{L^p(\Omega;\mathbb{R}^3)} = 0 \ \textit{for all } k \geq 2, \tag{5.14}$$

and

$$\lim_{k\to\infty}\limsup_{j\to\infty}\int\limits_{\Omega_k}\left|\left(\nabla_\alpha w_{k,j}\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 w_{k,j}\right)\right|^p dx=0. \tag{5.15}$$

Assuming that the claim holds, the usual diagonalization argument provides an increasing sequence $j(k)\nearrow\infty$ as $k\to\infty$, so that we have

$$\lim_{k\to\infty}\mathcal{L}^3(\{x\in\Omega : w_k(x)\neq u_{j(k)}(x)\})=0, \tag{5.16}$$

$$\lim_{k\to\infty}\|w_k-u_0\|_{L^p(\Omega;\mathbb{R}^3)}=0, \tag{5.17}$$

and

$$\lim_{k\to\infty}\int\limits_{\Omega_k}\left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx=0, \tag{5.18}$$

where we have denoted $w_k:=w_{k,j(k)}$ and $\varepsilon_{n_k}:=\overline{\varepsilon}_{j(k)}$. Note that the corresponding subsequence $\{v_{n_k}\}$ of $\{v_n\}$ is obtained by taking $v_{n_k}:=\overline{v}_{j(k)}$. Clearly (see (5.12)), we have that

$$w_{k|\partial\omega\times(-1,1)}=u_0. \tag{5.19}$$

In view of (5.2) and (5.16) we deduce that (i) holds, while (iii) follows from the particular way in which we have constructed the sequence $\{w_k\}$. To prove (ii), let A be an arbitrary Borel subset of Ω. We have

$$\begin{aligned}\int\limits_A\left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx &= \int\limits_{A\cap(\Omega\setminus\Omega_k)}\left|\left(\nabla_\alpha u_{j(k)}\Big|\frac{1}{\overline{\varepsilon}_{j(k)}}\nabla_3 u_{j(k)}\right)\right|^p dx\\ &\quad+\int\limits_{A\cap\Omega_k}\left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx\\ &\le \int\limits_A\left|\left(\nabla_\alpha u_{j(k)}\Big|\frac{1}{\overline{\varepsilon}_{j(k)}}\nabla_3 u_{j(k)}\right)\right|^p dx\\ &\quad+\int\limits_{\Omega_k}\left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx,\end{aligned}$$

and (ii) follows by (5.4) and (5.18).

We have

$$\begin{aligned}&\int\limits_\Omega W\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)dx\\ &=\int\limits_\Omega W\left(\nabla_\alpha u_{j(k)}\Big|\frac{1}{\overline{\varepsilon}_{j(k)}}\nabla_3 u_{j(k)}\right)dx+\int\limits_{\Omega_k}W\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)dx\\ &-\int\limits_{\Omega_k}W\left(\nabla_\alpha u_{j(k)}\Big|\frac{1}{\overline{\varepsilon}_{j(k)}}\nabla_3 u_{j(k)}\right)dx,\end{aligned}$$

and thus, in view of (ii), (1.1), (5.4) and (5.6), we obtain that

$$\lim_{k\to\infty}\int\limits_{\Omega} W\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)dx = 2\int\limits_{\omega} Q\overline{W}(\nabla_\alpha u_0(x_\alpha))dx_\alpha.$$

Taking now into account (5.17) and (5.19) we deduce that (iv) holds.

Proof of Claim: We have

$$\|w_{k,j}-u_j\|^p_{L^p(\Omega;\mathbb{R}^3)} \leq \int\limits_{\Omega_k^-}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^-_{k,j})|^p dx + \int\limits_{\Omega_k^+}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx,$$

and by Hölder's inequality we obtain that

$$\begin{aligned}
\int\limits_{\Omega_k^+}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx_\alpha dx_3 &= \int\limits_{\Omega_k^+}\left|\int\limits_{x_3}^{a^+_{k,j}}\frac{d}{ds}(u_j(x_\alpha,s))ds\right|^p dx_\alpha dx_3\\
&\leq \frac{1}{k^{\frac{p}{p'}}}\int\limits_{\Omega_k^+}\left(\int\limits_{1-\frac{1}{k}}^{1}\left|\frac{d}{ds}(u_j(x_\alpha,s))\right|^p ds\right)dx\\
&= \frac{1}{k^{\frac{p}{p'}+1}}\int\limits_{\Omega_k^+}|\nabla_3 u_j(x)|^p\,dx.
\end{aligned}$$

Thus, in view of (5.8) we have that

$$\begin{aligned}
&\limsup_{j\to\infty}\int\limits_{\Omega_k^+}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx\\
&\leq \frac{1}{k^{\frac{p}{p'}+1}}\limsup_{j\to\infty}\left(\overline{\varepsilon}_j^p\int\limits_{\Omega_k^+}\left|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j(x)\right|^p dx\right)\\
&\leq \frac{1}{k^{\frac{p}{p'}+1}}\left(\limsup_{j\to\infty}\overline{\varepsilon}_j^p\right)\limsup_{j\to\infty}\int\limits_{\Omega_k^+}\left|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j(x)\right|^p dx = 0,
\end{aligned}$$

for each $k\geq 2$ fixed. Similarly, we obtain that

$$\limsup_{j\to\infty}\int\limits_{\Omega_k^-}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^-_{k,j})|^p dx = 0 \text{ for all } k\geq 2.$$

Hence

$$\limsup_{j\to\infty}\|w_{k,j}-u_j\|^p_{L^p(\Omega;\mathbb{R}^3)} = 0 \text{ for all } k\geq 2,$$

which together with (5.3) yields (5.14).

It remains to prove (5.15). To this end, we first note that we have

$$
\begin{aligned}
&\frac{1}{\overline{\varepsilon}_j^p}\int\limits_{\Omega_k^+}\left|\varphi'_{k,j}(x_3)\right|^p|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx_\alpha dx_3\\
&\leq \frac{Ck^{2p}}{\overline{\varepsilon}_j^p}\int\limits_{a^+_{k,j}-\frac{1}{2k^2}}^{a^+_{k,j}}\int\limits_{\omega}|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx_\alpha dx_3\\
&\leq \frac{Ck^{2p}}{\overline{\varepsilon}_j^p}\int\limits_{a^+_{k,j}-\frac{1}{2k^2}}^{a^+_{k,j}}\int\limits_{\omega}\left|\int\limits_{x_3}^{a^+_{k,j}}\frac{d}{ds}\left(u_j(x_\alpha,x_3)\right)ds\right|^p dx_\alpha dx_3\\
&\leq \frac{Ck^{2p}}{k^{2\frac{p}{p'}}\overline{\varepsilon}_j^p}\int\limits_{a^+_{k,j}-\frac{1}{2k^2}}^{a^+_{k,j}}\int\limits_{\omega}\left(\int\limits_{a^+_{k,j}-\frac{1}{2k^2}}^{a^+_{k,j}}\left|\frac{d}{ds}u_j(x_\alpha,s)\right|^p dsdx_\alpha\right)dx_3\\
&\leq \frac{C}{\overline{\varepsilon}_j^p}\int\limits_{\Omega_k^+}|\nabla_3 u_j(x)|^p\,dx,
\end{aligned}
$$

where we have used again Hölder's inequality. Thus, in view of (5.8) we deduce that

$$
\lim_{k\to\infty}\limsup_{j\to\infty}\frac{1}{\overline{\varepsilon}_j^p}\int\limits_{\Omega_k^+}\left|\varphi'_{k,j}(x_3)\right|^p|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx_\alpha dx_3=0, \tag{5.20}
$$

and also

$$
\lim_{k\to\infty}\limsup_{j\to\infty}\frac{1}{\overline{\varepsilon}_j^p}\int\limits_{\Omega_k^-}\left|\varphi'_{k,j}(x_3)\right|^p|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^-_{k,j})|^p dx_\alpha dx_3=0. \tag{5.21}
$$

We have

$$
\begin{aligned}
\int\limits_{\Omega_k^+}\left|\left(\nabla_\alpha w_{k,j}\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 w_{k,j}\right)\right|^p dx \;\leq\; & C\int\limits_{\Omega_k^+}\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 u_j\right)\right|^p dx + C\int\limits_{\Omega_k^+}\left|\nabla_\alpha u_j(x_\alpha,a^+_{k,j})\right|^p dx_\alpha dx_3\\
&+\frac{C}{\overline{\varepsilon}_j^p}\int\limits_{\Omega_k^+}\left|\varphi'_{k,j}(x_3)\right|^p|u_j(x_\alpha,x_3)-u_j(x_\alpha,a^+_{k,j})|^p dx_\alpha dx_3,
\end{aligned}
$$

and therefore, by virtue of (5.8), (5.11) and (5.20) we obtain that

$$
\lim_{k\to\infty}\limsup_{j\to\infty}\int\limits_{\Omega_k^+}\left|\left(\nabla_\alpha w_{k,j}\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 w_{k,j}\right)\right|^p dx=0.
$$

Similarly, invoking (5.8), (5.10) and (5.21) we find

$$
\lim_{k\to\infty}\limsup_{j\to\infty}\int\limits_{\Omega_k^-}\left|\left(\nabla_\alpha w_{k,j}\Big|\frac{1}{\overline{\varepsilon}_j}\nabla_3 w_{k,j}\right)\right|^p dx=0,
$$

thus asserting (5.15).

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