RELAXATION RESULTS IN MICROMAGNETICS

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In memory of Ennio De Giorgi

Abstract. In this paper an integral representation formula is obtained for the relaxed energy of a large ferromagnetic body. With $\Omega \subset \mathbb{R}^N$ being the reference configuration of the body and $m : \Omega \to S^{N-1}$ the magnetization, consider the energy

$$F(m) := \int_{\mathbb{R}^N} f(x, \chi_\Omega(x)\, m(x), u(x), \nabla u(x))\, dx$$

where $(\chi_\Omega\, m, \nabla u)$ satisfies Maxwell's equations, i.e. $u \in H^1(\mathbb{R}^N)$ is the unique solution of $\Delta u + \operatorname{div}(\chi_\Omega\, m) = 0$ in $\mathbb{R}^N$. If f is a Carathéodory function satisfying very mild growth conditions then it is shown that the relaxation of F with respect to L^∞-$w*$ convergence in $L^\infty(\Omega; \overline{B(0,1)})$ is given by

$$\mathcal{F}(m) = \int_\Omega Q_M\, f(x, m(x), u(x), \nabla u(x))\, dx + \int_{\mathbb{R}^N \setminus \Omega} f(x, 0, u(x), \nabla u(x))\, dx$$

where $Q_M\, f$ is the quasiconvex envelope of f relative to the underlying partial differential equations. This class of integrands includes those of the type

$$f(x, m, u, h) = \varphi(x, m, u) + \psi(x, u, h)$$

with $\psi(x, u, \cdot)$ non convex, thus extending the available relaxation results in micromagnetics.

1. Introduction

In recent years *micromagnetics*, a continuum model of the macroscopic behavior of a ferromagnetic body, has attracted the attention of researchers in Calculus of Variations (see e.g. the work of [2, 6, 9, 11, 17, 18, 22, 24, 25, 26]). According to this theory, essentially developed by Brown [5], equilibrium states of a body subject to a given external magnetic field h_e correspond to absolute (or local) minimizers of the energy functional

$$F_e(m) := \frac{\varepsilon}{2} \int_\Omega |\nabla m(x)|^2\, dx + \int_\Omega \varphi(m(x))\, dx - \int_\Omega h_e(x) \cdot m(x)\, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h(x)|^2\, dx. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^3$ represents the region occupied by the body and the *magnetization* is a function $m : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$|m(x)| = m_s\, \chi_\Omega(x) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x \in \mathbb{R}^3, \tag{1.2}$$

where $m_s > 0$, the *saturation magnetization*, is a function of the temperature and of the specific material. We assume that the temperature is fixed, and, without loss of generality, in what follows we take

$$m_s = 1.$$

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The *induced magnetic field* is a function $h : \mathbb{R}^3 \to \mathbb{R}^3$ related to m through the *Maxwell's equations* for magnetostatics

$$\mathcal{A}(m,h) := \begin{pmatrix} m+h \\ \operatorname{curl} h \end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^3. \tag{1.3}$$

The existence of absolute minimizers for F_e has been established by Visintin [26], using the Direct Method of the Calculus of Variations, and their structure strongly depends on the competition between different terms in the energy functional (1.1). Precisely:

• the *exchange energy* $\frac{\varepsilon}{2}\int_\Omega |\nabla m(x)|^2\,dx$ penalizes spatial changes of m and thus models the tendency of the body to exhibit large regions of uniform magnetization (*magnetic domains*) separated by very thin transitions layers (*domain walls*). The parameter $\varepsilon > 0$ is obtained by a rescaling argument and is related to size of the body (see [11]);

• the *anisotropy energy* $\int_\Omega \varphi(m(x))\,dx$ drives m to preferred crystallographic directions (*easy axes*), where the anisotropy energy density $\varphi(m)$ is a nonnegative, even, continuous function, defined on the unit sphere S^2 and vanishing only on a finite set of unit vectors which represent the easy axes;

• the *external field energy* (or *interaction energy*) $\int_\Omega h_e(x)\cdot m(x)\,dx$ fosters the alignment of the magnetization m with the external field h_e;

• the *field energy* $\frac{1}{2}\int_\Omega |h(x)|^2\,dx$ vanishes only when $\operatorname{div}(m\chi_\Omega) = 0$ and thus models the tendency of the magnetization to be divergence-free.

In [17] James and Kinderlehrer proposed the following simplified version of the energy functional (1.1)

$$F(m) := \int_\Omega \varphi(m(x))\,dx - \int_\Omega h_e(x)\cdot m(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |h(x)|^2\,dx, \tag{1.4}$$

which is usually referred as the *no-exchange* functional. The main new feature of the functional F is that it is no longer semicontinuous with respect to the natural topology for m, the $L^\infty(\Omega;\mathbb{R}^3)$ weak-* topology. Minimizing sequences may develop oscillations and the existence of minimizers is no longer guaranteed. This is in agreement with the formation of small magnetics domains observed in the experiments. As an example, in the uniaxial case when $\{\varphi = 0\} = \{m_1, -m_1\}$ for some $m_1 \in S^2$, and for $h_e = 0$, James and Kinderlehrer [17] proved non-existence of minimizers by constructing minimizing sequences which develop finer and finer spatial oscillations, and characterized the Young measures generated by such sequences.

Tartar [24, 25] identified the limiting energy

$$\lim_{\varepsilon\to 0^+} F_\varepsilon,$$

by using a combination of Young measures and H-measures and a result obtained in collaboration with Murat on the relations between Young measures and H-measures [20]. In particular, he showed that the relaxed functional of F, which is defined by

$$\mathcal{F}(m) := \inf\left\{\liminf_{k\to\infty} F(m_k) : \, m_k \in L^\infty(\Omega;S^2), \quad m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(\Omega;\mathbb{R}^3)\right\}$$

where $|m(x)| \le 1$, admits the following integral representation formula

$$\mathcal{F}(m) = \int_\Omega \varphi^{**}(m(x))\,dx - \int_\Omega h_e(x)\cdot m(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |h(x)|^2\,dx,$$

where φ^{**} is the convex envelope of the function φ extended to be $+\infty$ outside the unit sphere S^2. The same representation formula for $\mathcal{F}(m)$ was later obtained by De Simone [11] using classical methods from the Calculus of Variations. De Simone also showed that when the volume of the body becomes infinite then minimizers and minimizing sequences of F represent the limit behavior of minimizers of the functional F_ε, thus asserting that the no-exchange model is a good approximation for large ferromagnetics bodies.

However, as emphasized by Tartar in [24, 25], there are quantitative discrepancies between minimizing sequences and experimental observations of formation of magnetic domains, and it is not clear how robust and accurate this model is (see also [17]). As a first step towards understanding the oscillatory behavior of minimizing sequences, we propose here to address the relaxation of a general class of functionals which includes F.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and define

$$\mathcal{M} := \{m : \mathbb{R}^N \to \mathbb{R}^N \text{ measurable} : |m(x)| = \chi_\Omega(x) \text{ for } \mathcal{L}^N \text{ a.e } x \in \mathbb{R}^N\}. \tag{1.5}$$

Consider a Carathéodory function

$$f : \mathbb{R}^N \times \big(S^{N-1} \cup \{0\}\big) \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$$

such that

$$0 \le f(x, m, u, h) \le a(x) + C(|u|^{2^*} + |h|^2) \tag{1.6}$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(m, u, h) \in \big(S^{N-1} \cup \{0\}\big) \times \mathbb{R} \times \mathbb{R}^N$, and where $a \in L^1(\mathbb{R}^N)$, $C > 0$ and

$$2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ \text{any } q \in [1, \infty) & \text{if } N = 2. \end{cases}$$

Given a function $m \in \mathcal{M}$ there exists a unique function $u \in H^1(\mathbb{R}^N)$ such that (see [17])

$$\Delta u + \operatorname{div} m = 0 \qquad \text{in} \quad \mathbb{R}^N \tag{1.7}$$

in the sense of distributions, and thus we define the energy functional

$$F(m) := \int_{\mathbb{R}^N} f(x, m(x), u(x), \nabla u(x))\, dx.$$

Set

$$\mathcal{M}^* := \{m : \mathbb{R}^N \to \mathbb{R}^N \text{ measurable} : |m(x)| \le \chi_\Omega(x) \quad \text{for } \mathcal{L}^N \text{ a.e } x \in \mathbb{R}^N\}, \tag{1.8}$$

and for $m \in \mathcal{M}^*$ define

$$\mathcal{F}(m) := \inf\left\{\liminf_{k\to\infty} F(m_k) : \, m_k \in \mathcal{M}, \quad m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(\Omega; \mathbb{R}^N)\right\}.$$

The first main result of the paper is given by the following theorem:

Theorem 1. *Under condition* (1.6), *for all* $m \in \mathcal{M}^*$ *we have*

$$\mathcal{F}(m) = \int_\Omega \mathcal{Q}_M f(x, m(x), u(x), \nabla u(x))\, dx + \int_{\mathbb{R}^N \setminus \Omega} f(x, 0, u(x), \nabla u(x))\, dx$$

where the function $\mathcal{Q}_M f : \Omega \times \overline{B} \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ *coincides with* f *when* $|m| = 1$, *while for* $|m| < 1$ *is given by*

$$\mathcal{Q}_M f(x, m, u, h) := \inf\left\{\liminf_{k\to\infty} \frac{1}{|B|} \int_B f(x, M_k(y), u, h + \nabla U_k(y))\, dy : \, M_k \in C^\infty(\overline{B}; S^{N-1}),\right.$$
$$\left. M_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(B; \mathbb{R}^N), \quad U_k(y) := -\int_B G(\xi, y) \operatorname{div} M_k(\xi)\, d\xi\right\},$$

where $B := B(0, 1)$ *and for* $\xi,\, y \in \overline{B}$, $\xi \neq y$,

$$G(\xi, y) := \Gamma\left(\sqrt{|\xi|^2 + |y|^2 - 2\xi \cdot y}\right) - \Gamma\left(\sqrt{|\xi|^2|y|^2 + 1 - 2\xi \cdot y}\right),$$

with

$$\Gamma(z) := \begin{cases} \dfrac{1}{N(2-N)|B|}\, |z|^{2-N} & N > 2, \\ \dfrac{1}{2\pi} \log|z| & N = 2. \end{cases}$$

Remark 1. (i) In the definition of $\mathcal{Q}_M f(x, m, u, h)$ for $|m| < 1$ the function U_k is the unique solution $w \in H_0^1(B)$ of the equation $\Delta w + \operatorname{div} M_k = 0$ in B.

(ii) In [8] we have recently given an integral representation formula for the relaxed energy F corresponding to general constant rank partial differential operators $\mathcal{A}$ which include (1.3), in the case of bounded domains and when no constraints of the type (1.2) are present. As in that paper we work within the general setting of A-quasiconvexity, as introduced by Fonseca and Müller in [14]. The approach is based on the use of Young measures (see also the work of Balder [3] and of Kristensen [19]) together with the blow-up method introduced by Fonseca and Müller in [12].

Corollary 1. *Assume that* $f(x,m,u,h)$ *is of the form*

$$f(x,m,u,h) := \varphi(x,m,u) + \psi(x,u,h),$$

where $\varphi : \mathbb{R}^N \times \left(S^{N-1} \cup \{0\}\right) \times \mathbb{R} \to [0,\infty)$ *and* $\psi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ *are Carathéodory functions such that*

$$0 \leq \varphi(x,m,u) \leq a(x) + C|u|^{2^*},$$

$$\psi(x,u,\cdot) \text{ is convex in } \mathbb{R}^N, \qquad 0 \leq \psi(x,u,h) \leq a(x) + C(|u|^{2^*} + |h|^2)$$

for $\mathcal{L}^N$ *a.e.* $x \in \mathbb{R}^N$ *and all* $(m,u,h) \in \left(S^{N-1} \cup \{0\}\right) \times \mathbb{R} \times \mathbb{R}^N$, *and where* $a \in L^1(\mathbb{R}^N)$ *and* $C > 0$. *If* $|m| < 1$ *then*

$$\mathcal{Q}_M f(x,m,u,h) = \varphi^{**}(x,m,u) + \psi(x,u,h),$$

where, for $\mathcal{L}^N$ *a.e.* $x \in \Omega$ *and for each* $u \in \mathbb{R}$, *the function* $\varphi^{**}(x,\cdot,u)$ *is the convex envelope of*

$$\hat{\varphi}(x,m,u) := \begin{cases} \varphi(x,m,u) & \text{if } m \in S^{N-1}, \\ \infty & \text{in } B(0,1). \end{cases}$$

Remark 2. (i) Corollary 1 is closely related to a recent result of Pedregal [22] where the relaxed functional is defined in terms of Young measures. The analysis in that paper is restricted to the two dimensional case $N = 2$, where

$$\Delta u + \operatorname{div} m = 0 \quad \text{if and only if} \quad m(x) = \nabla u(x) + T\,\nabla v(x)$$

for some function v and where T is a $\pi/2$ rotation in the plane. Clearly this argument cannot be extended to higher dimensions.

In the special case where $\psi(h) := \frac{1}{2}|h|^2$ and $\varphi(x,m) := \chi_\Omega(x)g(m)$ we recover the results of Tartar [24, 25] and De Simone [11].

(ii) To our knowledge Theorem 1.1 is the first relaxation result in micromagnetics which does not make use of the convexity of $\psi(x,u,\cdot)$ (see e.g [11, 17, 22, 24, 25]). In general, even in the simplest case where

$$f(x,m,h) := \chi_\Omega(x)\varphi(m) + \psi(h),$$

if we start from a function ψ which is not convex in h, then $\mathcal{Q}_M f(x,m,h)$ will not coincide with $\chi_\Omega(x)\varphi^{**}(m) + \psi^{**}(h)$. Indeed let h be such that $\psi^{**}(h) < \psi(h)$. By taking $x \in \Omega$ and $|m| = 1$ we get

$$\psi^{**}(h) + \varphi(m) < \mathcal{Q}_M f(x,m,h) = \psi(h) + \varphi(m).$$

This remark partially answers a question raised in [22].

2. Preliminaries

Here $\Omega \subset \mathbb{R}^N$ is an open, bounded subset with Lipschitz boundary, $\mathcal{L}^N$ is the N dimensional Lebesgue measure, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere and $B := B(0,1)$ is the unit ball centered at the origin.

We recall briefly some facts about Young measures which will be useful in the sequel (see e.g. [4], [21]). If D is an open set (not necessarily bounded), we denote by $C_c(D;\mathbb{R}^d)$ the set of continuous functions with compact support in D, endowed with the supremum norm. The dual of the closure of $C_c(D;\mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass $\mathcal{M}(D;\mathbb{R}^d)$, through the duality

$$\langle \nu, f \rangle := \int_D f(y)\,d\nu(y), \quad \nu \in \mathcal{M}(D;\mathbb{R}^d), \quad f \in C_c(D;\mathbb{R}^d).$$

A map $\nu : \Omega \to \mathcal{M}(D;\mathbb{R}^d)$ is said to be *weak-$*$ measurable* if $x \mapsto \langle \nu_x, f \rangle$ is measurable for all $f \in C_c(D;\mathbb{R}^d)$. We say that a function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a *normal integrand* if for $\mathcal{L}^N$ a.e. $x \in \Omega$ the function $f(x,\cdot)$ is lower semicontinuous and there exists a Borel integrand $\tilde{f} : \Omega \times \mathbb{R}^d \to [-\infty,\infty]$ such that $f(x,\cdot) = \tilde{f}(x,\cdot)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

A sequence $\{z_n\} \subset L^p(D;\mathbb{R}^d)$ is *p-equintegrable* if $\{|z_n|^p\}$ is equintegrable.

The following result is a corollary of the Fundamental Theorem on Young Measures (see [4, 7, 23])

Theorem 2. *Let $z_n : \Omega \to \mathbb{R}^d$ be measurable functions such that*

$$\sup_{n\in\mathbb{N}} \int_\Omega |z_n|^q \, dx < \infty,$$

for some $q > 0$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a weak-$*$ measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ with

$$\nu_x \geq 0, \qquad ||\nu_x||_{\mathcal{M}} = \int_{\mathbb{R}^d} d\nu_x = 1 \quad \text{for a.e. } x \in \Omega,$$

and such that:

(i) if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand bounded from below then

$$\liminf_{k\to\infty} \int_\Omega f(x, z_{n_k}(x))\, dx \geq \int_\Omega \overline{f}(x)dx,$$

where

$$\overline{f}(x) := \langle \nu_x, f(x,\cdot)\rangle = \int_{\mathbb{R}^d} f(x,y)\, d\nu_x(y);$$

(ii) if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function bounded from below then

$$\lim_{k\to\infty} \int_\Omega f\left(x, z_{n_k}(x)\right)\, dx = \int_\Omega \overline{f}(x)dx$$

if and only if $\{f(\cdot, z_{n_k}(\cdot))\}$ is equi-integrable.

The map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ is called the *Young measure* generated by the sequence $\{z_{n_k}\}$.

Proposition 1. *If $\{z_n\}$ generates a Young measure ν and $v_n \to 0$ in measure, then $\{z_n + v_n\}$ still generates the Young measure ν.*

The following regularity results are well known, the former being an immediate consequence of Tiesze Extension Theorem.

Theorem 3. *Let $f : C \to [0,\infty)$ be a continuous function, where C is a closed subset of $\mathbb{R}^d$. Then there exists a continuous function $\tilde{f} : \mathbb{R}^d \to [0,\infty)$ such that $\tilde{f} = f$ on C.*

Theorem 4. *(Scorza-Dragoni) Let $f : A \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, where A is a Borel subset of $\mathbb{R}^d$. Then for every $\varepsilon > 0$ and every compact set $K \subset A$ there exists a compact set $K_\varepsilon \subset\subset K$ such that $|K\backslash K_\varepsilon| < \varepsilon$ and $f : K_\varepsilon \times \mathbb{R}^N \to \mathbb{R}$ is continuous.*

Proposition 2. *Consider a Carathéodory function*

$$f : \mathbb{R}^N \times \left(S^{N-1} \cup \{0\}\right) \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty).$$

Then there exists a Carathéodory function

$$\tilde{f} : \mathbb{R}^{2N} \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$$

such that $f(x,m,u,h) = \tilde{f}(x,m,u,h)$ for a.e. $x \in \mathbb{R}^N$ and all $(m,u,h) \in \left(S^{N-1}\cup\{0\}\right) \times \mathbb{R} \times \mathbb{R}^N$.

Proof of Lemma 2. Since f is Carathéodory, by Theorem 4 for each n, $j \in$ there exist compact sets $K_j^n \subset B(0,n)$ with

$$K_j^n \subset K_{j+1}^n \subset K_{j+1}^{n+1}, \qquad |B(0,n)\backslash K_j^n| \leq 1/j, \tag{2.1}$$

such that $f : K_j^n \times \left(S^{N-1}\cup\{0\}\right) \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ is continuous. By Theorem 3 we can find a continuous function $f_{n,j} : \mathbb{R}^{2N} \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ such that $f_{n,j} = f$ on $K_j^n \times \left(S^{N-1}\cup\{0\}\right) \times \mathbb{R} \times \mathbb{R}^N$. Moreover, by (2.1) we have

$$f_{n,j} = f_{n,j+1} = f_{n+1,j+1} \qquad \text{on } K_j^n.$$

Set

$$G := \bigcup_n^\infty \bigcup_j^\infty K_j^n, \qquad \tilde{f}(x,m,u,h) := \begin{cases} f_{n,j}(x,m,u,h) & \text{if } x \in K_j^n, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus G. \end{cases}$$

Then $f=\tilde{f}$ on $\mathbb{R}^N\setminus G$ and

$$|\mathbb{R}^N\backslash G|=\lim_{n\to\infty}|B(0,n)\backslash G|\leq\limsup_{n\to\infty}\limsup_{j\to\infty}|B(0,n)\backslash K_j^n|=0.$$

□

If $1<q\leq\infty$ then $W^{-1,q}(D)$ is the dual of $W_0^{1,q'}(D)$, where q' is the Hölder conjugate exponent of q, that is $1/q+1/q'=1$ if $1<q<\infty$ and $q'=\infty$ if $q=1$. It is well known that $F\in W^{-1,q}(D)$ if and only if there exist $g_1,\ldots,g_N\in L^q(D)$ such that

$$\langle F,w\rangle=\sum_{i=1}^N\int_D g_i\frac{\partial w}{\partial x_i}\,dx\qquad\text{for all } w\in W_0^{1,q}(D).$$

The following Sobolev-Gagliardo-Nirenberg Inequality may be found in [16].

Theorem 5. *For any $v\in W^{1,q}(\mathbb{R}^N)$, with $1\leq q<N$,*

$$||v||_{L^{q^*}(\mathbb{R}^N)}\leq C(q,N)||\nabla v||_{L^q(\mathbb{R}^N)},\tag{2.2}$$

where $q^*:=qN/\left(N-q\right)$.

For a proof of the proposition below see Giaquinta [15].

Proposition 3. *Let $D\subset\mathbb{R}^N$ be an open bounded set with Lipschitz boundary. If $m\in L^\infty(D;\mathbb{R}^N)$ and if $u\in H^1(D)$ is a solution of $\Delta u+\operatorname{div}m=0$ in D then $\nabla u\in L^p_{\mathrm{loc}}(D;\mathbb{R}^N)$ and*

$$||\nabla u||_{L^p(D')}\leq C(D,D',N,p)\left(||\nabla u||_{L^2(D)}+||u||_{L^2(D)}+||m||_{L^\infty(D;\mathbb{R}^N)}\right)\tag{2.3}$$

for any $p\geq1$ and $D'\subset\subset D$.

Lemma 1. *Let $m\in\mathcal{M}^*$ and consider a sequence $\{m_k\}\subset\mathcal{M}^*$ such that*

$$m_k\overset{*}{\rightharpoonup}m\ \text{in}\ L^\infty(\Omega;\mathbb{R}^N).\tag{2.4}$$

Let u, $u_k\in H^1(\mathbb{R}^N)$ be such that $\Delta u+\operatorname{div}m=\Delta u_k+\operatorname{div}m_k=0$ in $\mathbb{R}^N$. Then

$$u_k\rightharpoonup u\ \text{in}\ H^1_{\mathrm{loc}}(\mathbb{R}^N).$$

Proof of Lemma 1. By (1.7) we have

$$\int_{\mathbb{R}^N}\nabla u_k\cdot\nabla v\,dx=-\int_{\mathbb{R}^N}\chi_\Omega m_k\cdot\nabla v\,dx=-\int_\Omega m_k\cdot\nabla v\,dx\tag{2.5}$$

for all $v\in H^1(\mathbb{R}^N)$. Taking $v:=u_k$ in (2.5) and applying Hölder's inequality we obtain

$$\int_{\mathbb{R}^N}|\nabla u_k|^2\,dx\leq\left(\int_\Omega|m_k|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N}|\nabla u_k|^2\,dx\right)^{1/2},$$

and thus

$$\int_{\mathbb{R}^N}|\nabla u_k|^2\,dx\leq|\Omega|\,,\tag{2.6}$$

and by (2.2) we have that

$$||u_k||_{L^{2^*}(\mathbb{R}^N)}\leq C(N)\,|\Omega|\,.\tag{2.7}$$

Thus we may find a subsequence $\{u_{k_n}\}$ of $\{u_k\}$ and a function $U\in L^{2^*}(\mathbb{R}^N)$ with $\nabla U\in L^2(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$u_{k_n}\rightharpoonup U\ \text{in}\ L^{2^*}(\mathbb{R}^N),\quad\nabla u_{k_n}\rightharpoonup\nabla U\ \text{in}\ L^2(\mathbb{R}^N;\mathbb{R}^N).\tag{2.8}$$

In particular

$$u_{k_n}\rightharpoonup U\ \text{in}\ H^1_{\mathrm{loc}}(\mathbb{R}^N),$$

and $\Delta U+\operatorname{div}m=0$ in $\mathbb{R}^N$. In view of (1.7) we conclude that $u\left(x\right)=U\left(x\right)$ a.e. □

Lemma 2. *Let $m \in \mathcal{M}^*$ and consider a sequence $\{m_k\} \subset \mathcal{M}^*$ such that*

$$m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(\Omega;\mathbb{R}^N).$$

Let u, $u_k \in H^1(\mathbb{R}^N)$ be such that $\Delta u + \operatorname{div} m = \Delta u_k + \operatorname{div} m_k = 0$ in $\mathbb{R}^N$. Then $\{\nabla u_k\}$ is 2-equintegrable over $\mathbb{R}^N$, $u_k \to u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$, and

$$\lim_{k\to\infty} ||u_k - u||_{L^{2^*}(\mathbb{R}^N)} = \lim_{k\to\infty} ||\nabla u_k - \nabla u||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})} = 0.$$

Proof of Lemma 2. Consider two balls $B_1 \supset\supset B \supset\supset \overline{\Omega}$ and fix any $p > N$. By (2.3)

$$||\nabla u_k||_{L^p(B)} \leq C(B, B_1, N, p)\left(||u_k||_{H^1(B_1)} + 1\right), \tag{2.9}$$

which, together with (2.7) and a standard bootstrap argument, yields that the sequence $\{u_k\}$ is uniformly bounded in $W^{1,p}(B;\mathbb{R})$. By Rellich-Kondrachov Theorem we have $u_k \to u$ in $L^\infty(B;\mathbb{R})$, and since B is any fixed ball we conclude that $u_k \to u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$. Moreover, since $\{\nabla u_k\}$ is bounded in $L^p(B;\mathbb{R}^N)$ it follows that it is 2-equintegrable over B. Hence for any $\eta > 0$ there exists $\delta > 0$ such that

$$\int_E |\nabla(u_k - u)|^2 \, dx \leq \eta, \tag{2.10}$$

for any measurable set $E \subset B$ with $|E| \leq \delta$. Fix $\Omega \subset\subset \Omega' \subset\subset B$ with $|\Omega' \setminus \Omega| \leq \delta$ and consider a cut-off function φ such that $\varphi \equiv 1$ in $\mathbb{R}^N \setminus \overline{\Omega'}$ and $\varphi \equiv 0$ in Ω. Taking as a test function in (2.5) the function

$$v(x) := \varphi(x)\left(u_k(x) - u(x)\right),$$

and using the fact that

$$\nabla v = (u_k - u)\nabla\varphi + \varphi\nabla(u_k - u),$$

we get

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\overline{\Omega'}} |\nabla(u_k - u)|^2 \, dx &\leq \int_{\mathbb{R}^N\setminus\overline{\Omega}} \varphi|\nabla(u_k - u)|^2 \, dx = -\int_{\mathbb{R}^N\setminus\overline{\Omega}} (u_k - u)\,\nabla\varphi\cdot\nabla(u_k - u)\, dx \\ &\leq ||\nabla\varphi||_{L^\infty} \int_{\Omega'\setminus\Omega} |u_k - u| \; |\nabla(u_k - u)\,| \, dx \to 0\end{aligned}$$

as $k \to \infty$, because $u_k \to u$ in $L^2(\Omega')$ and ∇u_k is bounded in $L^2(\mathbb{R}^N;\mathbb{R}^N)$. Since $|\Omega' \setminus \Omega| \leq \delta$ it follows from (2.10) that

$$||\nabla(u_k - u)||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})} \leq ||\nabla(u_k - u)||_{L^2(\mathbb{R}^N\setminus\overline{\Omega'})} + \eta,$$

and thus

$$\limsup_{k\to\infty} ||\nabla(u_k - u)||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})} \leq \eta.$$

If we let $\eta \to 0^+$ we obtain that $\lim_{k\to\infty} ||\nabla(u_k - u)||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})} = 0$. By (2.2)

$$||u_k - u||_{L^{2^*}(\mathbb{R}^N\setminus\overline{\Omega'})} \leq C||\nabla\left(\varphi(u_k - u)\right)||_{L^2(\mathbb{R}^N)} \leq C(||\nabla\varphi||_{L^\infty}||u_k - u||_{L^2(\Omega'\setminus\overline{\Omega})} + ||\nabla(u_k - u)||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})}),$$

so letting $k \to \infty$ and using the fact that u_k converges to u in $L^\infty(\Omega')$, it follows that $||u_k - u||_{L^{2^*}(\mathbb{R}^N\setminus\overline{\Omega})} \to 0$. This, together with Lemma 1 and Sobolev Embedding Theorem, implies that $||u_k - u||_{L^{2^*}(\mathbb{R}^N)} \to 0$.

Since $\{\nabla u_k\}$ is 2-equintegrable over $B \supset\supset \overline{\Omega}$ and $\nabla u_k \to \nabla u$ strongly in $L^2(\mathbb{R}^N \setminus \overline{\Omega})$ this implies that $\{\nabla u_k\}$ is 2-equintegrable over $\mathbb{R}^N$. □

Lemma 3. *Let D be an open bounded set of $\mathbb{R}^N$ and consider a sequence*

$$\{(m_k, u_k)\} \subset L^\infty(D;\overline{B}) \times (\varphi_k + H^1_0(D)),$$

such that $\Delta u_k + \operatorname{div} m_k = 0$ in D, where $\varphi_k \in H^1(D)$. If $\varphi_k \rightharpoonup \varphi$ in $H^1(D)$ and $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(\Omega;\mathbb{R}^N)$ then $u_k \rightharpoonup u$ in $H^1(D)$, where $u \in \varphi + H^1_0(D)$ is the unique solution of $\Delta u + \operatorname{div} m = 0$ in D. Moreover, if $\{\nabla\varphi_k\}$ is 2-equi-integrable over D then $\{\nabla u_k\}$ is also 2-equi-integrable over D. Finally, if in addition we also assume that $m_k \to m$ strongly in $L^2(\Omega;\mathbb{R}^N)$ then $u_k \to$ strongly in $H^1(D)$.

Proof of Lemma 3. Since

$$\int_D \nabla u_k \cdot \nabla v \, dx = -\int_D m_k \cdot \nabla v \, dx \tag{2.11}$$

for all $v \in H_0^1(D)$, by taking as test functions $u_k - \varphi_k$ we obtain

$$\int_D |\nabla u_k|^2 \, dx = \int_D \nabla u_k \cdot \nabla \varphi_k \, dx - \int_D m_k \cdot \nabla (u_k - \varphi_k) \, dx.$$

Hölder's inequality now yields

$$||\nabla u_k||_{L^2(D)} \leq C \left(||\nabla \varphi_k||_{L^2(D)} + ||m_k||_{L^2(D)} \right). \tag{2.12}$$

Since $\varphi_k \rightharpoonup \varphi$ in $H^1(D)$, by (2.12) we have that

$$||\nabla u_k||_{L^2(D)} \leq C \left(\sup_j ||\nabla \varphi_j||_{L^2(D)} + |D| \right) < \infty,$$

which, together with Poincaré Inequality, implies that the sequence $\{u_k - \varphi_k\}$ is uniformly bounded in $H_0^1(D)$. Hence there exists $w \in H_0^1(D)$ such that (up to a subsequence) $u_k - \varphi_k \rightharpoonup w$ in $H^1(D)$. Since $\varphi_k \rightharpoonup \varphi$ in $H^1(D)$ and and $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(\Omega; \mathbb{R}^N)$ we have that $u_k \rightharpoonup u := \varphi + w$ in $H^1(D)$, and letting $k \to \infty$ in (2.11) we conclude that $\Delta u + \operatorname{div} m = 0$.

To prove the second part of the lemma assume that $\{\nabla \varphi_k\}$ is 2-equiintegrable over D. Then for any $\eta > 0$ there exists $\delta > 0$ such that

$$\int_E (|\nabla \varphi_k|^2 + 1) \, dx \leq \eta^2 / 4C_1^2, \tag{2.13}$$

for any measurable set $E \subset D$ with $|E| \leq \delta$, and where $C_1 := \sup_k \left(\int_D (1 + |\nabla u_k|^2) \, dx \right)^{1/2}$. Fix S and $D' \subset\subset D \setminus \overline{S} \subset\subset D$ such that $|D \setminus D'| \leq \delta$ and consider a cut-off function φ such that $\varphi \equiv 1$ in S and $\varphi \equiv 0$ in D'. Taking as a test function in (2.11) the function

$$v(x) := \varphi(x) \left(u_k(x) - \varphi_k(x) \right),$$

and using the fact that

$$\nabla v = (u_k - \varphi_k) \nabla \varphi + \varphi \nabla (u_k - \varphi_k),$$

we get

$$\begin{aligned}
\int_S |\nabla u_k|^2 \, dx \leq \int_D \nabla u_k \cdot \varphi \nabla u_k \, dx &= -\int_D m_k \cdot [(u_k - \varphi_k)\nabla \varphi + \varphi \nabla (u_k - \varphi_k)] \, dx \\
&\quad - \int_D \nabla u_k \cdot (u_k - \varphi_k) \nabla \varphi \, dx + \int_D \nabla u_k \cdot \varphi \nabla \varphi_k \, dx \\
&\leq ||\nabla \varphi||_{L^\infty} \int_{D \setminus D'} |u_k - \varphi_k| \, (1 + |\nabla u_k|) dx + \int_{D \setminus D'} (1 + |\nabla u_k|)(1 + |\nabla \varphi_k|) \, dx \\
&\leq 2 ||\nabla \varphi||_{L^\infty} \left(\int_{D \setminus D'} |u_k - \varphi_k|^2 \, dx \right)^{1/2} \left(\int_D (1 + |\nabla u_k|^2) \, dx \right)^{1/2} \\
&\quad + 2 \left(\int_{D \setminus D'} (1 + |\nabla \varphi_k|^2) \, dx \right)^{1/2} \left(\int_D (1 + |\nabla u_k|^2) \, dx \right)^{1/2} \\
&\leq 2C_1 ||\nabla \varphi||_{L^\infty} \left(\int_{D \setminus D'} |u_k - \varphi_k|^2 \, dx \right)^{1/2} + \eta
\end{aligned} \tag{2.14}$$

by (2.13). Since $u_k - \varphi_k \to 0$ in $L^2(D \setminus D')$ the right hand side of the previous inequality can be made smaller than 2η for all k large. By eventually replacing S with a smaller set we can assume that

$$\int_S |\nabla u_k|^2 \, dx \leq 2\eta \quad \text{for } all \ k.$$

To conclude the first part of the proof it remains to show the 2-equi-integrability of $\{\nabla u_k\}$ in $D \setminus S$. Since $D \setminus \overline{S} \subset\subset D$, by (2.3) for any $p > 2$

$$||\nabla u_k||_{L^p(D\setminus\overline{S})} \leq C(D \setminus \overline{S}, N, p) \left(||u_k||_{H^1(D)} + 1\right).$$

The right hand side of the previous inequality is bounded uniformly in k, consequently $\{\nabla u_k\}$ is uniformly bounded in $L^p(D \setminus \overline{S}; \mathbb{R}^N)$, and thus 2-equintegrable over $D \setminus \overline{S}$, and the claim is proved.

It remains to show that

$$\lim_{k\to\infty} ||\nabla u_k - \nabla u||_{L^2(D)} = 0 \tag{2.15}$$

if $m_k \to m$ in $L^2(\Omega; \mathbb{R}^N)$. From (2.11)

$$\int_D \nabla (u_k - u) \cdot \nabla v \, dx = -\int_D (m_k - m) \cdot \nabla v \, dx \tag{2.16}$$

for all $v \in H^1_0(D)$. Since the sequence $\{\nabla u_k\}$ is 2-equintegrable over D by the first part of the lemma, for any $\eta > 0$ there exists $\delta > 0$ such that

$$\int_E (|\nabla u_k|^2 + |\nabla u|^2) \, dx \leq \eta \tag{2.17}$$

for any measurable set $E \subset D$ with $|E| \leq \delta$. Let $D' := \{x \in D : \operatorname{dist}(x, \partial D) \geq \varepsilon\}$ where $\varepsilon > 0$ is chosen so small that $|D \setminus D'| \leq \delta$. Consider a smooth cut-off function $\varphi \in C^1_c(D)$ such that $\varphi \equiv 1$ in D'. Taking as a test function in (2.16) the function $v(x) := \varphi(x)(u_k(x) - u(x))$, and using the fact that

$$\nabla v = (u_k - u)\nabla\varphi + \varphi\nabla(u_k - u),$$

we get

$$\begin{aligned}\int_{D'} |\nabla(u_k - u)|^2 \, dx &\leq \int_D \nabla(u_k - u) \cdot \varphi\nabla(u_k - u) \, dx \\ &= -\int_D (m_k - m) \cdot [(u_k - u)\nabla\varphi + \varphi\nabla(u_k - u)] \, dx - \int_D \nabla(u_k - u) \cdot (u_k - u) \nabla\varphi \, dx \to 0\end{aligned}$$

as $k \to \infty$, since $u_k \rightharpoonup u$ in $H^1(D)$ and $m_k \to m$ in $L^2(\Omega; \mathbb{R}^N)$. Thus the right hand side of the previous inequality can be made smaller than η for all k large, which, together with (2.17), yields

$$\int_D |\nabla (u_k - u)|^2 \, dx \leq 3\eta \quad \text{for all } k \text{ large},$$

thus asserting (2.15). □

3. Relaxation

To prove Theorem 1 we introduce a *localization* of $\mathcal{F}$. Let $\mathcal{O}(\Omega)$ denote the class of all open subsets of Ω and for $m \in \mathcal{M}^*$ and $D \in \mathcal{O}(\Omega)$ set

$$\begin{aligned}\mathcal{F}(m; D) := \inf \Bigg\{ \liminf_{k\to\infty} \int_D g(x, m_k, \nabla u_k) \, dx : \; m_k \in \mathcal{M}^*, \quad |m_k| = 1 \text{ in } D, \quad m_k = m \text{ outside } D, \\ m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(D; \mathbb{R}^N), \quad u_k \in H^1\left(\mathbb{R}^N\right), \quad \Delta u_k + \operatorname{div} m_k = 0 \quad \text{in } \mathbb{R}^N \Bigg\},\end{aligned}$$

where $g : \mathbb{R}^{3N} \to [0, \infty]$ is the Carathéodory function defined as

$$g(x, \tilde{m}, h) := f(x, \tilde{m}, u(x), h), \tag{3.1}$$

and $u \in H^1\left(\mathbb{R}^N\right)$ is the unique solution of $\Delta u + \operatorname{div} m = 0$ in $\mathbb{R}^N$.

Lemma 4. *For all $m \in \mathcal{M}^*$ we have*

$$\mathcal{F}(m) = \mathcal{F}(m; \Omega) + \int_{\mathbb{R}^N\setminus\Omega} f(x, 0, u(x), \nabla u(x)) \, dx.$$

Proof of Lemma 4. Let $\{m_k\} \subset \mathcal{M}$ be any sequence such that $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(\Omega;\mathbb{R}^N)$, and let $u_k \in H^1(\mathbb{R}^N)$ be the unique solution of the equation $\Delta u_k + \operatorname{div} m_k = 0$ in $\mathbb{R}^N$. Assume, in addition, that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} f(x, m_k, u_k, \nabla u_k)\, dx = \lim_{k\to\infty} \int_{\mathbb{R}^N} f(x, m_k, u_k, \nabla u_k)\, dx.$$

By Lemma 2 $\{\nabla u_k\}$ is 2-equintegrable over $\mathbb{R}^N$ and

$$\lim_{k\to\infty} ||u_k - u||_{L^{2^*}(\mathbb{R}^N)} = \lim_{k\to\infty} ||\nabla u_k - \nabla u||_{L^2(\mathbb{R}^N\setminus\overline{\Omega})} = 0.$$

In view of the growth condition (1.6) we have that $\{f(\cdot, m_k(\cdot), u_k(\cdot), \nabla u_k(\cdot))\}$ is equintegrable over $\mathbb{R}^N$ and

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} f(x, m_k, u_k, \nabla u_k)\, dx = \int_{\mathbb{R}^N\setminus\overline{\Omega}} f(x, 0, u, \nabla u)\, dx + \lim_{k\to\infty} \int_{\Omega} f(x, m_k, u_k, \nabla u_k)\, dx.$$

Since $||u_k - u||_{L^\infty(\Omega)} \to 0$, by Proposition 1, and upon extracting a subsequence if necessary the two sequences

$$\{(m_k, u_k, \nabla u_k)\} \qquad \text{and} \qquad \{(m_k, u, \nabla u_k)\}$$

generate the same Young measure $\{\nu_x\}$. By Proposition 2 and Theorem 2(i) we deduce that

$$\begin{aligned}
\lim_{k\to\infty} \int_{\Omega} f(x, m_k, u_k, \nabla u_k)\, dx &= \lim_{k\to\infty} \int_{\Omega} \tilde{f}(x, m_k, u_k, \nabla u_k)\, dx \\
&= \int_{\Omega} \int_{\mathbb{R}^{2N}\times\mathbb{R}\times\mathbb{R}^N} \tilde{f}(x, M, U, H) d\nu_x(M, U, H) dx \\
&= \lim_{k\to\infty} \int_{\Omega} \tilde{f}(x, m_k, u, \nabla u_k)\, dx = \lim_{k\to\infty} \int_{\Omega} f(x, m_k, u, \nabla u_k)\, dx,
\end{aligned} \tag{3.2}$$

where we have used and the facts that $\{(m_k, u_k, \nabla u_k)\}$ and $\{(m_k, u, \nabla u_k)\}$ generate the same Young measure $\{\nu_x\}$, and that $\{\tilde{f}(\cdot, m_k(\cdot), u_k(\cdot), \nabla u_k(\cdot))\}$ and $\{\tilde{f}(\cdot, m_k(\cdot), u(\cdot), \nabla u_k(\cdot))\}$ are equi-integrable over Ω by (1.6) and by the 2-equi-integrability of the sequences $\{(u_k, \nabla u_k)\}$ and $\{(u, \nabla u_k)\}$ over Ω. In conclusion, we have shown that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} f(x, m_k, u_k, \nabla u_k)\, dx = \int_{\mathbb{R}^N\setminus\overline{\Omega}} f(x, 0, u, \nabla u)\, dx + \lim_{k\to\infty} \int_{\Omega} f(x, m_k, u, \nabla u_k)\, dx$$

and taking the infimum over all sequences $\{m_k\}$ as above we get

$$\mathcal{F}(m) = \int_{\mathbb{R}^N\setminus\overline{\Omega}} f(x, 0, u, \nabla u)\, dx + \mathcal{F}(m; \Omega).$$

□

Lemma 5. *For any $D \in \mathcal{O}(\Omega)$ and $m \in \mathcal{M}^*$ we have*

$$\mathcal{F}(m; D) = \mathcal{F}_0(m; D),$$

where

$$\begin{aligned}
\mathcal{F}_0(m; D) := \inf \Big\{ &\liminf_{k\to\infty} \int_D g(x, m_k, \nabla u_k)\, dx :\ m_k \in L^\infty(D; S^{N-1}), \quad m_k \overset{*}{\rightharpoonup} m \ \textit{in}\ L^\infty(D;\mathbb{R}^N), \\
&u_k \in u + H_0^1(D), \quad \Delta u_k + \operatorname{div} m_k = 0 \quad \textit{in}\ D \Big\}.
\end{aligned}$$

Proof of Lemma 5. Let

$$\{(m_k, u_k)\} \subset L^\infty(D; S^{N-1}) \times (u + H_0^1(D)),$$

be such that $\Delta u_k + \operatorname{div} m_k = 0$ in D, $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(D;\mathbb{R}^N)$, and

$$\lim_{k\to\infty} \int_D g(x, m_k(x), \nabla u_k(x))\, dx = \liminf_{k\to\infty} \int_D g(x, m_k(x), \nabla u_k(x))\, dx.$$

Extend m_k to be m outside D and let $U_k \in H^1(\mathbb{R}^N)$ be the unique solution of $\Delta U_k + \operatorname{div} m_k = 0$ in $\mathbb{R}^N$. Define $v_k := u_k - U_k$ and $\varphi_k := u - U_k$. Then

$$\begin{cases} \Delta v_k = 0 \text{ in } D, \\ v_k \in \varphi_k + H_0^1(D), \end{cases}$$

and by Lemmas 1 and 2 we have $\varphi_k \rightharpoonup 0$ in $H^1(D)$ with $\{\nabla \varphi_k\}$ 2-equi-integrable over D. Lemma 3 now yields

$$v_k \to 0 \text{ in } H_0^1(D), \text{ i.e. } \lim_{k\to\infty} ||u_k - U_k||_{H^1(D)} = 0. \tag{3.3}$$

Proceeding as in Lemma 4,

$$\liminf_{k\to\infty} \int_D g(x, m_k(x), \nabla u_k(x))\, dx = \liminf_{k\to\infty} \int_D g(x, m_k(x), \nabla U_k(x))\, dx \geq \mathcal{F}(m; D)$$

and thus

$$\mathcal{F}(m; D) \leq \mathcal{F}_0(m; D).$$

To prove the converse inequality let $m_k \in \mathcal{M}^*$, $|m_k| = 1$ in D, $m_k = m$ outside D, $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(D; \mathbb{R}^N)$. Let $U_k \in H^1(\mathbb{R}^N)$ be the unique solution of $\Delta U_k + \operatorname{div} m_k = 0$ in $\mathbb{R}^N$ and let $u_k \in u + H_0^1(D)$ be the unique solution of $\Delta u_k + \operatorname{div} m_k = 0$ in D. Upon extracting a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} \int_D g(x, m_k(x), \nabla U_k(x))\, dx = \liminf_{k\to\infty} \int_D g(x, m_k(x), \nabla U_k(x))\, dx$$

and then proceed as above. □

Remark 3. Note that $\mathcal{F}_0$(m;D) is well defined. Indeed it is well known that for any $D \in \mathcal{O}(\Omega)$, $u \in H^1(D)$ and $m \in L^2(D; \mathbb{R}^N)$, the generalized Dirichlet problem

$$\begin{cases} \Delta v + \operatorname{div} m = 0 \text{ in } D, \\ v - u \in H_0^1(D) \end{cases}$$

is uniquely solvable (see e.g. by Theorem 8.3 in [16]).

Lemma 6. *$\mathcal{F}_0(m; \cdot)$ is the trace of a Radon measure.*

Proof of Lemma 6. As it is usual, it suffices to prove subadditivity (see e.g [1, 14]), i.e.

$$\mathcal{F}_0(m; D) \leq \mathcal{F}_0(m; D \setminus \overline{E}) + \mathcal{F}_0(m; B)$$

if $E \subset\subset B \subset\subset D \subset \Omega$. Fix $\eta > 0$. By the definition of $\mathcal{F}_0(m; \cdot)$ there exist two sequences of pairs of functions

$$\{(m_k, u_k)\} \subset L^\infty(D \setminus \overline{E}; S^{N-1}) \times \left(u + H_0^1(D \setminus \overline{E})\right), \quad \Delta u_k + \operatorname{div} m_k = 0 \quad \text{in } D \setminus \overline{E},$$
$$\{(w_k, z_k)\} \subset L^\infty(B; S^{N-1}) \times \left(u + H_0^1(B)\right), \quad \Delta z_k + \operatorname{div} w_k = 0 \quad \text{in } B,$$

such that

$$m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(D \setminus \overline{E}; \mathbb{R}^N), \qquad w_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(B; \mathbb{R}^N)$$

and

$$\lim_{k\to\infty} \int_{D\setminus\overline{E}} g(x, m_k, \nabla u_k)\, dx \leq \mathcal{F}_0(m; D \setminus \overline{E}) + \eta, \quad \lim_{k\to\infty} \int_B g(x, w_k, \nabla z_k)\, dx \leq \mathcal{F}_0(m; B) + \eta. \tag{3.4}$$

Since by Lemma 1

$$\int_{B\setminus\overline{E}} (|\nabla u_k|^2 + |\nabla z_k|^2)\, dx \leq K,$$

where $K > 0$ is independent of k, we can find an open set S, with $E \subset\subset S \subset\subset B$ such that for all j sufficiently large

$$\int_{S_j} \left(a(x) + C(1 + |u|^{2^*} + |\nabla u_k|^2 + |\nabla z_k|^2)\right)\, dx < \eta, \tag{3.5}$$

where a and C are given in (1.6) and

$$S_j := \left\{ x \in S : \quad \frac{1}{j+1} \le \operatorname{dist}(x, \partial S) \le \frac{1}{j} \right\}.$$

Let θ_j be smooth cut-off functions with compact support in S, which are equal to 1 when $\operatorname{dist}(x, \partial S) \ge \frac{1}{j}$ and 0 when $\operatorname{dist}(x, \partial S) \le \frac{1}{j+1}$. For $x \in D$ set

$$\hat{M}_{j,k} := (1-\theta_j) m_k + \theta_j w_k, \qquad M_{j,k} := \pi(\hat{M}_{j,k}), \qquad \hat{U}_{j,k} := (1-\theta_j) u_k + \theta_j z_k,$$

where

$$\pi(m) := \begin{cases} m/|m| & \text{if } m \ne 0, \\ e_1 & \text{if } m = 0. \end{cases}$$

It is clear that $\hat{M}_{j,k} \overset{*}{\rightharpoonup} m$ in $L^\infty(D; \mathbb{R}^N)$ as $k \to \infty$ and $j \to \infty$. For j fixed

$$\begin{aligned} \Delta \hat{U}_{j,k} + \operatorname{div} \hat{M}_{j,k} =& (1-\theta_j)\left(\Delta u_k + \operatorname{div} m_k\right) + \theta_j \left(\Delta z_k + \operatorname{div} w_k\right) \\ &+ \nabla \theta_j \cdot (-2\nabla u_k + 2\nabla z_k - m_k + h_k) + (-u_k + z_k)\Delta\theta_j \\ =& \nabla \theta_j \cdot (-2\nabla u_k + 2\nabla z_k - m_k + h_k) + (-u_k + z_k)\Delta\theta_j \to 0 \end{aligned} \tag{3.6}$$

in $H^{-1}(D)$ strong, because $m_k - h_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(B \backslash \overline{E}; \mathbb{R}^N)$ and $u_k - z_k \rightharpoonup 0$ in $H^1(B \backslash \overline{E})$. Let $U_{j,k} \in u + H^1_0(D)$ be the solutions of $\Delta U_{j,k} + \operatorname{div} M_{j,k} = 0$ in D. For any $p \ge 1$

$$\int_D |\hat{M}_{j,k} - M_{j,k}|^p \, dx \le 2^p |S_j| \to 0 \qquad \text{as } j \to \infty,$$

hence, by (3.6),

$$\lim_{j\to\infty} \lim_{k\to\infty} \Delta \left(\hat{U}_{j,k} - U_{j,k} \right) = \lim_{j\to\infty} \lim_{k\to\infty} \operatorname{div} \left(-\hat{M}_{j,k} + M_{j,k} \right) = 0 \quad \text{ in } H^{-1}(D),$$

and thus, since $\hat{U}_{j,k} - U_{j,k} \in H^1_0(D)$,

$$\lim_{j\to\infty} \lim_{k\to\infty} ||\nabla \left(\hat{U}_{j,k} - U_{j,k} \right)||_{L^2(D;\mathbb{R}^N)} = 0.$$

Moreover

$$\begin{aligned} \int_D g(x, M_{j,k}, \nabla \hat{U}_{j,k}) \, dx \le& \int_{D\backslash\overline{E}} g(x, m_k, \nabla u_k) \, dx + \int_B g(x, w_k, \nabla z_k) \, dx \\ &+ \int_{S_j} \left(a(x) + C(1 + |u|^{2^*} + |\nabla \hat{U}_{j,k}|^2) \right) \, dx \\ \le& \int_{D\backslash\overline{E}} g(x, m_k, \nabla u_k) \, dx + \int_B g(x, w_k, \nabla z_k) \, dx \\ &+ \int_{S_j} \left(a(x) + C(1 + |u|^{2^*} + |\nabla u_k|^2 + |\nabla z_k|^2) \right) \, dx + C \int_{S_j} |\nabla\theta_j|^2 |u_k - z_k|^2) \, dx \end{aligned}$$

where we have used (1.6). If we now let $k \to \infty$ and then $j \to \infty$ and use (3.4) and (3.5), we get

$$\liminf_{j\to\infty} \liminf_{k\to\infty} \int_D g(x, M_{j,k}, \nabla \hat{U}_{j,k}) \, dx \le \mathcal{F}_0(m; D \setminus \overline{E}) + \mathcal{F}_0(m; B) + 3\eta.$$

Diagonalize to get

$$M_j \overset{*}{\rightharpoonup} m \quad \text{in } L^\infty(D; \mathbb{R}^N), \quad ||\hat{U}_j - U_j||_{H^1(D)} \to 0, \quad \Delta U_j + \operatorname{div} M_j = 0 \quad \text{ in } D, \tag{3.7}$$

and

$$\liminf_{j\to\infty} \int_D g(x, M_j, \nabla \hat{U}_j) \, dx \le \mathcal{F}_0(m; D \setminus \overline{E}) + \mathcal{F}_0(m; B) + 3\eta. \tag{3.8}$$

By Proposition 1, the two sequences

$$(M_j, \nabla \hat{U}_j) \qquad \text{and} \qquad (M_j, \nabla U_j)$$

generate the same Young measure $\{\nu_x\}$, and thus we can proceed as in Lemma 4 to conclude that

$$\begin{aligned}\mathcal{F}_0(m;D) &\le \liminf_{j\to\infty}\int_D g(x,M_j,\nabla U_j)\,dx = \liminf_{j\to\infty}\int_D g(x,M_j,\nabla \hat{U}_j)\,dx\\ &= \mathcal{F}_0(m;D\setminus\overline{E}) + \mathcal{F}_0(m;B) + 3\eta.\end{aligned}$$

It now suffices to let $\eta\to 0^+$. $\square$

Since

$$g:\Omega\times\left(S^{N-1}\cup\{0\}\right)\times\mathbb{R}^N\to[0,\infty)$$

is Carathéodory, by Theorem 4 for each $j\in$ there exists a compact set $K_j\subset\Omega$, with $|\Omega\backslash K_j|\le 1/j$, such that $g:K_j\times\left(S^{N-1}\cup\{0\}\right)\times\mathbb{R}^N\to[0,\infty)$ is continuous. Let K_j^* be the set of Lebesgue points of χ_{K_j}, and set

$$\omega := \bigcup_j^\infty (K_j\cap K_j^*). \tag{3.9}$$

Then

$$|\Omega\backslash\omega| \le |\Omega\backslash K_j| \le \frac{1}{j}\to 0 \quad \text{as } j\to\infty.$$

Lemma 7. *For $\mathcal{L}^N$ a.e. $x_0\in\Omega$ we have*

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \ge \hat{g}(x_0,m(x_0),\nabla u(x_0))$$

where the function $\hat{g}=\hat{g}(x,\tilde{m},h)$ coincides with g when $|\tilde{m}|=1$, while for $|\tilde{m}|<1$ it is given by

$$\begin{aligned}\hat{g}(x,\tilde{m},h) := \inf\Bigg\{&\liminf_{k\to\infty}\frac{1}{|B|}\int_B g(x,m_k(y),h+\nabla u_k(y))\,dy :\ m_k\in L^\infty(B;S^{N-1}),\\ &m_k\overset{*}{\rightharpoonup}\tilde{m} \text{ in } L^\infty(B;\mathbb{R}^N),\quad u_k\in H_0^1(B),\quad \Delta u_k+\operatorname{div} m_k = 0 \quad \text{in } B\Bigg\}.\end{aligned}$$

Proof of Lemma 7. Fix $x_0\in\omega$, where ω is defined in (3.9), such that

$$\begin{aligned}&\lim_{r\to 0^+}\frac{1}{r^N}\int_{B(x_0,r)}|m(x)-m(x_0)|\,dx = \lim_{r\to 0^+}\frac{1}{r^N}\int_{B(x_0,r)}|\nabla u(x)-\nabla u(x_0)|^2\,dx = 0,\\ &\lim_{r\to 0^+}\frac{1}{r^{N+2}}\int_{B(x_0,r)}|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|^2\,dx = 0,\end{aligned} \tag{3.10}$$

and

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\frac{\mathcal{F}_0(m;B(x_0,r))}{|B(x_0,r)|} < \infty,$$

where we have chosen the radii $r\to 0^+$ such that $\mathcal{F}_0(m;\partial B(x_0,r))=0$, and we have used Lemma 6. For each $r>0$ fixed let

$$\{(m_k,u_k)\}\subset L^\infty(B(x_0,r);S^{N-1})\times(u+H_0^1(B(x_0,r)))$$

be such that $\Delta u_k+\operatorname{div} m_k=0$ in $B(x_0,r)$ and

$$m_k\overset{*}{\rightharpoonup} m \quad \text{in} \quad L^\infty(B(x_0,r);\mathbb{R}^N),\qquad \lim_{k\to\infty}\int_{B(x_0,r)} g(x,m_k,\nabla u_k)\,dx \le \mathcal{F}_0(m;B(x_0,r)) + r^{N+1}. \tag{3.11}$$

We have

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \ge \limsup_{r\to 0^+}\lim_{k\to\infty}\frac{1}{|B|}\int_B g(x_0+ry,m_{r,k}(y),\nabla u_{r,k}(y))\,dy,$$

where

$$m_{r,k}(y) := m_k(x_0+ry) \quad \text{and} \quad u_{r,k}(y) := \frac{u_k(x_0+ry)-u(x_0)}{r}.$$

We claim that $u_{r,k} \rightharpoonup u_0(y) := \nabla u(x_0)\, y$ in $H^1(B)$ and $m_{r,k} \overset{*}{\rightharpoonup} m(x_0)$ in $L^\infty(B;\mathbb{R}^N)$ if we first let $k \to \infty$ and then $r \to 0^+$. Indeed for any fixed $\varphi \in L^\infty(B;\mathbb{R}^N)$ belonging to a dense set of $L^1(B;\mathbb{R}^N)$

$$\left|\int_B \varphi(y)\,(m_{r,k}(y) - m(x_0))\,dy\right| \le \left|\frac{1}{r^N}\int_{B(x_0,r)} \varphi\left(\frac{x-x_0}{r}\right)(m_k(x) - m(x))\,dx\right| + C\,\|\varphi\|_{L^\infty}\,\frac{1}{r^N}\int_{B(x_0,r)} |m(x) - m(x_0)|\,dx.$$

If we now let $k \to \infty$ the first integral tends to zero, since $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(B(x_0,r);\mathbb{R}^N)$, while for the second integral we use $(3.10)_1$. The separability property of $L^1(B;\mathbb{R}^N)$ now yields $m_{r,k} \overset{*}{\rightharpoonup} m(x_0)$ in $L^\infty(B;\mathbb{R}^N)$.

To prove the second claim observe that by Lemma 3 $u_k \rightharpoonup u$ in $H^1(B(x_0,r))$ for any fixed $r > 0$. Hence, also by $(3.10)_3$,

$$\lim_{r\to 0^+}\lim_{k\to\infty} ||u_{r,k} - u_0||_{L^2(B)} = 0.$$

Let $\varphi \in L^2(B;\mathbb{R}^N)$. Using Holder's inequality and then making a change of variables we get

$$\begin{aligned}\left|\int_B \varphi(y)\,(\nabla u_{r,k}(y) - \nabla u(x_0))\,dy\right| &\le \left|\int_B \varphi(y)(\nabla u_k(x_0+ry) - \nabla u(x_0+ry))\,dy\right| \\ &\quad + \left|\int_B \varphi(y)(\nabla u(x_0+ry) - \nabla u(x_0))\,dy\right| \\ &\le \left|\frac{1}{r^N}\int_{B(x_0,r)} \varphi\left(\frac{x-x_0}{r}\right)(\nabla u_k(x) - \nabla u(x))\,dx\right| \\ &\quad + ||\varphi||_{L^2(B)}\left(\frac{1}{r^N}\int_{B(x_0,r)} |\nabla u(x) - \nabla u(x_0)|^2\,dx\right)^{1/2}.\end{aligned}$$

If we now let $k \to \infty$ the first integral tends to zero, since $\nabla u_k \rightharpoonup \nabla u(x_0)$ in $L^2(B(x_0,r);\mathbb{R}^N)$. The claim then follows by letting $r \to 0^+$, by $(3.10)_2$, and in view of the separability of $L^2(B;\mathbb{R}^N)$.

Moreover the sequence

$$\{(m_{r,k}, u_{r,k})\} \subset L^\infty(B;S^{N-1}) \times (u_r + H_0^1(B))$$

is such that

$$\Delta u_{r,k} + \operatorname{div} m_{r,k} = 0 \quad \text{in } B,$$

where

$$u_r(y) := \frac{u(x_0+ry) - u(x_0)}{r}.$$

Diagonalize to get subsequences $\hat{m}_j := m_{r_j,k_j}$ and $\hat{u}_j := u_{r_j,k_j}$ with

$$\{(\hat{m}_j, \hat{u}_j)\} \subset L^\infty(B;S^{N-1}) \times (u_{r_j} + H_0^1(B)), \quad \Delta \hat{u}_j + \operatorname{div} \hat{m}_j = 0 \text{ in } B,$$

and such that and

$$\begin{aligned}&\hat{m}_j \overset{*}{\rightharpoonup} m(x_0) \quad \text{in} \quad L^\infty(B;\mathbb{R}^N), \quad \hat{u}_j \rightharpoonup u_0 \quad \text{in} \quad H^1(B),\\ &\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \ge \lim_{j\to\infty}\frac{1}{|B|}\int_B g(x_0 + r_j y, \hat{m}_j(y), \nabla\hat{u}_j(y))\,dy.\end{aligned} \tag{3.12}$$

We now distinguish two cases:

Case 1: Assume first that $|m(x_0)| = 1$. Then for any $p > 1$

$$\int_B |\hat{m}_j(y)|^p\,dy = \int_B |m(x_0)|^p\,dy = 1$$

and thus, since $L^p(B)$ is uniformly convex, $\hat{m}_j$ converges *strongly* to $m(x_0)$ in $L^p(B;\mathbb{R}^N)$. By Lemma 3 and in view of the fact that $\nabla u_{r_j} \to \nabla u(x_0)$ in L^2 by (3.10), $\hat{u}_j$ converges *strongly* to u_0 in $H^1(B)$, and we conclude that

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{j\to\infty}\frac{1}{|B|}\int_B g(x_0 + r_j y, \hat{m}_j(y), \nabla\hat{u}_j(y))\,dy = g(x_0, m(x_0), \nabla u(x_0)).$$

This completes the proof in the case $|m(x_0)| = 1$.
CASE 2: If $|m(x_0)| < 1$ we claim that

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \geq \lim_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla \hat{u}_j(y))\, dy. \tag{3.13}$$

By Lemma 3 and since $\{\nabla u_{r_j}\}$ is 2-equi-integrable, we deduce that the sequence $\{\nabla \hat{u}_j\}$ is also 2-equi-integrable over the set B, and so there exists $\delta > 0$ such that

$$\int_E \left(a(x_0 + r_j y) + a(x_0) + C(1 + |u(x_0 + r_j y)|^{2^*} + |u(x_0)|^{2^*} + |\nabla \hat{u}_j(y)|^2)\right) dy < \eta \tag{3.14}$$

for any measurable set $E \subset B$, with $|E| < \delta$, and where a and C are given in (1.6), and there is an $L > 0$ such that

$$|B \setminus E_j| \leq \delta, \quad \text{where} \quad E_j := \{y \in B : |u(x_0 + r_j y)| \leq L, \quad |\nabla \hat{u}_j(y)| \leq L\}. \tag{3.15}$$

From the definition of the set ω there exists an integer l such that $x_0 \in K_l \cap K_l^*$. Since

$$g : K_l \times \left(S^{N-1} \cup \{0\}\right) \times \overline{B(0,L)} \to [0, \infty)$$

is uniformly continuous there exists $\rho > 0$ such that

$$|g(x, m, h) - g(x_1, m, h)| \leq \eta \tag{3.16}$$

for all x, $x_1 \in K_l$ with $|x - x_1| \leq \rho$ and all $(m, h) \in \left(S^{N-1} \cup \{0\}\right) \times \overline{B\,(0, L)}$. By (3.12), (3.15) and (3.16)

$$\begin{aligned}
\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) &\geq \lim_{j\to\infty} \frac{1}{|B|}\int_B g(x_0 + r_j y, \hat{m}_j(y), \nabla \hat{u}_j(y))\, dy \\
&= \lim_{j\to\infty} \frac{1}{|B(x_0, r_j)|}\int_{B(x_0,r_j)} g(x, \hat{m}_j((x - x_0)/r_j), \nabla \hat{u}_j((x - x_0)/r_j))\, dx \\
&\geq \limsup_{j\to\infty} \frac{1}{|B(x_0, r_j)|}\int_{(x_0+r_jE_j)\cap K_l} g(x, \hat{m}_j((x - x_0)/r_j), \nabla \hat{u}_j((x - x_0)/r_j))\, dx \\
&\geq \limsup_{j\to\infty} \frac{1}{|B(x_0, r_j)|}\int_{(x_0+r_jE_j)\cap K_l} g(x_0, \hat{m}_j((x - x_0)/r_j), \nabla \hat{u}_j((x - x_0)/r_j))\, dx - \eta.
\end{aligned} \tag{3.17}$$

Now, since $|\nabla \hat{u}_j(y)| \leq L$ for $y \in E_j$, by (1.6) we have that

$$\begin{aligned}
&\frac{1}{|B(x_0, r_j)|}\int_{(x_0+r_jE_j)\setminus K_l} g(x_0, \hat{m}_j((x - x_0)/r_j), \nabla \hat{u}_j((x - x_0)/r_j))\, dx \\
&\qquad\qquad \leq \left(a(x_0) + C(1 + |u(x_0)|^{2^*} + L^2)\right) \frac{|B(x_0, r_j) \setminus K_l|}{|B(x_0, r_j)|} \to 0
\end{aligned}$$

as $j \to \infty$, where we have used the fact that x_0 is a Lebesgue point of χ_{K_l}. Consequently, from (3.17) we get

$$\begin{aligned}
\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) &\geq -\eta + \limsup_{j\to\infty} \frac{1}{|B|}\int_{E_j} g(x_0, \hat{m}_j(y), \nabla \hat{u}_j(y))\, dy \\
&\geq -2\eta + \limsup_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla \hat{u}_j(y))\, dy
\end{aligned}$$

by (1.6) and (3.14). By letting $\eta \to 0^+$ we obtain that

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \geq \limsup_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla \hat{u}_j(y))\, dy.$$

Thus claim (3.13) follows and hence

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) \geq \limsup_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla u(x_0) + \nabla \hat{v}_j(y))\, dy,$$

where $\hat{v}_j(y) := \hat{u}_j(y) - u_0(y)$. Let $v_j \in H_0^1(B)$ be the unique solution of $\Delta v_j + \operatorname{div} \hat{m}_j = 0$ in B. Since

$$\begin{cases} \Delta\,(\hat{v}_j - v_j) = 0 & \text{in } B, \\ \hat{v}_j - v_j = \hat{u}_j - u_0 & \text{on } \partial B, \end{cases}$$

as $\hat{u}_j - u_0 \rightharpoonup 0$ in $H^1(B)$ and $\{\nabla \hat{u}_j\}$ is 2-equi-integrable over the set B, it follows from Lemma 3 that $||\hat{v}_j - v_j||_{H^1(B)} \to 0$, and so

$$\begin{aligned}\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) &\geq \limsup_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla u(x_0) + \nabla \hat{v}_j(y))\, dy \\ &= \limsup_{j\to\infty} \frac{1}{|B|}\int_B g(x_0, \hat{m}_j(y), \nabla u(x_0) + \nabla v_j(y))\, dy.\end{aligned}$$

□

Lemma 8. *Let $\tilde{m}$, $M \in \mathcal{M}$ be piecewise constant functions and $\lambda \in (0,1)$. Then there exists a sequence $\{m_n\} \subset \mathcal{M}$ such that $m_n \overset{*}{\rightharpoonup} \lambda\tilde{m} + (1-\lambda)M$ in $L^\infty(B;\mathbb{R}^N)$, $\nabla u_n \to \lambda\nabla\tilde{u} + (1-\lambda)\nabla U$ strongly in $L^2(B;\mathbb{R}^N)$ and*

$$\begin{aligned}\lim_{n\to\infty}\int_B g(x, m_n(y), h + \nabla u_n(y))\, dy &= \lambda \int_B g(x, \tilde{m}(y), h + \lambda\nabla\tilde{u}(y) + (1-\lambda)\nabla U(y))\, dy \\ &+(1-\lambda)\int_B g(x, M(y), h + \lambda\nabla\tilde{u}(y) + (1-\lambda)\nabla U(y))\, dy,\end{aligned}$$

where $\tilde{u}$, U, $u_n \in H_0^1(B)$ are the unique solutions of $\Delta\tilde{u} + \operatorname{div}\tilde{m} = 0$, $\Delta U + \operatorname{div} M = 0$, $\Delta u_n + \operatorname{div} m_n = 0$, respectively.

Proof of Lemma 8. Consider a partition of the unit ball B into a finite number of disjoint subsets B_i, $i = 1, \ldots, L$, such that $\tilde{m}$ and M are constants on each set B_i, say $\tilde{m}(x) \equiv m_i$ and $M(x) \equiv M_i$ in B_i. Using an idea of De Simone [11], we let $p_i \in S^{N-1}$ be such that

$$p_i \cdot (m_i - M_i) = 0, \tag{3.18}$$

and consider a 1-periodic function θ such that

$$\theta(t) := \begin{cases} +1 & t \in [0,\lambda] \\ -1 & t \in (\lambda, 1]. \end{cases}$$

Define

$$B_{i,n}^{\pm} := \{y \in B_i : \theta(ny\cdot p_i) = \pm 1\}, \qquad m_n(y) := \begin{cases} m_i & y \in B_{i,n}^{+}, \\ M_i & y \in B_{i,n}^{-}, \end{cases}$$

$i = 1, \ldots, L$. Then $m_n \overset{*}{\rightharpoonup} \lambda m_i + (1-\lambda)M_i$ in $L^\infty(B_i;\mathbb{R}^N)$ as $n \to \infty$, for each $i = 1, \ldots, L$, and, in turn, $m_n \overset{*}{\rightharpoonup} \lambda\tilde{m} + (1-\lambda)M$ in $L^\infty(B;\mathbb{R}^N)$. Let $m_{n,i}(y) := m_n(y) - (\lambda m_i + (1-\lambda)M_i)$ in B_i and $v_{n,i} \in H_0^1(B)$ be the unique solution of $\Delta v_{n,i} + \operatorname{div}(\chi_{B_i} m_{n,i}) = 0$. We claim that $\nabla v_{n,i} \to 0$ in $L^2(B;\mathbb{R}^N)$. Integration by parts gives

$$\begin{aligned}\int_B |\nabla v_{n,i}(y)|^2 dy &= -\int_{B_i} \nabla v_{n,i}(y)\cdot m_{n,i}\, dy \\ &= -\int_{\partial B_{i,n}^-} v_{n,i} m_{n,i}\cdot \nu_{i,n}^-\, d\mathcal{H}^{N-1} - \int_{\partial B_{i,n}^+} v_{n,i} m_{n,i}\cdot\nu_{i,n}^+\, d\mathcal{H}^{N-1} \\ &= -\int_{\partial B_i}\left(\chi_{\partial B_{i,n}^-} v_{n,i} m_{n,i}\cdot\nu_i^- + \chi_{\partial B_{i,n}^+} v_{n,i} m_{n,i}\cdot \nu_i^+\right) d\mathcal{H}^{N-1},\end{aligned} \tag{3.19}$$

where we have used the fact that, by construction of the sets $B_{i,n}^{\pm}$, on $(\partial B_{i,n}^- \cap \partial B_{i,n}^-) \setminus \partial B_i$ we have $\nu_{i,n}^+ = -\nu_{i,n}^- = p_i$ so that

$$\begin{aligned}\int_{\partial B_{i,n}^-\setminus\partial B_i} v_{n,i} m_{n,i}\cdot\nu_{i,n}^-\, d\mathcal{H}^{N-1} + \int_{\partial B_{i,n}^+\setminus\partial B_i} v_{n,i} m_{n,i}\cdot\nu_{i,n}^+\, d\mathcal{H}^{N-1} \\ = \int_{\partial B_{i,n}^+\setminus\partial B_{i,n}^-} v_{n,i}(m_i - M_i)\cdot p_i\, d\mathcal{H}^{N-1} = 0\end{aligned}$$

by (3.18). Since $\chi_{B_i} m_{n,i} \overset{*}{\rightharpoonup} 0$ in $L^\infty(B;\mathbb{R}^N)$ as $n \to \infty$, by Lemma 3 we have that $v_{n,i} \rightharpoonup 0$ in $H_0^1(B)$ and thus $v_{n,i} \to 0$ strongly in $L^2(\partial B_i)$. Hence, by (3.19) $\nabla v_{n,i} \to 0$ in $L^2(B;\mathbb{R}^N)$ and the claim is asserted.

Let $u_n \in H^1_0(B)$ be the unique solution of $\Delta u_n + \mathrm{div} m_n = 0$. By linearity we can write

$$u_n(y) = \lambda\, \tilde{u}(y) + (1-\lambda)U(y) + \sum_{i=1}^{L} v_{n,i},$$

and so $\nabla u_n \to \lambda \nabla \tilde{u} + (1-\lambda)\nabla U$ strongly in $L^2(B;\mathbb{R}^N)$. Hence, also by (1.6), the sequence $\{g(x, m_n(\cdot), h + \nabla u_n(\cdot))\}$ is equi-integrable and thus by Theorem 2(ii) and Proposition 1, extracting a subsequence if necessary, we have that

$$\lim_{n\to\infty} \int_{B_i} g(x, m_n(y), h + \nabla u_n(y))\, dy = \lim_{n\to\infty} \int_{B_i} g(x, m_n(y), h + \lambda \nabla u(y) + (1-\lambda)\nabla U(y))\, dy.$$

To conclude the proof it suffices to observe that

$$\int_B g(x, m_n(y), h + \lambda \nabla \tilde{u}(y) + (1-\lambda)\nabla U(y))\, dy = \int_B \chi_{\cup_i B^+_{i,n}}(y) g(x, \tilde{m}(y), h + \lambda \nabla \tilde{u}(y) + (1-\lambda)\nabla U(y))\, dy$$
$$+ \int_B \chi_{\cup_i B^-_{i,n}}(y) g(x, M(y), h + \lambda \nabla \tilde{u}(y) + (1-\lambda)\nabla U(y))\, dy,$$

where

$$\chi_{\cup_i B^+_{i,n}} \overset{*}{\rightharpoonup} \lambda, \qquad \chi_{\cup_i B^-_{i,n}} \overset{*}{\rightharpoonup} (1-\lambda) \qquad \text{in } L^\infty(B;\mathbb{R}^N).$$

□

Lemma 9. $\hat{g}(x,\cdot,\cdot)$ *is upper semicontinuos and satisfies the growth condition*

$$0 \le \hat{g}(x, \tilde{m}, h) \le a(x) + C(|u(x)|^{2^*} + |h|^2) \tag{3.20}$$

for $\mathcal{L}^N$ *a.e.* $x \in \mathbb{R}^N$ *and all* $(\tilde{m}, h) \in \times \overline{B(0,1)} \times \mathbb{R}^N$*, where* $a \in L^1(\mathbb{R}^N)$ *and* $C > 0$ *are given in* (1.6).

Proof of Lemma 9. For $\mathcal{L}^N$ a.e. $x \in \Omega$ and every $(\tilde{m}, h) \in \overline{B(0,1)} \times \mathbb{R}^N$ and $\varepsilon > 0$, consider a sequence $\{(m_k, u_k)\} \subset L^\infty(B; S^{N-1}) \times H^1_0(B)$ such that $m_k \overset{*}{\rightharpoonup} \tilde{m}$ in $L^\infty(B;\mathbb{R}^N)$, $\Delta u_k + \mathrm{div}\, m_k = 0$, and

$$\lim_{k\to\infty} \frac{1}{|B|} \int_B g(x, m_k(y), h + \nabla u_k(y))\, dy \le \hat{g}(x, \tilde{m}, h) + \varepsilon. \tag{3.21}$$

By a density argument we may assume that the functions m_k are piecewise constant. Note that by Lemma 3

$$u_k \rightharpoonup 0 \text{ in } H^1(B) \text{ and } \{\nabla u_k\} \text{ is 2-equi-integrable over } B. \tag{3.22}$$

Let $\{(\hat{m}_j, h_j)\} \subset \overline{B(0,1)} \times \mathbb{R}^N$ be any fixed sequence of vectors converging to $(\tilde{m}, h)$. Without loss of generality we may assume that

$$\limsup_{j\to\infty} \hat{g}(x, \hat{m}_j, h_j) = \lim_{j\to\infty} \hat{g}(x, \hat{m}_j, h_j).$$

Step 1: Suppose that all the $\hat{m}_j$ belong to the segment with endpoints $\pi(\tilde{m})$ and $-\pi(\tilde{m})$. Then there exist $\lambda_j \in [0,1]$ such that

$$\lambda_j \to 1 \quad \text{and} \quad \hat{m}_j = \lambda_j\, \tilde{m} + (1-\lambda_j)\pi_\pm(\tilde{m}),$$

where $\pi_\pm(\tilde{m})$ is either $\pi(\tilde{m})$ or $-\pi(\tilde{m})$. If there exists a subsequence $\{\hat{m}_{j_h}\}$ of $\{\hat{m}_j\}$ such that $|\hat{m}_{j_h}| = 1$ then $\lambda_{j_h} = 1$, and so $|\tilde{m}| = 1$ and we have

$$\limsup_{j\to\infty} \hat{g}(x, \hat{m}_j, h_j) = \lim_{h\to\infty} \hat{g}(x, \hat{m}_{j_h}, h_{j_h}) = \lim_{h\to\infty} g(x, \hat{m}_{j_h}, h_{j_h}) = g(x, \tilde{m}, h) = \hat{g}(x, \tilde{m}, h).$$

If $|\hat{m}_j| < 1$ for all j (but a finite number) then by Lemma 8 we may find a sequence $\{m_{n,k,j}\} \subset \mathcal{M}$ such that $m_{n,k,j} \overset{*}{\rightharpoonup} \lambda_j m_k + (1-\lambda_j)\pi_\pm(\tilde{m})$ in $L^\infty(B;\mathbb{R}^N)$, $\nabla u_{n,k,j} \to \lambda_j \nabla u_k$ strongly in $L^2(B;\mathbb{R}^N)$, and

$$\lim_{n\to\infty} \int_B g(x, m_{n,k,j}(y), h_j + \nabla u_{n,k,j}(y))\, dy = \lambda_j \int_B g(x, m_k(y), h_j + \lambda_j \nabla u_k(y))\, dy$$
$$+ (1-\lambda_j) \int_B g(x, \pi_\pm(\tilde{m}), h_j + \lambda_j \nabla u_k(y))\, dy,$$

where $u_{n,k,j} \in H_0^1(B)$ is the unique solution of $\Delta u_{n,k,j} + \operatorname{div} m_{n,k,j} = 0$. Since $m_{n,k,j} \overset{*}{\rightharpoonup} \lambda_j \tilde{m} + (1 - \lambda_j)\pi_\pm(\tilde{m}) = \hat{m}_j$ in $L^\infty(B;\mathbb{R}^N)$ as $n \to \infty$ and $k \to \infty$, in this order, we have

$$\begin{aligned}\hat{g}(x, \hat{m}_j, h_j) \leq &\liminf_{k\to\infty} \lim_{n\to\infty} \frac{1}{|B|} \int_B g(x, m_{n,k,j}(y), h_j + \nabla u_{n,k,j}(y))\, dy \\ \leq &\lambda_j \liminf_{k\to\infty} \frac{1}{|B|} \int_B g(x, m_k(y), h_j + \lambda_j \nabla u_k(y))\, dy \\ &+ (1 - \lambda_j) \limsup_{k\to\infty} \frac{1}{|B|} \int_B g(x, \pi_\pm(\tilde{m}), h_j + \lambda_j \nabla u_k(y))\, dy,\end{aligned}$$

and, in turn,

$$\limsup_{j\to\infty} \hat{g}(x, \hat{m}_j, h_j) \leq \limsup_{j\to\infty} \liminf_{k\to\infty} \frac{1}{|B|} \int_B g(x, m_k(y), h_j + \lambda_j \nabla u_k(y))\, dy, \tag{3.23}$$

where we have used (1.6), (3.22) and the fact that $\lambda_j \to 1$. By (3.22) and in view of

$$\limsup_{j\to\infty} \liminf_{k\to\infty} \|h_j + \lambda_j \nabla u_k(y) - (h + \nabla u_k(y))\|_{L^2(B;\mathbb{R}^N)} = 0,$$

we may conclude, as in the proof of Lemma 4 that

$$\limsup_{j\to\infty} \liminf_{k\to\infty} \frac{1}{|B|} \int_B g(x, m_k(y), h_j + \lambda_j \nabla u_k(y))\, dy = \liminf_{k\to\infty} \frac{1}{|B|} \int_B g(x, m_k(y), h + \nabla u_k(y))\, dy.$$

Hence, by (1.6), (3.21), (3.23),

$$\begin{aligned}\limsup_{j\to\infty} \hat{g}(x, \hat{m}_j, h_j) &\leq \limsup_{j\to\infty} \liminf_{k\to\infty} \frac{1}{|B|} \int_{E_k} g(x, m_k(y), h_j + \lambda_j \nabla u_k(y))\, dy + \varepsilon \\ &\leq \liminf_{k\to\infty} \frac{1}{|B|} \int_{E_k} g(x, m_k(y), h + \nabla u_k(y))\, dy + 2\varepsilon \\ &\leq \hat{g}(x, \tilde{m}, h) + 3\varepsilon.\end{aligned}$$

By letting $\varepsilon \to 0^+$ we obtain upper semicontinuity.

Step 2: In the general case, given a sequence of vectors $\{(m_j, h_j)\} \subset \overline{B(0,1)} \times \mathbb{R}^N$ converging to $(\tilde{m}, h)$, we rotate m_j of an angle α_j until it intersects the segment of end points $\pi(\tilde{m})$ and $-\pi(\tilde{m})$ and we denote by $\hat{m}_j$ the point of intersection. Then $|m_j - \hat{m}_j| \to 0$ as $j \to \infty$ (note that when $\tilde{m} = 0$ this follows from the fact that $|m_j| = |\hat{m}_j| \to 0$). Applying Step 1 we can find a sequence $\{m_{n,j}\} \subset \mathcal{M}$ such that $m_{n,j} \overset{*}{\rightharpoonup} \hat{m}_j$ in $L^\infty(B;\mathbb{R}^N)$ as $n \to \infty$, and

$$\limsup_{j\to\infty} \lim_{n\to\infty} \frac{1}{|B|} \int_B g(x, m_{n,j}(y), h_j + \nabla u_{n,j}(y))\, dy \leq \hat{g}(x, \tilde{m}, h),$$

where $u_{n,j} \in H_0^1(B)$ is the solution of $\Delta u_{n,j} + \operatorname{div} m_{n,j} = 0$. By rotating (backwards) $m_{n,j}$ of an angle $-\alpha_j$ we obtain a new sequence $\{\tilde{m}_{n,j}\} \subset \mathcal{M}$ such that $\tilde{m}_{n,j} \overset{*}{\rightharpoonup} m_j$ in $L^\infty(B;\mathbb{R}^N)$ as $n \to \infty$ and $||\tilde{m}_{n,j} - m_{n,j}||_\infty \to 0$ as $j \to \infty$ *uniformly* in n. Hence by Lemma 3 $\|\nabla \tilde{u}_{n,j} - \nabla u_{n,j}\|_{L^2(B)} \to 0$, where $\tilde{u}_{n,j} \in H_0^1(B)$ is the solution of $\Delta \tilde{u}_{n,j} + \operatorname{div} \tilde{m}_{n,j} = 0$, and so

$$\begin{aligned}\limsup_{j\to\infty} \hat{g}(x, m_j, h_j) \leq &\limsup_{j\to\infty} \lim_{n\to\infty} \frac{1}{|B|} \int_B g(x, \tilde{m}_{n,j}(y), h_j + \nabla \tilde{u}_{n,j}(y))\, dy \\ = &\limsup_{j\to\infty} \lim_{n\to\infty} \frac{1}{|B|} \int_B g(x, m_{n,j}(y), h_j + \nabla u_{n,j}(y))\, dy \\ \leq &\hat{g}(x, \tilde{m}, h).\end{aligned}$$

Step 3: To prove (3.20) observe that by (1.6) and the definition of $\hat{g}$ we have

$$\hat{g}(x,\tilde{m},h) \leq a(x) + C(|u(x)|^{2^*} + |h|^2) + \inf \left\{ \liminf_{k\to\infty} \frac{1}{|B|}\int_B |\nabla u_k(y)|^2\,dy : m_k \in L^\infty(B;S^{N-1}), \right.$$

$$\left. m_k \overset{*}{\rightharpoonup} \tilde{m} \text{ in } L^\infty(B;\mathbb{R}^N), \quad u_k \in H_0^1(B), \quad \Delta u_k + \operatorname{div} m_k = 0 \quad \text{in } B \right\}.$$

By Lemma 8 there exists a sequence $\{(M_n,U_n)\} \subset \mathcal{M} \times H_0^1$ such that $m_n \overset{*}{\rightharpoonup} \lambda\pi(\tilde{m}) - (1-\lambda)\pi(\tilde{m})$ in $L^\infty(B;\mathbb{R}^N)$, with $\lambda \in (0,1)$ such that $\lambda\pi(\tilde{m}) - (1-\lambda)\pi(\tilde{m}) = \tilde{m}$, $\nabla U_n \to 0$ strongly in $L^2(B;\mathbb{R}^N)$, and thus

$$\hat{g}(x,\tilde{m},h) \leq a(x) + C(|u(x)|^{2^*} + |h|^2) + \lim_{k\to\infty} \frac{1}{|B|}\int_B |\nabla U_k(y))|^2\,dy = a(x) + C(|u(x)|^{2^*} + |h|^2).$$

□

Lemma 10. *For $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have*

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) = \hat{g}(x_0, m(x_0), \nabla u(x_0)).$$

Proof of Lemma 10. Assume first that

$$m(x) = \sum_{i=1}^K \xi_i \chi_{D_i}(x), \tag{3.24}$$

where $|\xi_i| \leq 1$ and the sets $D_i \in \mathcal{O}(\Omega)$ are pairwise disjoint, $|\partial D_i| = 0$, and $\cup_{i=1}^K \overline{D_i} = \Omega$. Without loss of generality we may assume that $x_0 \in \cup_{i=1}^K D_i$, say $x_0 \in D_1$, and that

$$\frac{u\left(x_0 + ry\right) - u\left(x_0\right)}{r} - \nabla u\left(x_0\right) y \to 0 \text{ in } H^1\left(B\right), \text{ there exists } \frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0). \tag{3.25}$$

Then

$$m(x) \equiv \xi_1 \quad \text{for all } x \in B(x_0,r)$$

for all $r > 0$ sufficiently small. Fix $\eta > 0$ and consider a sequence $\{(m_k, v_k)\}$ as in the definition of the function $\hat{g}(x_0, m(x_0), \nabla u(x_0))$ such that

$$\lim_{k\to\infty} \frac{1}{|B|}\int_B g(x_0, m_k(y), \nabla u(x_0) + \nabla v_k(y))\,dy \leq \hat{g}(x_0, m(x_0), \nabla u(x_0)) + \eta, \tag{3.26}$$

where $m_k \equiv m(x_0)$ and $v_k \equiv 0$ if $|m(x_0)| = 1$. For any fixed $r > 0$ sufficiently small define

$$\hat{m}_k(x) := m_k((x - x_0)/r).$$

Then $\hat{m}_k \in L^\infty(B(x_0,r);S^{N-1})$, $\hat{m}_k \overset{*}{\rightharpoonup} m$ in $L^\infty(B(x_0,r);\mathbb{R}^N)$, where we used the fact that m restricted to $B\left(x_0,r\right)$ is identically equal to $m(x_0)$. Let $\tilde{u}_k \in u + H_0^1((B(x_0,r))$ be the unique solution of $\Delta\tilde{u}_k + \operatorname{div}\hat{m}_k = 0$ in $B(x_0,r)$. Then

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to0^+} \frac{\mathcal{F}_0(m;B(x_0,r))}{|B(x_0,r)|} \leq \lim_{r\to0^+} \liminf_{k\to\infty} \frac{1}{|B|}\int_B g(x_0 + ry, m_{r,k}(y), \nabla u_{r,k}(y))\,dy$$

where

$$m_{r,k}(y) := \hat{m}_k(x_0 + ry) = m_k(y) \quad \text{and} \quad u_{r,k}(y) := \frac{\tilde{u}_k(x_0 + ry) - u(x_0)}{r}.$$

Setting $\hat{u}_k\left(y\right) := \nabla u(x_0)y + v_k\left(y\right)$ then

$$\Delta\left(\hat{u}_k - u_{r,k}\right) = 0 \text{ in } B, \quad \left(\hat{u}_k - u_{r,k}\right)|_{\partial B} = \nabla u(x_0)y - \frac{u(x_0 + ry) - u(x_0)}{r},$$

so by Lemma 3 and (3.25)

$$\left\|\hat{u}_k - u_{r,k}\right\|_{H^1(B)} \to 0$$

as $r \to 0^+$, and, in particular,

$$\lim_{r\to0^+} \liminf_{k\to\infty} \int_B \left|\nabla u_{r,k}\left(y\right) - \left(\nabla u(x_0) + \nabla v_k\left(y\right)\right)\right|^2 dy = 0.$$

We can now continue exactly as in the proof of Lemma 7 to obtain (3.13), that is

$$\lim_{r\to0^+}\liminf_{k\to\infty}\int_B g(x_0+ry,m_{r,k}(y),\nabla u_{r,k}(y))\,dy=\lim_{i\to\infty}\int_B g(x_0,m_{k_i}(y),\nabla u(x_0)+\nabla v_{k_i}(y))\,dy$$

for some subsequence $k_i\to\infty$. Hence by (3.26)

$$\frac{d\mathcal{F}_0(m;\cdot)}{d\mathcal{L}^N}(x_0)\le\hat{g}(x_0,m(x_0),\nabla u(x_0))+\eta$$

and we can now let $\eta\to0^+$. Together with Lemma 7 this yields that

$$\mathcal{F}_0(m;D)=\int_D\hat{g}(x,m(x),\nabla u(x))\,dx \tag{3.27}$$

for all functions $m\in\mathcal{M}^*$ of the form (3.24).

For a general $m\in\mathcal{M}^*$ construct a sequence $m_k\in M^*$ of the form (3.24) such that m_k converges to m *strongly* in $L^\infty(D;\mathbb{R}^N)$. A simple diagonalization argument and (3.27) show that

$$\mathcal{F}_0(m;D)\le\liminf_{k\to\infty}\mathcal{F}_0(m_k;D)=\liminf_{k\to\infty}\int_D\hat{g}(x,m_k(x),\nabla u_k(x))\,dx.$$

Extracting a subsequence, if necessary, we have that

$$\liminf_{k\to\infty}\int_D\hat{g}(x,m_k(x),\nabla u_k(x))\,dx=\lim_{k\to\infty}\int_D\hat{g}(x,m_k(x),\nabla u_k(x))\,dx,$$

and, since by Lemma 3 ∇u_k converges to ∇u strongly in $L^2(B;\mathbb{R}^N)$, in view of the growth condition (3.20) we may apply Fatou's Lemma to $a(\cdot)+C(|u(\cdot)|^{2^*}+|\nabla u_k(\cdot)|^2)-\hat{g}(\cdot,m_k(\cdot),\nabla u_k(\cdot))$ to obtain

$$\begin{aligned}\limsup_{k\to\infty}\int_D\hat{g}(x,m_k,\nabla u_k)\,dx&\le\int_D\limsup_{k\to\infty}\hat{g}(x,m_k,\nabla u_k)\,dx\\&\le\int_D\hat{g}(x,m,\nabla u(x))\,dx,\end{aligned}$$

where we have used the upper semicontinuity of $\hat{g}(x,\cdot,\cdot)$ (see Lemma 9). Together with Lemma 7 this yields the desired result. $\square$

Proof of Corollary 1. Fix $m\in\mathcal{M}^*$, $(x,h)\in\Omega\times\mathbb{R}^N$, and define

$$\varphi_1(\tilde{m}):=\varphi(x,\tilde{m},u(x)),\qquad\psi_1(h):=\psi(x,u(x),h),$$

where $u\in H^1\left(\mathbb{R}^N\right)$ is the unique solution of $\Delta u+\operatorname{div}m=0$ in $\mathbb{R}^N$. We claim that the function $\hat{g}(x,\cdot,h)$ is convex in $\overline{B(0,1)}$. Indeed given m_1, $m_2\in\overline{B(0,1)}$, $\lambda\in(0,1)$ and $\varepsilon>0$, consider two sequences $\{(m_k,u_k)\}$ and $\{(M_k,U_k)\}$ in $L^\infty(B;S^{N-1})\times H_0^1(B)$, such that $m_k\overset{*}{\rightharpoonup}m_1$, $M_k\overset{*}{\rightharpoonup}m_2$ in $L^\infty(B;\mathbb{R}^N)$, and

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{|B|}\int_B g(x,m_k(y),h+\nabla u_k(y))\,dy\le\hat{g}(x,m_1,h)+\varepsilon,\\&\lim_{k\to\infty}\frac{1}{|B|}\int_B g(x,M_k(y),h+\nabla U_k(y))\,dy\le\hat{g}(x,m_2,h)+\varepsilon.\end{aligned}$$

By a density argument we may assume that the functions m_k, M_k are piecewise constant, and thus by Lemma 8 we may find a sequence $\{m_{n,k}\}\subset\mathcal{M}$ such that $m_{n,k}\overset{*}{\rightharpoonup}\lambda m_k+(1-\lambda)M_k$ in $L^\infty(B;\mathbb{R}^N)$, $\nabla u_{n,k}\to\lambda\nabla u_k+(1-\lambda)\nabla U_k$ strongly in $L^2(B;\mathbb{R}^N)$, and

$$\begin{aligned}\lim_{n\to\infty}\int_B g(x,m_{n,k}(y),h+\nabla u_{n,k}(y))\,dy&=\lambda\int_B g(x,m_k(y),h+\lambda\nabla u_k(y)+(1-\lambda)\nabla U_k(y))\,dy\\&\quad+(1-\lambda)\int_B g(x,M_k(y),h+\lambda\nabla u_k(y)+(1-\lambda)\nabla U_k(y))\,dy\\&=\lambda\int_B\varphi_1(m_k(y))\,dy+(1-\lambda)\int_B\varphi_1(M_k(y))\,dy+\int_B\psi_1(h+\lambda\nabla u_k(y)+(1-\lambda)\nabla U_k(y))\,dy\\&\le\lambda\int_B g(x,m_k(y),h,\nabla u_k(y))\,dy+(1-\lambda)\int_B g(x,M_k(y),h+\nabla U_k(y))\,dy\end{aligned}$$

where $u_{n,k} \in H_0^1(B)$ is the unique solution of $\Delta u_{n,k} + \operatorname{div} m_{n,k} = 0$, and where we have used the convexity of ψ_1. Since $m_{n,k} \overset{*}{\rightharpoonup} \lambda m_1 + (1-\lambda)m_2$ in $L^\infty(B;\mathbb{R}^N)$ as $n \to \infty$ and $k \to \infty$, in this order, we have

$$\begin{aligned}\hat{g}(x, \lambda m_1 + (1-\lambda)m_2, h) \leq& \liminf_{k\to\infty} \lim_{n\to\infty} \frac{1}{|B|}\int_B g(x, m_{n,k}(y), h + \nabla u_{n,k}(y))\,dy \\ \leq& \lambda \lim_{k\to\infty} \frac{1}{|B|}\int_B g(x, m_k(y), h, \nabla u_k(y))\,dy \\ &+ (1-\lambda)\lim_{k\to\infty} \frac{1}{|B|}\int_B g(x, M_k(y), h + \nabla U_k(y))\,dy \\ \leq& \lambda \hat{g}(x, m_1, h) + (1-\lambda)\hat{g}(x, m_2, h) + 2\varepsilon.\end{aligned}$$

The claim follows by letting $\varepsilon \to 0^+$.

Since $\hat{g}(x, \cdot, h)$ is convex in $\overline{B(0,1)}$ we have that

$$\hat{g}(x, m, h) \leq (\varphi_1(\cdot) + \psi_1(h))^{**}(m) = \varphi_1^{**}(m) + \psi_1(h).$$

On the other hand, by classical lower semicontinuity results and, in view of the convexity of ψ_1, for any sequence $\{(m_k, u_k)\} \subset L^\infty(B; S^{N-1}) \times H_0^1(B)$ such that $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(B;\mathbb{R}^N)$ and $\Delta u_k + \operatorname{div} m_k = 0$ in B, we have that

$$\liminf_{k\to\infty} \frac{1}{|B|}\int_B g(x, m_k, h + \nabla u_k)\,dy \geq \liminf_{k\to\infty} \frac{1}{|B|}\int_B (\varphi_1^{**}(m_k) + \psi_1(h + \nabla u_k))\,dy \geq \varphi_1^{**}(m) + \psi_1(h)$$

and thus

$$\hat{g}(x, m, h) \geq \varphi_1^{**}(m) + \psi_1(h).$$

This concludes the proof. $\square$

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