Motion of Islands of Elastic Thin Films in the Dewetting Regime

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Abstract

This paper addresses a two-dimensional sharp interface variational model for solid-state dewetting of thin films with surface energies, introduced by Wang, Jiang, Bao, and Srolovitz in [1]. Using the $\boldsymbol{H^{-1}}$-gradient flow structure of the evolution law, short-time existence for a surface diffusion evolution equation with curvature regularization is established in the context of epitaxially strained two-dimensional films. The main novelty, as compared to the study of the wetting regime, is the presence of moving contact lines.

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1 Introduction

Understanding the dewetting process and its underlying mechanisms is essential for controlling the morphology and properties of thin films, which have many applications in microelectronics, optics, and other fields (see [2]). This type of dewetting is distinct from liquid dewetting and is primarily driven by surface diffusion-controlled mass transport at temperatures below the melting point of the film. The movement of the contact line, where the film, substrate, and vapor phases meet, plays a crucial role in this process. Mathematically modeling the morphology evolution of solid-state dewetting involves treating it as a surface-tracking problem. The minimization of interfacial energy guides the process and consists of a combination of surface diffusion-controlled mass transport and the movement of the contact line. While moving contact line problems have been extensively studied in fluid mechanics, incorporating surface diffusion-based geometric evolution equations with moving contact lines presents significant challenges in materials science, applied mathematics, and scientific computing.

This paper studies a two-dimensional sharp interface variational model for simulating solid-state dewetting of thin films with surface energies, proposed by Wang, Jiang, Bao, and Srolovitz in [1], (see also [3]). The morphology evolution is driven by surface diffusion and contact points migration, coupled with elastic deformation. We restrict our consideration to a single island whose profile is given by a function $h : [\alpha, \beta] \to [0, \infty)$, where α and β are the contact points, $h(\alpha) = h(\beta) = 0$, and $h(x) > 0$ for every $x \in (\alpha, \beta)$. The region occupied by the island is

$$\Omega_h := \{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta,\, 0 < y < h(x)\}\,.$$

We assume that the region $\mathbb{R} \times (-\infty, 0]$ is filled by a rigid substrate and $(\mathbb{R} \times (0, \infty)) \setminus \Omega_h$ by a vapor.

The elastic displacement within the island is described by a function $u : \Omega_h \to \mathbb{R}^2$, which satisfies the boundary condition

$$u(x, 0) = (e_0 x, 0) \;\text{ for }\; x \in (\alpha, \beta)\,. \tag{1}$$

The parameter $e_0 \neq 0$ reflects the mismatch between the crystalline structures of the thin film and the substrate. The underlying energy is

$$\mathcal{W}(u, \Omega_h) + \gamma\, \mathrm{length}(\Gamma_h) - \gamma_0(\beta - \alpha) + \frac{\nu_0}{2} \int_{\Gamma_h} \kappa^2 ds\,, \tag{2}$$

where $\mathcal{W}(u, \Omega_h)$ is the linearized elastic energy of the displacement u, Γ_h is the graph of h, κ is the curvature, and s is the arclength parameter on Γ_h. The constant $\gamma = \gamma_{FV} > 0$ is the surface energy density between film and vapor, while $\gamma_0 = \gamma_{VS} - \gamma_{FS}$, where γ_{VS} is the surface energy density between vapor and substrate, and γ_{FS} between film and substrate. The constant $\nu_0 > 0$ is a small parameter in the curvature regularization, which is commonly used in the literature ([4], [5]). The dewetting regime (Volmer–Weber) is characterized by the inequality $\gamma > \gamma_0$, which favors the exposure of the substrate.

We will assume that at each time the displacement u satisfies the elastic equilibrium problem on Ω_h with natural boundary conditions on Γ_h and the Dirichlet boundary condition (1) in the rest of the boundary.

The time evolution of (α, β, h) is obtained as gradient flow of the energy (2) with the area constraint $|\Omega_h| = A_0 > 0$, where for the dynamics of h we use a type of

$H^{-1}(\Gamma_h)$ norm (see [6], [7]). To describe the equations of this evolution system, we introduce the chemical potential ζ defined in terms of the arclength parameter s of Γ_h by

$$\zeta = -\gamma\kappa + \nu_0\Big(\partial_{ss}\kappa + \frac{\kappa^3}{2}\Big) + \widetilde{W}\,, \tag{3}$$

where $\widetilde{W}$ is the value of the elastic energy density of u at the point of Γ_h corresponding to s.

The equation for h is given by

$$\widetilde{V} = \rho_0 \partial_{ss}\zeta \quad \text{on } \Gamma_h\,, \tag{4}$$

where $\widetilde{V}$ denotes the normal velocity of the time dependent curve Γ_h at the point corresponding to s and $\rho_0 > 0$ is a material constant.

The equations for the contact points migration are

$$\begin{aligned} \sigma_0\dot{\alpha} &= \gamma\cos\theta_\alpha - \gamma_0 + \nu_0\partial_s\kappa_\alpha \sin\theta_\alpha\,, \\ \sigma_0\dot{\beta} &= -\gamma\cos\theta_\beta + \gamma_0 - \nu_0\partial_s\kappa_\beta \sin\theta_\beta\,, \end{aligned} \tag{5}$$

where $\sigma_0 > 0$ is a material constant, the dot denotes the time derivative, θ_α and θ_β are the oriented angles between the x-axis and the tangent to Γ_h at $(\alpha, 0)$ and $(\beta, 0)$, both oriented with increasing values of x, while $\partial_s\kappa_\alpha$ and $\partial_s\kappa_\beta$ are the derivatives with respect to s of the curvature κ of Γ_h at the values of s corresponding to $(\alpha, 0)$ and $(\beta, 0)$. Observe that if we neglect the curvature regularization, that is, we take $\nu_0 = 0$, then the resulting equations reduce to the usual Young's law ([8], [9], [2]).

The main result of this paper is that for every initial condition (α_0, β_0, h_0) there exists a small time $T > 0$ such that the evolution equations (4) and (5) admit a weak solution on $[0, T]$. Our approach does not allow us to obtain the uniqueness of solutions.

The existence proof relies on a minimizing movements argument: we consider a time discretization and construct an approximate solution via incremental minimum problems involving the energy (2). While this approach is not novel for epitaxial growth (see [10], [11], [12]), the major challenge here is that the domain Ω_h has evolving corners. This requires a delicate $W^{2,p}(\Omega_h)$ estimate of the solution for the Lamé system, with a precise dependence on the time step of the discretization. This estimate plays a central role in the study of the convergence of the discretized solutions for the generalized Young's law (5). We refer to [13] for a stress-driven grain boundary diffusion problem, where the analysis of the singularities of the solutions to the Lamé system near triple junctions plays a crucial role.

There is an extensive body of literature in two and three dimensions for the static problem both in the wetting ($\gamma < \gamma_0$) and dewetting ($\gamma > \gamma_0$) regimes. We refer to [14], [15], [16], [17], [18], [19], [20], [21], [22], [23],[24], [25], [26], [27], [28], [29], and the references therein for the wetting regime; and [8], [30] for the dewetting regime. We also refer to [31] for a study of the Lamé system in the presence of cracks.

For the evolution case in the wetting regime, we refer to [10], [11], [32], [33], [12], [34]. Observe that in the papers [32], [33] the curvature regularization is omitted.

While moving contact line problems have been studied in the fluid mechanics community (see, e.g., [35],[36], [37] and the references therein), to our knowledge our work is the first to prove the generalized Young's law in a problem involving elasticity.

2 Preliminaries

Throughout this paper, we fix the physical parameters γ, γ_0, σ_0, A_0, $e_0 \in \mathbb{R}$, with $\gamma > 0$, $\gamma > \gamma_0$, $\sigma_0 > 0$, $A_0 > 0$, $e_0 \neq 0$, the Lamé coefficients λ and μ, with $\mu > 0$ and $\lambda + \mu > 0$, and the regularizing parameters $L_0 \geq 1$ and $\nu_0 > 0$. We renormalize the parameter ρ_0 in (4) to be one. We use standard notation for Lebesgue and Sobolev spaces, as well as for spaces of Hölder continuous and differentiable functions.

We introduce the class of admissible surface profiles, $\mathcal{A}_s$, as the set of all (α, β, h) such that $\alpha < \beta$, $h \in H^2((\alpha,\beta)) \cap H^1_0((\alpha,\beta))$, with $h \geq 0$ in (α, β), $\operatorname{Lip} h \leq L_0$, and

$$\int_\alpha^\beta h(x)\,dx = A_0\,. \tag{6}$$

Moreover, if $(\alpha,\beta,h) \in \mathcal{A}_s$ then $\check{h}$ is the extension of h by zero outside of $[\alpha,\beta]$, and we set

$$H(x;\alpha,\beta,h) := \int_{-\infty}^x \check{h}(\rho)\,d\rho = \int_\alpha^x \check{h}(\rho)\,d\rho\,, \quad x \in \mathbb{R}\,, \tag{7}$$

and

$$\Omega_h := \{(x,y) \in \mathbb{R}^2 :\ \alpha < x < \beta,\ 0 < y < h(x)\}\,. \tag{8}$$

Furthermore, the admissible class $\mathcal{A}_e(\alpha,\beta,h)$ of elastic displacements in Ω_h is defined as

$$\mathcal{A}_e(\alpha,\beta,h) := \{u \in H^1(\Omega_h;\mathbb{R}^2) :\ u(x,0) = (e_0 x, 0) \text{ for a.e. } x \in (\alpha,\beta)\}\,. \tag{9}$$

Finally the admissible class $\mathcal{A}$ for the total energy is

$$\mathcal{A} := \{(\alpha,\beta,h,u) :\ (\alpha,\beta,h) \in \mathcal{A}_s,\ u \in \mathcal{A}_e(\alpha,\beta,h)\}\,. \tag{10}$$

In what follows we will use the result below.

Lemma 1. *We have*

$$\beta - \alpha \geq \sqrt{\frac{2A_0}{L_0}} \tag{11}$$

for every $(\alpha,\beta,h) \in \mathcal{A}_s$.

Proof. Since $h(\alpha) = h(\beta) = 0$ and $\operatorname{Lip} h \leq L_0$, we have $h(x) \leq L_0 \frac{\beta-\alpha}{2}$ for every $x \in (\alpha,\beta)$. Hence, by (6),

$$A_0 = \int_\alpha^\beta h(x)\,dx \leq \frac{L_0}{2}(\beta-\alpha)^2,$$

which concludes the proof. □

For $(\alpha, \beta, h) \in \mathcal{A}_s$ we define the surface energy as

$$\mathcal{S}(\alpha, \beta, h) := \gamma \int_\alpha^\beta \sqrt{1 + (h'(x))^2} dx - \gamma_0(\beta - \alpha) + \frac{\nu_0}{2} \int_\alpha^\beta \frac{(h''(x))^2}{(1 + (h'(x))^2)^{5/2}} dx \,. \quad (12)$$

Note that since $\gamma > 0$ and $\gamma > \gamma_0$, we have

$$\gamma \int_\alpha^\beta \sqrt{1 + (h'(x))^2} dx - \gamma_0(\beta - \alpha) \geq (\gamma - \gamma_0)(\beta - \alpha) \geq 0 \quad (13)$$

and so

$$\mathcal{S}(\alpha, \beta, h) \geq 0 \,. \quad (14)$$

For $(\alpha, \beta, h) \in \mathcal{A}_s$ and $u \in \mathcal{A}_e(\alpha, \beta, h)$ we define the elastic energy as

$$\mathcal{E}(\alpha, \beta, h, u) := \int_{\Omega_h} W(Eu(x, y)) \, dxdy \,, \quad (15)$$

where $W : \mathbb{R}^{2\times 2} \to [0, \infty)$ is given by

$$W(\xi) := \frac{1}{2} \mathbb{C}\xi \cdot \xi \,, \quad \text{with} \quad \mathbb{C}\xi := \mu(\xi + \xi^T) + \lambda(\operatorname{tr} \xi) I \,, \quad (16)$$

where μ, $\lambda \in \mathbb{R}$ are the Lamé coefficients and I is the 2×2 identity matrix. Note that $\mathbb{C}\xi = \mathbb{C}\xi_{\text{sym}} \in \mathbb{R}^{2\times 2}_{\text{sym}}$ for every $\xi \in \mathbb{R}^{2\times 2}$, where $\xi_{\text{sym}} := (\xi + \xi^T)/2$.

We assume that $\mu > 0$ and $\lambda + \mu > 0$ so that there exists a constant $C_W > 0$ such that

$$\frac{1}{C_W} |\xi|^2 \leq W(\xi) \leq C_W |\xi|^2 \quad (17)$$

for all $\xi \in \mathbb{R}^{2\times 2}_{\text{sym}}$.

In order to study the incremental problem, we introduce the following functionals. Given $\tau > 0$ and $(h^0, \alpha^0, \beta^0) \in \mathcal{A}_s$, for every $(h, \alpha, \beta) \in \mathcal{A}_s$ we define

$$\mathcal{T}_\tau(\alpha, \beta, h; \alpha^0, \beta^0, h^0) := \frac{1}{2\tau} \int_{\mathbb{R}} (H - H^0)^2 \sqrt{1 + ((\check{h}^0)')^2} dx + \frac{\sigma_0}{2\tau} (\alpha - \alpha^0)^2 + \frac{\sigma_0}{2\tau} (\beta - \beta^0)^2, \quad (18)$$

where we abbreviate

$$H(x) := H(x; \alpha, \beta, h) \,, \quad H^0(x) := H(x; \alpha^0, \beta^0, h^0) \,, \quad (19)$$

and H is given in (7). Observe that

$$\frac{1}{2\tau} \int_{\mathbb{R}} (H - H^0)^2 \sqrt{1 + ((\check{h}^0)')^2} dx = \frac{1}{2\tau} \int_{\min\{\alpha, \alpha^0\}}^{\max\{\beta, \beta^0\}} (H - H^0)^2 \sqrt{1 + ((\check{h}^0)')^2} dx \quad (20)$$

since by construction and (6), $H - H^0 = 0$ for $x \notin (\min\{\alpha, \alpha^0\}, \max\{\beta, \beta^0\})$.

The existence of a minimizer for the incremental problem will be a consequence of the following result.

Theorem 2. *For every $\tau > 0$ and every $(\alpha^0, \beta^0, h^0) \in \mathcal{A}_s$ there exists a minimizer $(\alpha, \beta, h, u) \in \mathcal{A}$ of the total energy functional*

$$\mathcal{F}^0(\alpha, \beta, h, u) := \mathcal{S}(\alpha, \beta, h) + \mathcal{E}(\alpha, \beta, h, u) + \mathcal{T}_\tau(\alpha, \beta, h; \alpha^0, \beta^0, h^0)\,. \tag{21}$$

Moreover, there exists a constant $C > 0$, depending only on the structural parameters A_0, e_0, λ, μ, and L_0, such that

$$\|u\|_{H^1(\Omega_h)} \le C \tag{22}$$

for every minimizer (α, β, h, u) of $\mathcal{F}^0$ in $\mathcal{A}$.

The proof of the theorem relies on the following Korn's inequality.

Lemma 3. *Let $(\alpha, \beta, h) \in \mathcal{A}_s$, let Ω_h be as in (8), and let $1 < p < \infty$. Then there exists a constant $C > 0$, depending only on p and L_0, such that*

$$\int_{\Omega_h} |\nabla u|^p \, dxdy \le C \int_{\Omega_h} |Eu|^p \, dxdy + C e_0^p A_0$$

for every $u \in W^{1,p}(\Omega_h; \mathbb{R}^2)$ such that $u(x, 0) = (e_0 x, 0)$ for $x \in (\alpha, \beta)$ (in the sense of traces).

Proof. By a translation we can assume that $\alpha = 0$. Define $v(x, y) := u(x, y) - (e_0 x, 0)$ in Ω_h and $v := 0$ in $(0, \beta) \times (0, -\infty)$, and

$$w(x, y) = v(\beta x, \beta y)$$

for $(x, y) \in D_{h_\beta}$, where

$$D_{h_\beta} := \{(x, y) \in \mathbb{R}^2 : 0 < x < 1\,,\ y < h_\beta(x)\}$$

and $h_\beta(x) := h(\beta x)/\beta$. Note that

$$|h_\beta(x_1) - h_\beta(x_2)| \le \frac{1}{\beta} |h(\beta x_1) - h(\beta x_2)| \le L_0 |x_1 - x_2|$$

for all $x_1, x_2 \in (0, 1)$. Applying Theorem 4.2 in [23] to w we can find a constant C depending only on p and L_0 such that

$$\int_{D_{h_\beta}} |\nabla w|^p \, dxdy \le C \int_{D_{h_\beta}} |Ew|^p \, dxdy\,.$$

By the change of variables $(x', y') = (\beta x, \beta y)$ we have

$$\int_{\Omega_h} |\nabla v|^p \, dxdy \le C \int_{\Omega_h} |Ev|^p \, dxdy\,,$$

where we used the fact that $v = 0$ in $(0, \beta) \times (-\infty, 0)$. Recalling that $v = u - w_0$, it follows that

$$\int_{\Omega_h} |\nabla u|^p \, dxdy \leq C \int_{\Omega_h} |Eu|^p \, dxdy + Ce_0^p \mathcal{L}^2(\Omega_h) = C \int_{\Omega_h} |Eu|^p \, dxdy + Ce_0^p A_0$$

by (6). □

Proof of Theorem 2. Let $\{(\alpha_n, \beta_n, h_n, u_n)\}_n$ be a minimizing sequence in $\mathcal{A}$ for (21). Then there exists a constant $M > 0$ such that

$$\mathcal{F}^0(\alpha_n, \beta_n, h_n, u_n) \leq M \tag{23}$$

for all n. Since all the terms in $\mathcal{F}^0$ are nonnegative (see (14)), by (18) and (23) there exist two constants $M_l \in \mathbb{R}$ and $M_r \in \mathbb{R}$ such that

$$M_l \leq \alpha_n < \beta_n \leq M_r \tag{24}$$

for every n. Hence, up to a subsequence, not relabeled, we can assume that

$$\alpha_n \to \alpha \quad \text{and} \quad \beta_n \to \beta \tag{25}$$

as $n \to \infty$, with $\alpha \leq \beta$. Since $\operatorname{Lip} h_n \leq L_0$ for every n, the extension $\check{h}_n$ by zero of h_n outside of $[\alpha_n, \beta_n]$ satisfies $\operatorname{Lip} \check{h}_n \leq L_0$. Using (25), up to a further subsequence, not relabeled, there exists a nonnegative Lipschitz continuous function $\check{h} : \mathbb{R} \to \mathbb{R}$ such that $\check{h}_n \to \check{h}$ uniformly and $\check{h} = 0$ outside (α, β). In particular this implies that the restriction h of $\check{h}$ to (α, β) belongs to $H_0^1((\alpha, \beta))$ and that (6) is satisfied. Since $A_0 > 0$ we deduce that $\alpha < \beta$. By (12), (23), and the fact that $\operatorname{Lip} \check{h}_n \leq L_0$ for every n,

$$\sup_n \int_{\alpha_n}^{\beta_n} |h_n''|^2 dx < \infty \, . \tag{26}$$

By (25) this implies that $h \in H^2((\alpha, \beta))$, and so $(h, \alpha, \beta) \in \mathcal{A}_s$ and $h \in C^1([\alpha, \beta])$. Moreover, since $\check{h} = 0$ outside (α, β), from (25) and (26) we deduce that

$$\check{h}_n'(x) \to \check{h}'(x) \tag{27}$$

for every $x \in \mathbb{R} \setminus \{\alpha, \beta\}$.

By (15), (17), and (23),

$$\int_{\Omega_{h_n}} |Eu_n(x,y)|^2 dxdy \leq MC_W \tag{28}$$

for every n. By Korn's inequality (see Lemma 3)

$$\int_{\Omega_{h_n}} |\nabla u_n|^2 \, dxdy \leq C \int_{\Omega_{h_n}} |Eu_n|^2 \, dxdy + Ce_0^2 A_0 \leq CMC_W + Ce_0^2 A_0 \, . \tag{29}$$

Since $u_n(x,0) = (e_0 x, 0)$ for $x \in (\alpha, \beta)$, by Poincaré's inequality we find that

$$\int_{\Omega_{h_n}} |u_n(x,y)|^2 \, dxdy \leq C \tag{30}$$

where $C > 0$ is independent of n. By (29), (30), and a standard diagonal argument, using the increasing sequence of sets Ω_{h^ε}, where $h^\varepsilon := (h - \varepsilon) \vee 0$, we conclude that there exist a subsequence, not relabeled, and a function $u \in H^1(\Omega_h; \mathbb{R}^2)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega_{h^\varepsilon}; \mathbb{R}^2)$ for every ε. Note that the trace of u satisfies $u(x,0) = (e_0 x, 0)$ for a.e. $x \in \{h > 0\}$. In conclusion, we have shown that $(h, u, \alpha, \beta) \in \mathcal{A}$, where $\mathcal{A}$ is given in (10).

Next we claim that

$$\liminf_{n \to \infty} \mathcal{S}(\alpha_n, \beta_n, h_n) \geq \mathcal{S}(\alpha, \beta, h) \,, \tag{31}$$

where $\mathcal{S}$ is given in (12). It is convenient to write

$$\begin{aligned} &\mathcal{S}(\alpha, \beta, h) + \gamma_0 (M_r - M_l) \\ &\qquad = \int_{M_l}^{M_r} g(\check{h}(x)) \sqrt{1 + (\check{h}'(x))^2} dx + \frac{\nu_0}{2} \int_\alpha^\beta \frac{(h''(x))^2}{(1 + (h'(x))^2)^{5/2}} dx \,, \end{aligned} \tag{32}$$

where

$$g(y) := \begin{cases} \gamma & \text{if } y > 0 \,, \\ \gamma_0 & \text{if } y = 0 \,, \end{cases}$$

and similarly for $\mathcal{S}(\alpha_n, \beta_n, h_n) + \gamma_0 (M_r - M_l)$.

By (27), $\check{h}_n'(x) \to \check{h}'(x)$ for a.e. $x \in \mathbb{R}$. Moreover,

$$\liminf_{n \to \infty} g(\check{h}_n(x)) \geq g(\check{h}(x))$$

for every $x \in \mathbb{R}$, since $\check{h}_n \to \check{h}$ uniformly and g is lower semicontinuous in view of the inequality $\gamma > \gamma_0$. Therefore, by Fatou's lemma we have

$$\liminf_{n \to \infty} \int_{M_l}^{M_r} g(\check{h}_n(x)) \sqrt{1 + (\check{h}_n'(x))^2} dx \geq \int_{M_l}^{M_r} g(\check{h}(x)) \sqrt{1 + (\check{h}'(x))^2} dx \,. \tag{33}$$

On the other hand for every $[a,b] \subset (\alpha, \beta)$, by (25), (26), and (27) we can apply a weak-strong lower semicontinuity theorem (see [38, Theorem 2.3.1]) to get

$$\begin{aligned} \liminf_{n \to \infty} \int_{\alpha_n}^{\beta_n} \frac{(h_n''(x))^2}{(1 + (h_n'(x))^2)^{5/2}} dx &\geq \liminf_{n \to \infty} \int_a^b \frac{(h_n''(x))^2}{(1 + (h_n'(x))^2)^{5/2}} dx \\ &\geq \int_a^b \frac{(h''(x))^2}{(1 + (h'(x))^2)^{5/2}} dx \,. \end{aligned}$$

Taking the supremum over all $[a,b] \subset (\alpha,\beta)$, we obtain

$$\liminf_{n\to\infty} \int_{\alpha_n}^{\beta_n} \frac{(h_n''(x))^2}{(1+(h_n'(x))^2)^{5/2}} dx \geq \int_{\alpha}^{\beta} \frac{(h''(x))^2}{(1+(h'(x))^2)^{5/2}} dx \,.$$

Therefore, from (32) and (33) we deduce (31).

Now we prove that

$$\liminf_{n\to\infty} \mathcal{E}(\alpha_n,\beta_n,h_n,u_n) \geq \mathcal{E}(\alpha,\beta,h,u)\,, \tag{34}$$

where $\mathcal{E}$ is defined in (15). Fix an open set $U \Subset \Omega_h$. Since $u_n \rightharpoonup u$ weakly in $H^1(U;\mathbb{R}^2)$, by (16) and (17) we have

$$\begin{aligned} \liminf_{n\to\infty} \int_{\Omega_{h_n}} W(Eu_n(x,y))\, dxdy &\geq \liminf_{n\to\infty} \int_U W(Eu_n(x,y))\, dxdy \\ &\geq \int_U W(Eu(x,y))\, dxdy \end{aligned}$$

and letting $U \nearrow \Omega_h$ we obtain (34).

Finally, by (7), (19), (20), and (25), the fact that $h_n \to h$ uniformly, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \mathcal{T}_\tau(\alpha_n,\beta_n,h_n;\alpha^0,\beta^0,h^0) = \mathcal{T}_\tau(\alpha,\beta,h;\alpha^0,\beta^0,h^0)\,.$$

This, together with (21), (31), and (34), allows us to conclude that

$$\liminf_{n\to\infty} \mathcal{F}^0(\alpha_n,\beta_n,h_n,u_n) \geq \mathcal{F}^0(\alpha,\beta,h,u)\,.$$

Since $(\alpha,\beta,h,u) \in \mathcal{A}$ and $\{(\alpha_n,\beta_n,h_n,u_n)\}_n$ is a minimizing sequence, we deduce that (α,β,h,u) is a minimizer.

The proof of (22) can be obtained from Korn's and Poincaré's inequalities, arguing as in the estimates for the minimizing sequence. □

3 Euler–Lagrange Equations

Given $(\alpha,\beta,h) \in \mathcal{A}_s$ and $\alpha \leq a < b \leq \beta$, we define

$$\Omega_h^{a,b} := \{(x,y) \in \mathbb{R}^2 : \, a < x < b\,,\, 0 < y < h(x)\}\,, \tag{35}$$

Theorem 4 (Euler-Lagrange equations)**.** *Let $\tau > 0$, let $(\alpha^0,\beta^0,h^0) \in \mathcal{A}_s$, and let $(\alpha,\beta,h,u) \in \mathcal{A}$ be a minimizer of the total energy functional*

$$\mathcal{F}^0(\alpha,\beta,h,u) := \mathcal{S}(\alpha,\beta,h) + \mathcal{E}(\alpha,\beta,h,u) + \mathcal{T}_\tau(\alpha,\beta,h;\alpha^0,\beta^0,h^0)\,. \tag{36}$$

Assume that

$$\operatorname{Lip} h < L_0 \quad and \quad h(x) > 0 \quad for\ all\ x \in (\alpha, \beta)\,. \tag{37}$$

Then

$$h \in C^{4,1}((\alpha,\beta)) \cap C^5((\alpha,\beta) \setminus \{\alpha^0, \beta^0\})\,, \tag{38}$$

$$u \in C^{3,1/2}(\overline{\Omega}_h^{a,b};\mathbb{R}^2) \quad for\ every\ \alpha < a < b < \beta\,, \tag{39}$$

and u satisfies the elliptic boundary value problem

$$\begin{cases} -\operatorname{div} \mathbb{C}Eu(x,y) = 0 & in\ \Omega_h\,, \\ \mathbb{C}Eu(x,h(x))\nu^h(x) = 0 & for\ x \in (\alpha,\beta)\,, \\ u(x,0) = (e_0 x, 0) & for\ x \in (\alpha,\beta)\,, \end{cases} \tag{40}$$

where $\nu^h(x)$ denotes the outer unit normal to $\partial\Omega_h$ at $(x, h(x))$. Moreover,

$$\frac{1}{\tau}(h - h^0) = \Big[-\gamma \frac{1}{J^0}\Big(\frac{h'}{J}\Big)'' + \nu_0 \frac{1}{J^0}\Big(\frac{h''}{J^5}\Big)''' + \frac{5}{2}\nu_0 \frac{1}{J^0}\Big(\frac{h'(h'')^2}{J^7}\Big)'' + \frac{1}{J^0}\overline{W}'\Big]'\,, \tag{41}$$

for every $x \in (\alpha,\beta) \setminus \{\alpha^0, \beta^0\}$, where

$$J(x) := \sqrt{1 + (h'(x))^2}\,, \quad J^0(x) := \sqrt{1 + ((\check{h}^0)'(x))^2}\,, \\ and \quad \overline{W}(x) := W(Eu(x, h(x)))\,. \tag{42}$$

Proof. **Step 1:** We first observe that standard variations with respect to u in Ω_h lead to the weak form of (40). Since $h \in C^{1,1/2}((\alpha,\beta))$ and $h > 0$ in (α,β), by elliptic regularity (see [39, Theorem 9.3]), we have that

$$u \in C^{1,1/2}(\overline{\Omega}_h^{a,b};\mathbb{R}^2) \quad \text{for every } \alpha < a < b < \beta\,.$$

Fix $\alpha < a < b < \beta$ and extend u to a function defined on $\Omega_h \cup ([a,b] \times \mathbb{R})$, still denoted by u, such that $u \in C^1([a,b] \times \mathbb{R};\mathbb{R}^2)$. Let $\varphi \in C_c^\infty((\alpha,\beta))$ be such that $\operatorname{supp}\varphi \subset [a,b]$ and

$$\int_\alpha^\beta \varphi(x)\,dx = 0\,. \tag{43}$$

Since $\operatorname{Lip} h < L_0$ and h is bounded away from zero on $[a,b]$, we have that $(\alpha, \beta, h + \varepsilon\varphi) \in \mathcal{A}_s$ and $h(x) + \varepsilon\varphi(x) > 0$ for all $x \in (\alpha,\beta)$, if $|\varepsilon|$ is sufficiently small. In turn, $(\alpha, \beta, h + \varepsilon\varphi, u|_{\Omega_{h+\varepsilon\varphi}}) \in \mathcal{A}$. Taking the derivative with respect to ε of $\mathcal{F}^0(\alpha, \beta, h + \varepsilon\varphi, u|_{\Omega_{h+\varepsilon\varphi}})$ at $\varepsilon = 0$, we obtain

$$\gamma \int_\alpha^\beta \frac{h'(x)\varphi'(x)}{(1 + (h'(x))^2)^{1/2}}\,dx + \nu_0 \int_\alpha^\beta \frac{h''(x)\varphi''(x)}{(1 + (h'(x))^2)^{5/2}}\,dx$$

$$
\begin{aligned}
&-\frac{5}{2}\nu_0\int_\alpha^\beta \frac{h'(x)(h''(x))^2\varphi'(x)}{(1+(h'(x))^2)^{7/2}}dx+\int_\alpha^\beta W(Eu(x,h(x)))\varphi(x)\,dx \\
&+\frac{1}{\tau}\int_{\mathbb{R}}(H(x)-H^0(x))\sqrt{1+((\check{h}^0)'(x))^2}\Big(\int_{-\infty}^x \varphi(s)\,ds\Big)dx=0\,.
\end{aligned} \tag{44}
$$

Since φ belongs to $C_c^\infty(\mathbb{R})$ and satisfies (43), using integration by parts the integral over $\mathbb{R}$ becomes

$$
-\frac{1}{\tau}\int_\alpha^\beta\Big(\int_{-\infty}^x(H(s)-H^0(s))\sqrt{1+((\check{h}^0)'(s))^2}\,ds\Big)\varphi(x)\,dx\,. \tag{45}
$$

This allows us to rewrite (44) as

$$
\int_\alpha^\beta(Ah''\varphi''+Bh'\varphi'+f\varphi)\,dx=0
$$

for all $\varphi\in C_c^\infty((\alpha,\beta))$ with $\operatorname{supp}\varphi\subset[a,b]$ satisfying (43), where

$$
A(x):=\frac{\nu_0}{(1+(h'(x))^2)^{5/2}}\,, \tag{46}
$$

$$
B(x):=\gamma\frac{1}{(1+(h'(x))^2)^{1/2}}-\frac{5}{2}\nu_0\frac{(h''(x))^2}{(1+(h')^2)^{7/2}}\,, \tag{47}
$$

$$
f(x):=W(Eu(x,h(x)))-\frac{1}{\tau}\int_\alpha^x(H(s)-H^0(s))\sqrt{1+((\check{h}^0)'(s))^2}ds\,. \tag{48}
$$

By introducing a Lagrange multiplier m for (43), we obtain

$$
\int_a^b(Ah''\varphi''+Bh'\varphi'+f\varphi)\,dx=m\int_a^b\varphi\,dx
$$

for all $\varphi\in C_c^\infty((a,b))$.

Note that $A\in C^{0,1/2}([\alpha,\beta])$, $A\geq a_0$ for some constant $a_0>0$, $B\in L^1((\alpha,\beta))$, and $f\in C^0([a,b])$ since $u\in C^1([a,b]\times\mathbb{R})$. Let us fix $x_0\in(a,b)$. Integrating by parts once we get

$$
\int_a^b(Ah''\varphi''+(Bh'-F_m)\varphi')\,dx=0\,, \tag{49}
$$

where

$$
F_m(x):=\int_{x_0}^x(f(s)-m)\,ds\,. \tag{50}
$$

For every $\psi\in C_c^\infty((a,b))$ with $\int_a^b\psi\,dx=0$, setting $\varphi(x):=\int_a^x\psi\,ds$, from (49) we obtain

$$
\int_a^b(Ah''\psi'+(Bh'-F_m)\psi)\,dx=0\,.
$$

By the arbitrariness of ψ we obtain

$$A(x)h''(x) = \int_{x_0}^{x} (B(s)h'(s) - F_m(s))\, ds + p_1 \tag{51}$$

for all $x \in [a,b]$, where p_1 is a polynomial of degree one associated with the constraint $\int_a^b \psi\, dx = 0$. Observe that since A, B, and F_m are independent of a and b, with $\alpha < a \leq x_0 \leq b < \beta$, so is p_1, hence (51) holds for all $x \in (\alpha,\beta)$. Since $A \in C^{0,1/2}([\alpha,\beta])$, $A \geq a_0$, $B \in L^1((\alpha,\beta))$, and $F_m \in C^1((\alpha,\beta))$, (51) implies that $h'' \in C^0((\alpha,\beta))$. In turn, this gives $A \in C^1((\alpha,\beta))$ and $B \in C^0((\alpha,\beta))$. Hence, the right-hand side of (51) is $C^1((\alpha,\beta))$, therefore $h'' \in C^1((\alpha,\beta))$. In turn, $A \in C^2((\alpha,\beta))$ and $B \in C^1((\alpha,\beta))$ by (46) and (47), and so $h'' \in C^2((\alpha,\beta))$ by (51).

By elliptic regularity ([39, Theorem 9.3]) it follows that $u \in C^{3,1/2}(\overline{\Omega}_h^{a,b})$ for every $\alpha < a < b < \beta$, which gives (39). In turn, (40) hold in the classical sense.

By (39) and (48) we have $f \in C^{0,1}((\alpha,\beta)) \cap C^1((\alpha,\beta) \setminus \{\alpha^0,\beta^0\})$. Hence, $F_m \in C^{1,1}((\alpha,\beta)) \cap C^2((\alpha,\beta) \setminus \{\alpha^0,\beta^0\})$ by (50). Moreover, again by (46) and (47), $A \in C^3((\alpha,\beta))$ and $B \in C^2((\alpha,\beta))$ and, by (51), $h'' \in C^{2,1}((\alpha,\beta)) \cap C^3((\alpha,\beta) \setminus \{\alpha^0,\beta^0\})$, which proves (38).

Step 2: We prove (41). Fix $\alpha < a < b < \beta$ and let $\varphi \in C_c^\infty((\alpha,\beta))$ be such that $\operatorname{supp}\varphi \subset [a,b]$ and (43) holds. Define

$$\overline{H}(x) := \int_{-\infty}^{x} (H(s) - H^0(s))\sqrt{1 + ((\check{h}^0)'(s))^2}\, ds\,, \tag{52}$$

By (42), (44), and (45), we have

$$\gamma \int_a^b \frac{h'\varphi'}{J} dx + \nu_0 \int_a^b \frac{h''\varphi''}{J^5} dx - \frac{5}{2}\nu_0 \int_a^b \frac{h'(h'')^2\varphi'}{J^7} dx + \int_a^b \overline{W}\varphi\, dx - \frac{1}{\tau}\int_a^b \overline{H}\varphi dx = 0\,,$$

where J is defined in (42). We integrate by parts once the integrals containing φ' and twice the integral containing φ'' to obtain

$$-\gamma \int_a^b \Big(\frac{h'}{J}\Big)'\varphi dx + \nu_0 \int_a^b \Big(\frac{h''}{J^5}\Big)''\varphi\, dx + \frac{5}{2}\nu_0 \int_a^b \Big(\frac{h'(h'')^2}{J^7}\Big)'\varphi dx + \int_a^b \overline{W}\varphi\, dx - \frac{1}{\tau}\int_a^b \overline{H}\varphi dx = 0\,,$$

By the arbitrariness of $\varphi \in C_c^\infty((a,b))$ satisfying (43) we obtain

$$-\gamma\Big(\frac{h'}{J}\Big)' + \nu_0\Big(\frac{h''}{J^5}\Big)'' + \frac{5}{2}\nu_0\Big(\frac{h'(h'')^2}{J^7}\Big)' + \overline{W} - \frac{1}{\tau}\overline{H} = m \tag{53}$$

where m is a Lagrange multiplier due to (43). Using (52) and differentiating

$$-\gamma\Big(\frac{h'}{J}\Big)'' + \nu_0\Big(\frac{h''}{J^5}\Big)''' + \frac{5}{2}\nu_0\Big(\frac{h'(h'')^2}{J^7}\Big)'' + \overline{W}' - \frac{1}{\tau}(H - H^0)J^0 = 0\,,$$

where J_0 is defined in (42). Dividing by J^0 and differentiating once again we obtain (41). $\square$

Remark 1. *From the special form of $\mathbb{C}$, it follows that*

$$\mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u = 0 \quad \text{in } \Omega_h\,,$$
$$2\mu(Eu)\nu^h + \lambda(\operatorname{div} u)\nu^h = 0 \quad \text{on } \Gamma_h\,,$$

where Γ_h is the graph of h in (α,β) and ν^h is the outward unit normal.

Given $(\alpha,\beta,h)\in\mathcal{A}_s$, $\alpha\le a<b\le\beta$, and $\eta>0$ we define

$$\Omega^{a,b}_{h,\eta} := \{(x,y)\in\mathbb{R}^2 : \; a<x<b\,,\; h(x)-\eta<y<h(x)\}\,.$$

Theorem 5. *Let $(\alpha,\beta,h)\in\mathcal{A}_s$, let $u\in\mathcal{A}_e(\alpha,\beta,h)$ be the minimizer of the functional $\mathcal{E}(\alpha,\beta,h,\cdot)$ defined in (15), let $\delta>0$, $\eta>0$, and $M>1$. Assume that there exist $\alpha<a<b<\beta$, with $b-a>4\delta$, such that*

$$h(x)\ge 2\eta \quad \text{for all } x\in[a,b]\,, \tag{54}$$
$$\int_\alpha^\beta |h''(x)|^2 dx \le M\,. \tag{55}$$

Then there exists a constant $C=C\,(\delta,\eta,M)>0$ (independent of α, β, a, b, h, h_0, and τ) such that

$$\|u\|_{C^{1,1/2}(\overline{\Omega}^{a+\delta,b-\delta}_{h,\eta})}\le C\,. \tag{56}$$

Proof. In what follows C denotes a positive constant whose value changes from formula to formula and which depends only on δ, η, M and the fixed parameters λ, μ, γ, γ_0, σ_0, A_0, e_0, L_0, and ν_0 of the problem. Let $\Omega_{-h}:=\{(x,y)\in\mathbb{R}^2 : \; \alpha<x<\beta\,,\; -h(x)<y<0\}$ and let v be the function defined by $v(x,y):=u(x,y+h(x))$ for every $(x,y)\in\Omega_{-h}$. Then $v\in H^1(\Omega_{-h};\mathbb{R}^2)$ and by (22) we have

$$\|v\|_{H^1(\Omega_{-h})}\le C\,. \tag{57}$$

It can be shown that v is a weak solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\mathbb{A}\nabla v)=0 \text{ in } \Omega_{-h}, \\ (\mathbb{A}\nabla v)e_2 = 0 \quad \text{on } (\alpha,\beta)\times\{0\}\,, \end{cases} \tag{58}$$

where

$$\begin{aligned}
(\mathbb{A}\xi)_{11} &= 2\mu(\xi_{11}-\xi_{12}h') + \lambda(\xi_{11}-\xi_{12}h'+\xi_{22})\,,\\
(\mathbb{A}\xi)_{12} &= -2\mu(\xi_{11}-\xi_{12}h')h' + \lambda(\xi_{11}-\xi_{12}h'+\xi_{22})h' + \mu\left(\xi_{12}+\xi_{21}-\xi_{22}h'\right)\,,\\
(\mathbb{A}\xi)_{21} &= \mu\left(\xi_{21}+\xi_{12}-\xi_{22}h'\right)\,,\\
(\mathbb{A}\xi)_{22} &= -\mu\left(\xi_{21}+\xi_{12}-\xi_{22}h'\right)h' + 2\mu\xi_{22} + \lambda(\xi_{11}+\xi_{22}-\xi_{12}h')\,.
\end{aligned} \tag{59}$$

Define $\Omega^{a,b}_{-h} := \{(x,y) \in \mathbb{R}^2 :\, a < x < b\,,\ -h(x) < y < 0\}$. Since $h \in C^{1,1/2}([\alpha,\beta])$ in view of (55), we have that

$$\|\mathbb{A}\|_{C^{0,1/2}(\overline{\Omega}^{a,b}_{-h})} \leq C\,. \tag{60}$$

Moreover, recalling that $\mathbb{C}$ satisfies the Legendre–Hadamard condition and the latter is preserved by changes of variables, we deduce that $\mathbb{A}$ satisfies the Legendre–Hadamard condition. Hence, we are in a position to apply Theorem 9.3 in [39] in each rectangle of the form $(x-2\delta, x+2\delta)\times(-2\eta,0)$ with $a+2\delta \leq x \leq b-2\delta$. Using (57) we obtain that $v \in C^{1,1/2}([x-\delta,x+\delta]\times[-\eta,0];\mathbb{R}^2)$ with $\|v\|_{C^{1,1/2}([x-\delta,x+\delta]\times[-\eta,0])} \leq C$. This implies that $\|v\|_{C^{1,1/2}([a+\delta,b-\delta]\times[-\eta,0])} \leq C$ and, in turn, (56). □

Let us define

$$B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0) := 1 + \frac{1}{\tau}\int_{\mathbb{R}} |H-H^0|\,dx + \frac{1}{\tau}|\alpha-\alpha^0| + \frac{1}{\tau}|\beta-\beta^0|\,, \tag{61}$$

where H and H^0 are defined in (19).

Theorem 6. *Under the assumptions of Theorem 4, suppose in addition that there exist $0<\eta_0<1$, $0<\eta_1<1$, and $M>1$ such that*

$$2\eta_0 \leq h'(\alpha)\,, \quad h'(\beta) \leq -2\eta_0\,, \tag{62}$$

$$h(x) \geq 2\eta_1 \quad \text{for all } x \in [\alpha+\delta_0, \beta-\delta_0]\,, \tag{63}$$

$$\int_\alpha^\beta |h''(x)|^2 dx \leq M\,, \tag{64}$$

where

$$\delta_0 := \eta_0^2/(4M) < 1/4\,. \tag{65}$$

Then $W(Eu(\cdot,h(\cdot))) \in L^1((\alpha,\beta))$, $h \in W^{4,1}((\alpha,\beta))$, and there exists a constant $c_0 = c_0(\eta_0,\eta_1,M) > 1$ (independent of α, β, h, h_0, and τ) such that

$$\int_\alpha^\beta W(Eu(x,h(x)))\,dx \leq c_0 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,, \tag{66}$$

$$\|h''\|_{L^\infty((\alpha,\beta))} \leq c_0 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,, \tag{67}$$

$$\|h'''\|_{L^\infty((\alpha,\beta))} \leq c_0 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^2, \tag{68}$$

$$\|h'''\|_{L^1((\alpha,\beta))} \leq c_0 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,, \tag{69}$$

$$\|h^{(\mathrm{iv})}\|_{L^1((\alpha,\beta))} \leq c_0 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^2. \tag{70}$$

Unless otherwise indicated, in the proofs of the rest of the paper C denotes a constant depending only on the constants η_0, η_1, and M of the previous theorem (and on the structural constants e_0, λ, μ, and L_0, and possibly on the exponents considered in the statements). The value of C can change from formula to formula.

In the proof of Theorem 6 we use the following estimate.

Lemma 7. *Let $\eta_0 > 0$, $M > 0$, $\alpha < \beta$, and $h \in H^2((\alpha,\beta)) \cap H^1_0((\alpha,\beta))$. Assume that*

$$2\eta_0 \le h'(\alpha)\,, \quad h'(\beta) \le -2\eta_0\,, \tag{71}$$

$$\int_\alpha^\beta |h''(x)|^2 dx \le M\,. \tag{72}$$

Then

$$\beta - \alpha \ge 16\eta_0^2/M\,. \tag{73}$$

Proof. By the fundamental theorem of calculus and by (71) for every $x \in (\alpha,\beta)$ we have

$$h'(x) = h'(\alpha) + \int_\alpha^x h''(s)\,ds \ge 2\eta_0 - \int_\alpha^x |h''(s)|\,ds\,. \tag{74}$$

We claim that there exists $x_\alpha \in (\alpha,\beta)$ such that

$$\int_\alpha^{x_\alpha} |h''(s)|\,ds = 2\eta_0 \quad \text{and} \quad \int_\alpha^x |h''(s)|\,ds < 2\eta_0 \quad \text{for } x \in (\alpha, x_\alpha)\,. \tag{75}$$

If not, by (74) we would have $h'(x) > 0$ for every $x \in (\alpha,\beta)$, which contradicts the assumption $h \in H^1_0((\alpha,\beta))$ and proves the claim. By (74) and (75) we have $h'(x) > 0$ for every $x \in (\alpha, x_\alpha)$.

In the same way we prove that there exists $x_\beta \in (\alpha,\beta)$ such that

$$\int_{x_\beta}^\beta |h''(s)|\,ds = 2\eta_0 \quad \text{and} \quad h'(x) < 0 \quad \text{for } x \in (x_\beta, \beta)\,. \tag{76}$$

Since $x_\alpha \le x_\beta$, from (75) and (76) we deduce that

$$4\eta_0 \le \int_\alpha^\beta |h''(s)|\,ds\,.$$

By Hölder's inequaliy we get

$$4\eta_0 \le M^{1/2}(\beta-\alpha)^{1/2}\,,$$

which gives (73). □

Proof of Theorem 6. **Step 1: A Variation of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h}, \boldsymbol{u})$.** Extend h to a function $h \in H^2((\alpha-1,\beta))$ by setting $h(x) := h'(\alpha)(x-\alpha)$ for $x \in (\alpha-1,\alpha]$. Using Hölder's inequality for $x \in (\alpha,\beta)$ we have

$$h'(x) \ge h'(\alpha) - (x-\alpha)^{1/2}\left(\int_\alpha^x |h''(s)|^2 ds\right)^{1/2} \ge 2\eta_0 - (x-\alpha)^{1/2}M^{1/2} \ge \eta_0 \tag{77}$$

provided $x - \alpha \le \eta_0^2/M$.

Then, using also the fact that $h(x) := h'(\alpha)(x-\alpha)$ for $x \in (\alpha-1, \alpha]$, we have

$$\eta_0 \le h'(x) \le L_0 \quad \text{for every } x \in [\alpha-\delta_0, \alpha+2\delta_0]\,. \tag{78}$$

Since $h(\alpha) = 0$ and $\alpha + 4\delta_0 < \beta$ by Lemma 7, we obtain

$$\eta_0(x-\alpha) \le h(x) \quad \text{for every } x \in [\alpha, \alpha+4\delta_0]\,, \tag{79}$$

$$|h(x)| \le L_0|x-\alpha| \quad \text{for every } x \in [\alpha-\delta_0, \alpha+2\delta_0]\,. \tag{80}$$

Take $\varphi_0 \in C^\infty(\mathbb{R})$ with $\varphi_0(0) = 1$, $\varphi_0(x) \ge 1/2$ for every $x \in [-\delta_0/2, \delta_0/2]$, $\int_0^{\delta_0} \varphi_0\, dx = 0$, and $\operatorname{supp}\varphi_0 \subset (-\delta_0, \delta_0)$. Define

$$\varphi(x) := \varphi_0(x-\alpha)\,, \quad x \in \mathbb{R}\,. \tag{81}$$

Let $\varepsilon_0 := \min\{1, \frac{1}{2}\delta_0\eta_0/\|\varphi_0\|_{C^1}\}$. Then for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \le \varepsilon_0$ we have

$$\begin{gathered} h'(x) + \varepsilon\varphi'(x) \ge \eta_0/2 \quad \text{for all } x \in [\alpha-\delta_0, \alpha+\delta_0]\,, \\ h(\alpha-\delta_0) + \varepsilon\varphi(\alpha-\delta_0) < 0 < h(\alpha+\delta_0) + \varepsilon\varphi(\alpha+\delta_0)\,. \end{gathered} \tag{82}$$

This implies that there exists a unique α_ε such that

$$\alpha_\varepsilon \in (\alpha-\delta_0, \alpha+\delta_0) \quad \text{and} \quad h(\alpha_\varepsilon) + \varepsilon\varphi(\alpha_\varepsilon) = 0\,. \tag{83}$$

Moreover, by the Implicit Function Theorem the function $\varepsilon \mapsto \alpha_\varepsilon$ is of class C^1 and so we can differentiate the previous identity to get

$$\frac{d\alpha_\varepsilon}{d\varepsilon} = -\frac{\varphi(\alpha_\varepsilon)}{h'(\alpha_\varepsilon) + \varepsilon\varphi'(\alpha_\varepsilon)}\,. \tag{84}$$

By (82),

$$\left|\frac{d\alpha_\varepsilon}{d\varepsilon}\right| \le \frac{2\|\varphi_0\|_{C^0}}{\eta_0} \quad \text{for all } \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,,$$

and, since $\alpha^0 = \alpha$, this implies that

$$|\alpha_\varepsilon - \alpha| \le \frac{2\|\varphi_0\|_{C^0}}{\eta_0}|\varepsilon|\,. \tag{85}$$

Since $\varphi_0(x) \ge 1/2$ for $|x| \le \delta_0/2$, we have
$\varphi(x) \ge 1/2$ for $|x-\alpha| \le \delta_0/2$. This, together with (85) implies that $\varphi(\alpha_\varepsilon) \ge \frac{1}{2}$ for $|\varepsilon| \le \varepsilon_1$, where $\varepsilon_1 := \min\{\delta_0\eta_0/(4\|\varphi_0\|_{C^0}), \varepsilon_0\}$. In turn, by (82) and (84),

$$\frac{d\alpha_\varepsilon}{d\varepsilon} < 0 \quad \text{for } |\varepsilon| \le \varepsilon_1\,. \tag{86}$$

Observe that (82) implies that $h + \varepsilon\varphi > 0$ in $(\alpha_\varepsilon, \alpha+\delta_0)$. On the other hand $h + \varepsilon\varphi = h \ge 0$ on $[\alpha+\delta_0, \beta]$.

In order to satisfy the area constraint (6) we fix a function $\psi_0 \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\psi_0 \subset (\delta_0, 2\delta_0)$ and $\int_{\delta_0}^{2\delta_0} \psi_0\, dx = 1$, and we define $\psi(x) := \psi_0(x-\alpha)$, $x \in \mathbb{R}$. Consider the function $h_\varepsilon \in H^2((\alpha_\varepsilon, \beta)) \cap H_0^1((\alpha_\varepsilon, \beta))$ defined by

$$h_\varepsilon(x) := h(x) + \varepsilon\varphi(x) + \omega_\varepsilon \psi(x)\,, \tag{87}$$

where $\omega_\varepsilon \in \mathbb{R}$ is the unique constant such that

$$\int_{\alpha_\varepsilon}^{\beta} h_\varepsilon\, dx = A_0\,. \tag{88}$$

Since $\int_\alpha^\beta \varphi\, dx = 0$, from (6) and (88) it follows that

$$\int_{\alpha_\varepsilon}^{\beta} (h(x) + \varepsilon\varphi(x) + \omega_\varepsilon\psi(x))\, dx = A_0 = \int_\alpha^\beta (h(x) + \varepsilon\varphi(x))\, dx\,.$$

Hence, using the fact that $\int_{\alpha_\varepsilon}^{\beta} \psi\, dx = 1$, we get

$$\int_\alpha^\beta (h(x) + \varepsilon\varphi(x))\, dx + \int_{\alpha_\varepsilon}^{\alpha} (h(x) + \varepsilon\varphi(x))\, dx + \omega_\varepsilon = \int_\alpha^\beta (h(x) + \varepsilon\varphi(x))\, dx\,,$$

and so

$$\frac{\omega_\varepsilon}{\varepsilon} = -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha} h(x)\, dx - \int_{\alpha_\varepsilon}^{\alpha} \varphi(x)\, dx\,.$$

To estimate the right-hand side we use (85) to get

$$\left|\int_{\alpha_\varepsilon}^{\alpha} \varphi(x)\, dx\right| \le \frac{2\|\varphi_0\|_{C^0}^2}{\eta_0}|\varepsilon|\,,$$

while, using also (80), we obtain

$$\left|\int_{\alpha_\varepsilon}^{\alpha} h(x)\, dx\right| \le \frac{4\|\varphi_0\|_{C^0}^2}{\eta_0^2}\varepsilon^2(L_0 + 2^{1/2}\delta_0^{1/2}M^{1/2})\,.$$

Combining these inequalities we have

$$\left|\frac{\omega_\varepsilon}{\varepsilon}\right| \le C|\varepsilon|\,. \tag{89}$$

Let $\varepsilon_2 := \min\{\frac{\eta_0}{2\|\varphi_0\|_{C^1} + 2C\|\psi_0\|_{C^1}}, \varepsilon_1\}$. We claim that for all $|\varepsilon| \le \varepsilon_2$ we have

$$h'_\varepsilon \ge \eta_0/2 \quad \text{in } (\alpha_\varepsilon, \alpha + 2\delta_0)\,. \tag{90}$$

Fix $|\varepsilon| \le \varepsilon_2$. Then by (78),

$$h'_\varepsilon = h' + \varepsilon\varphi' + \omega_\varepsilon \psi' \ge \eta_0 - |\varepsilon| \|\varphi_0\|_{C^1} - C\varepsilon^2 \|\psi_0\|_{C^1} \ge \eta_0 - |\varepsilon|(\|\varphi_0\|_{C^1} + C\|\psi_0\|_{C^1}) \ge \eta_0/2\,,$$

which proves the claim.

Since $\operatorname{supp}\psi \subset (\alpha+\delta_0, \alpha+2\delta_0)$ and $\alpha_\varepsilon \in (\alpha-\delta_0, \alpha+\delta_0)$, we have $\psi(\alpha_\varepsilon) = 0$. By (83) and (87), this gives $h_\varepsilon(\alpha_\varepsilon) = 0$, hence (90) implies that $h_\varepsilon > 0$ in $(\alpha_\varepsilon, \alpha+2\delta_0)$. Moreover, since $\operatorname{supp}\varphi \subset (\alpha-\delta_0, \alpha+\delta_0)$ and $\operatorname{supp}\psi \subset (\alpha+\delta_0, \alpha+2\delta_0)$ we have that

$$h_\varepsilon = h \quad \text{on } [\alpha+2\delta_0, \beta]\,. \tag{91}$$

We conclude that $h_\varepsilon \ge 0$ in $(\alpha_\varepsilon, \beta)$. In turn, also by (88), $(\alpha_\varepsilon, \beta, h_\varepsilon) \in \mathcal{A}_s$.

Let U be the interior of the set $\overline{\Omega}_h \cup ([\alpha-\delta_0, \beta] \times [-1, 0])$ and let $\hat{u}\colon \overline{U} \to \mathbb{R}^2$ be the function defined by

$$\hat{u}(x,y) := \begin{cases} u(x,y) & \text{if } (x,y) \in \overline{\Omega}_h\,, \\ (e_0 x, 0) & \text{if } (x,y) \in [\alpha-\delta_0, \beta] \times [-1,0]\,. \end{cases}$$

Since the two definitions match on $[\alpha,\beta] \times \{0\}$, we have $\hat{u} \in H^1(U;\mathbb{R}^2)$. Recalling that U has Lipschitz boundary, we can extend $\hat{u}$ to a function defined on $\mathbb{R}^2$, still denoted by $\hat{u}$, such that $\hat{u} \in H^1(\mathbb{R}^2;\mathbb{R}^2)$.

Hence, if we let u_ε be the restriction of $\hat{u}$ to Ω_{h_ε}, we have $u_\varepsilon(x,0) = (e_0 x, 0)$ for a.e. $x \in (\alpha_\varepsilon, \beta)$, which gives $(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) \in \mathcal{A}$.

Step 2: Evaluation of derivatives. We claim that

$$\begin{aligned} \liminf_{\varepsilon\to 0} & \frac{\mathcal{S}(\alpha_\varepsilon, \beta, h_\varepsilon) - \mathcal{S}(\alpha,\beta,h)}{\varepsilon} \\ & \ge \gamma \int_\alpha^\beta \frac{h'\varphi'}{\sqrt{1+(h')^2}}dx + \gamma\sqrt{1+(h'(\alpha))^2}\frac{1}{h'(\alpha)} - \gamma_0 \frac{1}{h'(\alpha)} \\ & \quad + \nu_0 \int_\alpha^\beta \frac{h''\varphi''}{(1+(h')^2)^{5/2}}dx - \frac{5}{2}\nu_0 \int_\alpha^\beta \frac{h'(h'')^2\varphi'}{(1+(h')^2)^{7/2}}dx\,. \end{aligned} \tag{92}$$

Since $(\alpha_\varepsilon, \beta, h_\varepsilon) \in \mathcal{A}_s$ and $\varphi(\alpha) = 1$, by (84), (87), and (89) we have

$$\begin{aligned} \liminf_{\varepsilon\to 0} \frac{\mathcal{S}(\alpha_\varepsilon, \beta, h_\varepsilon) - \mathcal{S}(\alpha,\beta,h)}{\varepsilon} &= \gamma \int_\alpha^\beta \frac{h'\varphi'}{\sqrt{1+(h')^2}}dx + \gamma\sqrt{1+(h'(\alpha))^2}\frac{1}{h'(\alpha)} \\ & \quad - \gamma_0 \frac{1}{h'(\alpha)} + \frac{\nu_0}{2} \liminf_{\varepsilon\to 0} I_\varepsilon\,, \end{aligned} \tag{93}$$

where

$$I_\varepsilon := \frac{1}{\varepsilon}\left[\int_{\alpha_\varepsilon}^\beta \frac{(h''_\varepsilon)^2}{(1+(h'_\varepsilon)^2)^{5/2}}dx - \int_\alpha^\beta \frac{(h'')^2}{(1+(h')^2)^{5/2}}dx\right].$$

Write

$$I_\varepsilon = \frac{1}{\varepsilon}\left[\int_{\alpha_\varepsilon}^{\beta} \frac{(h_\varepsilon'')^2}{(1+(h_\varepsilon')^2)^{5/2}}dx - \int_{\alpha_\varepsilon}^{\beta} \frac{(h'')^2}{(1+(h')^2)^{5/2}}dx\right]$$
$$+ \frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha} \frac{(h'')^2}{(1+(h')^2)^{5/2}}dx =: I_{\varepsilon,1} + I_{\varepsilon,2}\,.$$

By (87) and (89),

$$\lim_{\varepsilon\to 0} I_{\varepsilon,1} = 2\int_\alpha^\beta \frac{h''\varphi''}{(1+(h')^2)^{5/2}}dx - 5\int_\alpha^\beta \frac{h'(h'')^2\varphi'}{(1+(h')^2)^{7/2}}dx\,. \tag{94}$$

By (86), $\alpha_\varepsilon < \alpha$ for $\varepsilon > 0$ and $\alpha_\varepsilon > \alpha$ for $\varepsilon < 0$, provided $|\varepsilon| \le \varepsilon_2$. This implies that

$$\liminf_{\varepsilon\to 0} I_{\varepsilon,2} \ge 0\,.$$

From this inequality and from (93) and (94) we conclude that (92) holds. Since $\operatorname{Lip} h \le L_0$, from (64), (65), and (73), we deduce that

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{S}(\alpha_\varepsilon,\beta,h_\varepsilon) - \mathcal{S}(\alpha,\beta,h)}{\varepsilon}$$
$$\ge -(\gamma L_0\delta_0 + \nu_0 M^{1/2}\delta_0^{1/2} + (5/2)\nu_0 L_0 M)\|\varphi_0\|_{C^2} - \gamma/(2\eta_0)\,. \tag{95}$$

For simplicity of notation we abbreviate

$$\mathcal{T}^0(\cdot,\cdot,\cdot) := \mathcal{T}_\tau(\cdot,\cdot,\cdot;\alpha^0,\beta^0,h^0)$$

To evaluate $\frac{d}{d\varepsilon}\mathcal{T}^0(\alpha_\varepsilon,\beta,h_\varepsilon)\big|_{\varepsilon=0}$ we define

$$H_\varepsilon(x) := \int_{\alpha_\varepsilon}^{x} \check{h}_\varepsilon(\rho)\,d\rho\,, \quad \Phi(x) := \int_\alpha^x \check{\varphi}(\rho)\,d\rho\,, \tag{96}$$

where $\check{h}_\varepsilon$ and $\check{\varphi}$ are the extensions of h_ε and φ by zero outside $(\alpha_\varepsilon,\beta)$ and (α,β), respectively. Since $\int_0^{\delta_0}\varphi_0\,dx = 0$ and $\operatorname{supp}\varphi_0 \subset (-\delta_0,\delta_0)$, we have $\int_\alpha^\beta \varphi\,dx = 0$, hence

$$\Phi(x) = 0 \quad \text{for every } x \notin (\alpha,\beta)\,. \tag{97}$$

Observe that if $\alpha < x < \beta$ then $\alpha_\varepsilon < x < \beta$ for all $|\varepsilon|$ sufficiently small, and so by (84), (89), and the fact that $h(\alpha) = 0$,

$$\frac{d}{d\varepsilon}H_\varepsilon(x)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{\alpha_\varepsilon}^{x}(\hat{h} + \varepsilon\varphi + \omega_\varepsilon\psi)\,d\rho\bigg|_{\varepsilon=0} = \int_\alpha^x \varphi\,d\rho = \Phi(x)\,. \tag{98}$$

On the other hand, if $x < \alpha$, then $x < \alpha_\varepsilon$ for all $|\varepsilon|$ sufficiently small. Hence, $H_\varepsilon(x) = 0$ and so $\frac{d}{d\varepsilon}H_\varepsilon(x)\big|_{\varepsilon=0} = 0$. Moreover, $H_\varepsilon(x) = A_0$ for $x \ge \beta$ by (88), and so again $\frac{d}{d\varepsilon}H_\varepsilon(x)\big|_{\varepsilon=0} = 0$. By (97) it follows that (98) holds for all $x \in \mathbb{R} \setminus \{\alpha\}$.

By the regularity of h it follows from the Lebesgue dominated convergence theorem, (18), (84), and (98) that

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{T}^0(\alpha_\varepsilon, \beta, h_\varepsilon)\Big|_{\varepsilon=0} &= \frac{1}{\tau}\int_{\mathbb{R}}(H - H^0)\Phi\sqrt{1 + ((\check{h}^0)')^2}dx - \frac{\sigma_0}{\tau}(\alpha - \alpha^0)\frac{1}{h'(\alpha)} \qquad (99)\\ &\ge -(1 + L_0)\|\varphi_0\|_{C^0}2\delta_0\frac{1}{\tau}\int_{\mathbb{R}}|H - H^0|\,dx - \frac{\sigma_0}{\eta_0}\frac{1}{\tau}|\alpha - \alpha^0|\,,\end{aligned}$$

where in the second inequality we used (62) and the fact that $\operatorname{supp}\varphi_0 \subset (-\delta_0, \delta_0)$. Integrating by parts, and using (52), (96), and (97), we obtain

$$\int_{\mathbb{R}}(H - H^0)\Phi\sqrt{1 + ((\check{h}^0)')^2}dx = -\int_\alpha^\beta \overline{H}\varphi dx\,. \qquad (100)$$

Step 3: Proof of (66). In what follows C denotes a positive constant whose value changes from formula to formula and which depends only on η_0, η_1, M, and the fixed parameters λ, μ, γ, γ_0, σ_0, A_0, e_0, L_0, and ν_0 of the problem.

By Step 1, $(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) \in \mathcal{A}$ and because $(\alpha, \beta, h, u) \in \mathcal{A}$ is a minimizer of the total energy functional $\mathcal{F}^0$, we have that

$$\mathcal{F}^0(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) - \mathcal{F}^0(\alpha, \beta, h, u) \ge 0\,.$$

Then

$$\limsup_{\varepsilon\to 0^-}\frac{\mathcal{F}^0(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) - \mathcal{F}^0(\alpha, \beta, h, u)}{\varepsilon} \le 0$$

and so, by (21), (95), and (99)) we have

$$\limsup_{\varepsilon\to 0^-}\frac{\mathcal{E}(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) - \mathcal{E}(\alpha, \beta, h, u)}{\varepsilon} \le CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,. \qquad (101)$$

By (85), (86), and the fact that $\varepsilon < 0$ we have that $\alpha < \alpha_\varepsilon < \alpha + \delta_0/2$ for all $-\varepsilon_1 < \varepsilon < 0$. Since

$$\mathcal{E}(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) = \int_{\alpha_\varepsilon}^\beta\Big(\int_0^{h_\varepsilon(x)} W(E\hat{u}(x, y))\,dy\Big)dx\,,$$

$\operatorname{supp}\varphi \subset (\alpha - \delta_0, \alpha + \delta_0)$, and $\operatorname{supp}\psi \subset (\alpha + \delta_0, \alpha + 2\delta_0)$, we have that $h_\varepsilon = h$ in $(\alpha + 2\delta_0, \beta)$, and so for $-\varepsilon_1 < \varepsilon < 0$ we can write

$$\frac{\mathcal{E}(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) - \mathcal{E}(\alpha, \beta, h, u)}{\varepsilon} = -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x, y))\,dy\Big)dx$$

$$-\frac{1}{\varepsilon}\int_{\alpha}^{\alpha_\varepsilon}\Big(\int_0^{h(x)} W(Eu(x,y))\,dy\Big)dx \geq -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy\Big)dx\,. \tag{102}$$

By (79) we have that $h(x) \geq \eta_0\,\delta_0/4$ for all $x \in [\alpha+\delta_0/4, \alpha+4\delta_0]$. Hence, we can apply Theorem 5 to obtain that $W(Eu(x,h(x))) \leq C$ for all $x \in [\alpha+\delta_0/2, \alpha+2\delta_0]$. By (87), this implies that

$$\lim_{\varepsilon\to 0^-}\frac{1}{\varepsilon}\int_{\alpha+\delta_0/2}^{\alpha+2\delta_0}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy\Big)dx = -\int_{\alpha+\delta_0/2}^{\alpha+2\delta_0} W(Eu(x,h(x)))\varphi(x)\,dx \leq C. \tag{103}$$

Together with (101) and (102) this implies that

$$\limsup_{\varepsilon\to 0^-}\Big(-\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+\delta_0/2}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy\Big)dx\Big) \leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,. \tag{104}$$

Using the facts that $\varphi \geq 1/2$ in $(\alpha, \alpha+\delta_0/2)$ and $\operatorname{supp}\psi \subset (\alpha+\delta_0, \alpha+2\delta_0)$ we obtain that $h_\varepsilon = h + \varepsilon\varphi \leq h$ in $(\alpha, \alpha+\delta_0/2)$ for every $-\varepsilon_1 < \varepsilon < 0$. Hence,

$$-\frac{1}{\varepsilon}\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy \geq 0$$

and

$$-\frac{1}{\varepsilon}\chi_{(\alpha_\varepsilon,\alpha+\delta_0/2)}(x)\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy \to W(Eu(x,h(x)))\varphi(x)$$

for every $x \in (\alpha, \alpha+\delta_0/2)$. By Fatou's lemma, the fact that $\varphi \geq 1/2$ in $(\alpha, \alpha+\delta_0/2)$, and (104),

$$\begin{aligned}\frac{1}{2}\int_{\alpha}^{\alpha+\delta_0/2} W(Eu(x,h(x)))\,dx &\leq \liminf_{\varepsilon\to 0^-}\Big(-\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+\delta_0/2}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy\Big)dx\Big)\\ &\leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,.\end{aligned}$$

A similar argument gives the corresponding estimate over the interval $[\beta-\delta_0/2, \beta]$. To prove the estimate over the interval $[\alpha+\delta_0/2, \beta-\delta_0/2]$, we apply Theorem 5 with $a = \alpha+\delta_0/4$, $b = \beta-\delta_0/4$, and $\delta = \delta_0/4$. We observe that (54) is satisfied with $\eta = \min\{\eta_0\delta_0/8, \eta_1\}$ thanks to (63), (79), and a similar estimate near β. By (17) and (56) we obtain

$$\int_{\alpha+\delta_0/2}^{\beta-\delta_0/2} W(Eu(x,h(x)))\,dx \leq C \leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,,$$

where we used the fact that $1 \leq B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)$.

Combining the inequalities on $[\alpha, \alpha+\delta_0/2]$, $[\alpha+\delta_0/2, \beta-\delta_0/2]$, and $[\beta-\delta_0/2, \beta]$ we obtain (66).

Step 4: Regularity of h. Observe that by (6) and (63), we have that $\beta-\alpha-2\delta_0\leq A_0/(2\eta_1)$, and so

$$\beta-\alpha\leq A_0/(2\eta_1)+2\delta_0\,. \tag{105}$$

Let us fix $x_0\in(\alpha+\delta_0,\beta-\delta_o)$. By (51) we have

$$A(x)h''(x)=\int_{x_0}^{x}(B(s)h'(s)-\,F_m(s))\,ds+m_1 \tag{106}$$

for all $x\in(\alpha,\beta)$, where $A,\ B,\ F_m$ are defined in (46), (47), and (50). In particular we have

$$F_m(x):=F(x)-m(x-x_0)\,,\quad\text{ where }\quad F(x):=\int_{x_0}^{x}f(s)\,ds\,. \tag{107}$$

By (66) we have that $F_m\in W^{1,1}((\alpha,\beta))$. Since $A\in C^{0,1/2}([\alpha,\beta])$, $A\geq\nu_0/(1+L_0^2)^{5/2}$, $B\in L^1((\alpha,\beta))$, and $F_m\in C^0([\alpha,\beta])$, (106) implies that $h''\in C^0([\alpha,\beta])$. In turn, this gives $A\in C^1([\alpha,\beta])$ and $B\in C^0([\alpha,\beta])$ by (46) and (47). Hence, the right-hand side of (106) is $C^1([\alpha,\beta])$, therefore $h''\in C^1([\alpha,\beta])$. In turn, $A\in C^2([\alpha,\beta])$ and $B\in C^1([\alpha,\beta])$, and so by (106), $h''\in W^{2,1}((\alpha,\beta))$. By differentiating (106) we get

$$A(x)h'''(x)=-A'(x)h''(x)+B(x)h'(x)-\,F_m(x)\,. \tag{108}$$

By (46), (47), and (105) we have

$$\|A\|_{H^1((\alpha,\beta))}+\|B\|_{L^1((\alpha,\beta))}\leq C\,. \tag{109}$$

To estimate F_m, we first obtain bounds for the function f defined in (48). In view of (66) and (105),

$$\|f\|_{L^1((\alpha,\beta))}\leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,. \tag{110}$$

Hence, by (107),

$$\|F\|_{L^\infty((\alpha,\beta))}\leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)\,. \tag{111}$$

Next we estimate the constant m in (107). Let $\zeta_0\in C_c^\infty((-\delta_0/2,\delta_0/2))$ be such that $\int_{-\delta_0/2}^{\delta_0/2}\zeta_0(x)\,dx=0$ and $\int_{-\delta_0/2}^{\delta_0/2}\zeta_0(x)x^2dx=1$, and let $\zeta(x):=\zeta_0(x-x_0)$. Since $\alpha+\delta_0<x_0<\beta-\delta_0$, we have that $\zeta\in C_c^\infty((\alpha+\delta_0/2,\beta-\delta_0/2))$. Multiplying (106) by ζ and integrating over $(\alpha+\delta_0/2,\beta-\delta_0/2)$ we obtain

$$\int_{\alpha+\delta_0/2}^{\beta-\delta_0/2}A(x)h''(x)\zeta(x)\,dx=\int_{\alpha+\delta_0/2}^{\beta-\delta_0/2}\zeta(x)\int_{x_0}^{x}(B(s)h'(s)-\,F(s))\,ds\,dx-\frac{m}{2}\,,$$

where we used the facts that $\int_{\alpha+\delta_0/2}^{\beta-\delta_0/2} \zeta(x)\,dx = 0$ and $\int_{\alpha+\delta_0/2}^{\beta-\delta_0/2} \zeta(x)(x-x_0)^2 dx = 1$. From (64), (109), (111), and Hölder's inequality, it follows that

$$|m| \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,. \tag{112}$$

Together with (105), (107), and (111), this gives

$$\begin{aligned} &\| F_m\|_{L^\infty((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0) \\ &\quad \text{and} \quad \| F_m\|_{L^1((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,. \end{aligned} \tag{113}$$

Using the fact that $A \geq \nu_0/(1+L_0^2)^{5/2}$, (64), (108), (109), (113), and Hölder's inequality, we obtain

$$\|h'''\|_{L^1((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,. \tag{114}$$

This proves (69).

For every $x \in (\alpha, \beta)$ we have

$$|h''(x)| \leq (\beta-\alpha)^{-1}\|h''\|_{L^1((\alpha,\beta))} + \|h'''\|_{L^1((\alpha,\beta))}\,.$$

By (11), (64), and (114), this inequality yields

$$\|h''\|_{L^\infty((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,, \tag{115}$$

and proves (67).

By (46), (47), (105), (114), and (115) we have

$$\|A'\|_{L^\infty((\alpha,\beta))}+\|A''\|_{L^1((\alpha,\beta))}+\|B\|_{L^\infty((\alpha,\beta))}+\|B'\|_{L^1((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)\,. \tag{116}$$

Using again the fact that $A \geq \nu_0/(1+L_0^2)^{5/2}$ and (108), from (113), (115), and (116) we get

$$\|h'''\|_{L^\infty((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^2\,,$$

which proves (68).

Differentiating (108) gives

$$A(x)h^{(\mathrm{iv})}(x) = -2A'(x)h'''(x) + (B(x) - A''(x))h''(x) + B'(x)h'(x) - f(x) + m\,.$$

Using the fact that $A \geq \nu_0/(1+L_0^2)^{5/2}$, (112), (105), (110), (114), (115), and (116) we obtain

$$\|h^{(\mathrm{iv})}\|_{L^1((\alpha,\beta))} \leq CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^2\,, \tag{117}$$

which proves (70) and concludes the proof of the theorem. □

Remark 2. *Since $h \in C^2([\alpha,\beta])$ in view of Step 4 in the previous proof, the limit inferior of I_ε in (93) is actually a limit and, by (84) and (94), it is given by*

$$\lim_{\varepsilon\to 0} I_\varepsilon = 2\int_\alpha^\beta \frac{h''\varphi''}{(1+(h')^2)^{5/2}}dx - 5\int_\alpha^\beta \frac{h'(h'')^2\varphi'}{(1+(h')^2)^{7/2}}dx + \frac{(h''(\alpha))^2}{(1+(h'(\alpha))^2)^{5/2}}\frac{1}{h'(\alpha)}\,.$$

Hence, (93) becomes

$$\frac{d}{d\varepsilon}\mathcal{S}(\alpha_\varepsilon,\beta,h_\varepsilon)\Big|_{\varepsilon=0} = \gamma\int_\alpha^\beta \frac{h'\varphi'}{\sqrt{1+(h')^2}}dx + \gamma\sqrt{1+(h'(\alpha))^2}\frac{1}{h'(\alpha)} - \gamma_0\frac{1}{h'(\alpha)}$$
$$+\,\nu_0\int_\alpha^\beta \frac{h''\varphi''}{(1+(h')^2)^{5/2}}dx - \frac{5}{2}\nu_0\int_\alpha^\beta \frac{h'(h'')^2\varphi'}{(1+(h')^2)^{7/2}}dx + \frac{\nu_0}{2}\frac{(h''(\alpha))^2}{(1+(h'(\alpha))^2)^{5/2}}\frac{1}{h'(\alpha)}\,. \tag{118}$$

4 Flattening the Boundary

In this section we transform the intersection of a neighborhood of $(\alpha,0)$ with Ω_h into the triangle

$$A_r^m := \{(x,y)\in\mathbb{R}^2 : 0<x<r\,,\,0<y<mx\}\,, \tag{119}$$

for some $r>0$ and $m>0$. We fix $0<\eta_0<1$ and $M>1$ and assume that

$$\alpha<\beta,\quad h\in W^{4,1}((\alpha,\beta))\subset C^3([\alpha,\beta])\,, \tag{120}$$
$$h>0 \text{ in } (\alpha,\beta)\,,\quad h(\alpha)=h(\beta)=0\,, \tag{121}$$
$$h'(\alpha)\ge 2\eta_0\,,\quad h'(\beta)\le -2\eta_0\,,\quad \operatorname{Lip} h\le L_0\,, \tag{122}$$
$$\int_\alpha^\beta |h''(x)|^2dx \le M\,. \tag{123}$$

For simplicity, in this section we assume that $\alpha=0$ and we write $h_0' := h'(0)$, $h_0'' := h''(0)$, and $h_0''' := h'''(0)$.

Given $r>0$ we set $I_r := (0,r)$ and $\Omega_h^{0,r} := \Omega_h \cap (I_r\times\mathbb{R})$. Assume that

$$0<r\le\delta_0 = \frac{\eta_0^2}{4M} < \frac{1}{4}\,. \tag{124}$$

By Lemma 7 we have $64r\le\beta-\alpha$. Define for $x\in I_r$,

$$\sigma(x) := \frac{h_0' x}{h(x)} \tag{125}$$

and the diffeomorphisms $\Phi : I_r\times\mathbb{R}\to I_r\times\mathbb{R}$ and $\Psi : I_r\times\mathbb{R}\to I_r\times\mathbb{R}$ by

$$\Phi(x,y) := (x, y/\sigma(x)) \quad\text{and}\quad \Psi(x,y) := (x,\sigma(x)y)\,. \tag{126}$$

Throughout tis section we set $m = h_0'$. Observe that

$$\Psi(\Omega_h^{0,r}) = A_r^m \,, \tag{127}$$

where A_r^m is the triangle introduced in (119). By (120) and a direct computation, we have that $\sigma \in C^2(\bar{I}_r)$ and that

$$\sigma'(x) = \frac{h_0' h(x) - h_0' x h'(x)}{(h(x))^2} \,, \tag{128}$$

$$\sigma''(x) = -\frac{h_0' x h''(x) h(x) + 2h_0' h(x) h'(x) - 2h_0' x (h'(x))^2}{(h(x))^3} \,, \tag{129}$$

if $x \in I_r$ and

$$\sigma(\alpha) = 1 \,, \quad \sigma'(\alpha) = -\frac{h_0''}{2h_0'} \,, \quad \sigma''(\alpha) = \frac{(h_0'')^2}{2(h_0')^2} - \frac{h_0'''}{3h_0'} \,. \tag{130}$$

In turn,

$$\Phi \in C^2(\overline{A_r^m}; \mathbb{R}^2) \quad \text{and} \quad \Psi \in C^2(\overline{\Omega_h^{0,r}}; \mathbb{R}^2) \,. \tag{131}$$

Lemma 8. *Under the assumptions (120)–(123), let r be as in (124). Then there exists a constant $C = C(\eta_0, M) > 0$, independent of r, such that*

$$\|\sigma - 1\|_{L^\infty(I_r)} \le \frac{r^{1/2}}{\eta_0} \|h''\|_{L^2(I_r)} \le Cr \|h''\|_{L^\infty(I_r)} \,, \tag{132}$$

$$\|\sigma'\|_{L^\infty(I_r)} \le C|h_0''| + Cr\big(|h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) \,, \tag{133}$$

$$\|\sigma''\|_{L^\infty(I_r)} \le C\big(|h_0''|^2 + |h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) + Cr^2\big(|h_0'''|^2 + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)}\big) \,. \tag{134}$$

Moreover,

$$\sup_{(x,y)\in\Omega_h^{0,r}} |y\sigma'(x)| \le Cr|h_0''| + Cr^2\big(|h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) \,, \tag{135}$$

$$\sup_{(x,y)\in\Omega_h^{0,r}} |y\sigma''(x)| \le Cr\big(|h_0''|^2 + |h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) + Cr^3\big(|h_0'''|^2 + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)}\big) \,, \tag{136}$$

$$\sup_{(x,y)\in A_r^m} |y\sigma'(x)| \le Cr|h_0''| + Cr^2\big(|h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) \,, \tag{137}$$

$$\sup_{(x,y)\in A_r^m} |y\sigma''(x)| \le Cr\big(|h_0''|^2 + |h_0'''| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) + Cr^3\big(|h_0'''|^2 + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)}\big) \,. \tag{138}$$

Proof. Using Taylor's formula with integral remainder for $x \in I_r$ we can write

$$\sigma(x) - 1 = \frac{h_0' x - h(x)}{h(x)} = -\frac{\int_0^x h''(s)(x-s)\,ds}{h(x)} \,.$$

If $x \in I_r$ by Hölder's inequality, (122), (123), and (124) we have

$$h'(x) = h'_0 + \int_0^x h''(s)\,ds \geq 2\eta_0 - M^{1/2}x^{1/2} \geq \eta_0\,.$$

Hence,

$$h(x) = \int_0^x h'(s)\,ds \geq \eta_0 x\,. \tag{139}$$

In turn, by Hölder's inequality,

$$|\sigma(x) - 1| \leq \frac{x^{3/2}}{\eta_0 x}\|h''\|_{L^2(I_r)} \leq \frac{r^{1/2}}{\eta_0}\|h''\|_{L^2(I_r)}$$

for every $x \in I_r$, which gives the estimate (132).

To prove (133), in view of (128), we use Taylor's formula with integral remainder to get

$$\begin{aligned} h'_0 h(x) - h'_0 x h'(x) &= -\frac{1}{2}h'_0 h''_0 x^2 - \frac{1}{3}h'_0 h'''_0 x^3 \\ &\quad - \frac{1}{2}h'_0 x \int_0^x h^{(\mathrm{iv})}(s)(x-s)^2 ds + \frac{1}{6}h'_0 \int_0^x h^{(\mathrm{iv})}(s)(x-s)^3 ds\,. \end{aligned}$$

Hence,

$$|h'_0 h(x) - h'_0 x h'(x)| \leq \frac{1}{2}h'_0|h''_0|x^2 + \frac{1}{3}h'_0|h'''_0|x^3 + \frac{2}{3}h'_0 x^3 \int_0^x |h^{(\mathrm{iv})}(s)|\,ds\,.$$

Using (139), if follows from (128) that

$$|\sigma'(x)| \leq \frac{L_0}{\eta_0^2}\left(\frac{1}{2}|h''_0| + \frac{1}{3}r|h'''_0| + \frac{2}{3}r\|h^{(\mathrm{iv})}\|_{L^1(I_r)}\right),$$

which proves (133).

To prove (134), in view of (129), we use Taylor's formula with integral remainder and the inequality $2ab \leq a^2 + b^2$ to estimate

$$\begin{aligned} &|h'_0 x h''(x)h(x) + 2h'_0 h(x)h'(x) - 2h'_0 x(h'(x))^2| \\ &\leq C\big[|h''_0|^2 x^3 + |h'''_0|x^3 + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}x^3 + |h'''_0|^2 x^5 + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)}x^5\big]\,, \end{aligned}$$

where $C = C(L_0) > 0$. Using (129) and (139), for $x \in I_r$ we get

$$|\sigma''(x)| \leq \frac{C}{\eta_0^3}\big[|h''_0|^2 + |h'''_0| + \|h^{(\mathrm{iv})}\|_{L^1(I_r)} + r^2(|h'''_0|^2 + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\big]\,,$$

which proves (134).

To prove (135) and (136), in view of (133) and (134), it is enough to observe that if $(x,y) \in \Omega_h^{0,r}$, then $0 < y < L_0 r$.

Finally, if $(x,y) \in A_r^m$, then $0 < y < mr \le L_0 r$, where in the last inequality we used the fact that $m = h_0' \le L_0$ by (122)) Together with (133) and (134), this inequality proves (137) and (138). This concludes the proof. □

Remark 3. *By (123), (124), and (132) we have that*

$$\|\sigma - 1\|_{L^\infty(I_r)} \le 1/2\,.$$

Lemma 9. *Under the assumptions (120)–(123), let r be as in (124), and let $1 \le p < \infty$. If $f \in L^p(\Omega_h^{0,r})$ and $w \in W^{2,p}(\Omega_h^{0,r})$, then $f \circ \Phi \in L^p(A_r^m)$ and $w \circ \Phi \in W^{2,p}(A_r^m)$. Moreover the following estimates hold:*

$$\begin{aligned}
\|f\circ\Phi\|_{L^p(A_r^m)} &\le C_p \|f\|_{L^p(\Omega_h^{0,r})}\\
\|\nabla(w\circ\Phi)\|_{L^p(A_r^m)} &\le C_p\big(1 + \sup_{(x,y)\in A_r^m} y|\sigma'(x)|\big)\|\nabla w\|_{L^p(\Omega_h^{0,r})}\\
\|\nabla^2(w\circ\Phi)\|_{L^p(A_r^m)} &\le C_p\big(1 + \sup_{(x,y)\in A_r^m} y^2(\sigma'(x))^2\big)\|\nabla^2 w\|_{L^p(\Omega_h^{0,r})}\\
+C_p\big(1 + \sup_{x\in I_r}|\sigma'(x)| &+ \sup_{(x,y)\in A_r^m} y(\sigma'(x))^2 + \sup_{(x,y)\in A_r^m} y|\sigma''(x)|\big)\|\nabla w\|_{L^p(\Omega_h^{0,r})}\,,
\end{aligned}$$

where the constant C_p depends only on p.

Proof. In this proof C is an absolute constant, independent of all other parameters, whose value can change from formula to formula. Since $\det \nabla\Phi(x,y) = \frac{1}{\sigma(x)}$, by a change of variables and Remark 3 we have

$$\int_{A_r^m} |f\circ\Phi|^p dxdy = \int_{\Omega_h^{0,r}} \sigma |f|^p dxdy \le \frac{3}{2}\int_{\Omega_h^{0,r}} |f|^p dxdy\,. \tag{140}$$

By (126) and Remark 3, it follows by the chain rule that

$$|\nabla(w\circ\Phi)(x,y)| \le C(1 + y|\sigma'(x)|)|\nabla w(\Phi(x,y))|\,. \tag{141}$$

Hence, by (140), we have

$$\|\nabla(w\circ\Phi)\|_{L^p(A_r^m)} \le C\big(1 + \sup_{(x,y)\in A_r^m} y|\sigma'(x)|\big)\|\nabla w\|_{L^p(\Omega_h^{0,r})}\,.$$

Similarly, again by Remark 3 and the chain rule

$$\begin{aligned}
|\nabla^2(w\circ\Phi)(x,y)| \le{}& C(1 + y^2(\sigma'(x))^2)|\nabla^2 w(\Phi(x,y))|\\
&+ C(1 + |\sigma'(x)| + y(\sigma'(x))^2 + y|\sigma''(x)|)|\nabla w(\Phi(x,y))|\,.
\end{aligned}$$

Hence, by (140) we obtain

$$\|\nabla^2(w\circ\Phi)\|_{L^p(A_r^m)} \le C\big(1+\sup_{(x,y)\in A_r^m} y^2(\sigma'(x))^2\big)\|\nabla^2 w\|_{L^p(\Omega_h^{0,r})}$$
$$+C\big(1+\sup_{x\in I_r}|\sigma'(x)|+\sup_{(x,y)\in A_r^m} y(\sigma'(x))^2+\sup_{(x,y)\in A_r^m} y|\sigma''(x)|\big)\|\nabla w\|_{L^p(\Omega_h^{0,r})}\,,$$

which concludes the proof. □

Remark 4. *Similarly, one can show that if $f\in L^p(A_r^m)$ and $w\in W^{2,p}(A_r^m)$, then $f\circ\Psi\in L^p(\Omega_h^{0,r})$ and $w\circ\Psi\in W^{2,p}(\Omega_h^{0,r})$ and the following estimates hold*

$$\|f\circ\Psi\|_{L^p(\Omega_h^{0,r})}\le C_p\|f\|_{L^p(A_r^m)}$$
$$\|\nabla(w\circ\Psi)\|_{L^p(\Omega_h^{0,r})}\le C_p\big(1+\sup_{(x,y)\in\Omega_h^{0,r}} y|\sigma'(x)|\big)\|\nabla w\|_{L^p(A_r^m)}$$
$$\|\nabla^2(w\circ\Psi)\|_{L^p(\Omega_h^{0,r})}\le C_p\big(1+\sup_{(x,y)\in\Omega_h^{0,r}} y^2(\sigma'(x))^2\big)\|\nabla^2 w\|_{L^p(A_r^m)}$$
$$+C_p\big(1+\sup_{x\in I_r}|\sigma'(x)|+\sup_{(x,y)\in\Omega_h^{0,r}} y(\sigma'(x))^2+\sup_{(x,y)\in\Omega_h^{0,r}} y|\sigma''(x)|\big)\|\nabla w\|_{L^p(A_r^m)}\,,$$

where the constant C_p depends only on p.

Given $r>0$ as in (124), let $\Gamma_h^{0,r}:=\Gamma_h\cap(I_r\times\mathbb{R})$, where Γ_h is the graph of h, and let $\Gamma:=\{(x,mx):\,0<x<r\}\subset\partial A_r^m$. For $x\in I_r$ let $\nu^h(x):=(-h'(x),1)/\sqrt{1+(h'(x))^2}$ be the outer unit normal to Ω_h on Γ_h, let $\nu^0:=(-h_0',1)/\sqrt{1+(h_0')^2}$ be the outer unit normal to A_r^m on Γ, and let $\omega_i\colon I_r\to\mathbb{R}$ be the functions defined by

$$\begin{aligned}
\omega_1(x)&:=\nu_1^0-\nu_1^h(x)=-\frac{h_0'}{\sqrt{1+|h_0'|^2}}+\frac{h'(x)}{\sqrt{1+|h'(x)|^2}}\,,\\
\omega_2(x)&:=\nu_2^0-\nu_2^h(x)=\frac{1}{\sqrt{1+|h_0'|^2}}-\frac{1}{\sqrt{1+|h'(x)|^2}}\,,\\
\omega_3(x)&:=-\sigma'(x)h(x)\nu_1^h(x)=\sigma'(x)h(x)\frac{h'(x)}{\sqrt{1+|h'(x)|^2}}\,,\\
\omega_4(x)&:=-(\sigma(x)-1)\nu_1^h(x)=(\sigma(x)-1)\frac{h'(x)}{\sqrt{1+|h'(x)|^2}}\,,\\
\omega_5(x)&:=\sigma'(x)h(x)\nu_2^h(x)=\sigma'(x)h(x)\frac{1}{\sqrt{1+|h'(x)|^2}}\,,\\
\omega_6(x)&:=(\sigma(x)-1)\nu_2^h(x)=(\sigma(x)-1)\frac{1}{\sqrt{1+|h'(x)|^2}}\,.
\end{aligned}\tag{142}$$

Lemma 10. *Under the assumptions (120)–(123), let r be as in (124), let $1\le p<\infty$, let $u\in W^{2,p}_{\rm loc}(\Omega_h^{0,r};\mathbb{R}^2)$, with $u\in W^{2,p}(\overline{\Omega}_h^{\rho,r};\mathbb{R}^2)$ for every $0<\rho<r$, let $m=h_0'$, let $v\colon A_r^m\to\mathbb{R}^2$ be defined by $v(x,y)=u(x,y/\sigma(x))$, and let $g\in W^{1,p}(\Omega_h^{0,r};\mathbb{R}^2)$. Assume*

that

$$(\mathbb{C}Eu)\nu^h = 2\mu(Eu)\nu^h + \lambda(\operatorname{div} u)\nu^h = g \quad on\ \Gamma_h^{0,r}\,. \tag{143}$$

Then

$$(\mathbb{C}Ev)\nu = 2\mu(Ev)\nu + \lambda(\operatorname{div} v)\nu = g\circ\Phi + \hat{g}^v + \check{g}^v \quad on\ \Gamma_r^m\,, \tag{144}$$

where $\hat{g}^v = (\hat{g}_1^v, \hat{g}_2^v)$ *and* $\check{g}^v = (\check{g}_1^v, \check{g}_2^v)$ *are defined by*

$$\hat{g}_1^v := (2\mu\omega_1 + \lambda\omega_1)\partial_x v_1 + \mu\omega_2\partial_x v_2\,, \tag{145}$$

$$\hat{g}_2^v := \lambda\omega_2\partial_x v_1 + \mu\omega_1\partial_x v_2\,. \tag{146}$$

$$\check{g}_1^v := (\mu\omega_2 + 2\mu\omega_3 + \lambda\omega_3 - \mu\omega_6)\partial_y v_1 + (\lambda\omega_1 + \lambda\omega_4 - \mu\omega_5)\partial_y v_2\,, \tag{147}$$

$$\check{g}_2^v := (\mu\omega_1 + \mu\omega_4 - \lambda\omega_5)\partial_y v_1 + (2\mu\omega_2 + \lambda\omega_2 + \mu\omega_3 - 2\mu\omega_6 - \lambda\omega_6)\partial_y v_2\,. \tag{148}$$

Remark 5. *If* $u = 0$ *on* $\partial\Omega_h^{0,r} \setminus \Omega_h^{0,r}$*, then* $\hat{g}^v = 0$ *on* $(0,r)\times\{0\}$ *and* $\check{g}^v = 0$ *on* $\{r\}\times(0,mr)$*.*

Proof of Lemma 10. Since $u(x,y) = v(x,\sigma(x)y)$, the partial derivatives of u are

$$\partial_x u(x,y) = \partial_x v(x,\sigma(x)y) + \partial_y v(x,\sigma(x)y)\sigma'(x)y \quad \text{and} \quad \partial_y u(x,y) = \partial_y v(x,\sigma(x)y)\sigma(x)\,.$$

So at $(x,h(x))$ the first component of $2\mu(Eu)\nu^h + \lambda(\operatorname{div} u)\nu^h$ is

$$2\mu\left(\partial_x v_1 + \partial_y v_1\sigma' y\right)\nu_1^h + \mu(\partial_x v_2 + \partial_y v_2\sigma' y + \partial_y v_1\sigma)\nu_2^h + \lambda\left(\partial_x v_1 + \partial_y v_1\sigma' y + \partial_y v_2\sigma\right)\nu_1^h\,,$$

while the second component is

$$\mu\left(\partial_x v_2 + \partial_y v_2\sigma' y + \partial_y v_1\sigma\right)\nu_1^h + 2\mu\partial_y v_2\sigma\nu_2^h + \lambda\left(\partial_x v_1 + \partial_y v_1\sigma' y + \partial_y v_2\sigma\right)\nu_2^h$$

where v_1 and v_2 are computed at $(x,\sigma(x)h(x)) = (x, h_0' x)$ and σ and σ' and ν^h at x. By adding and subtracting some terms containing ν_1^0 and ν_2^0, the first component can be written as

$$\begin{aligned}
&2\mu\partial_x v_1\nu_1^0 + \mu(\partial_x v_2 + \partial_y v_1)\nu_2^0 + \lambda\left(\partial_x v_1 + \partial_y v_2\right)\nu_1^0\\
&- 2\mu\partial_x v_1(\nu_1^0 - \nu_1^h) - \mu(\partial_x v_2 + \partial_y v_1)(\nu_2^0 - \nu_2^h) - \lambda\left(\partial_x v_1 + \partial_y v_2\right)(\nu_1^0 - \nu_1^h)\\
&+ 2\mu\partial_y v_1\sigma' y\nu_1^h + \mu(\partial_y v_2\sigma' y + \partial_y v_1(\sigma - 1))\nu_2^h + \lambda\left(\partial_y v_1\sigma' y + \partial_y v_2(\sigma - 1)\right)\nu_1^h
\end{aligned}$$

and the second component as

$$\begin{aligned}
&\mu\left(\partial_x v_2 + \partial_y v_1\right)\nu_1^0 + 2\mu\partial_y v_2\nu_2^0 + \lambda\left(\partial_x v_1 + \partial_y v_2\right)\nu_2^0\\
&- \mu\left(\partial_x v_2 + \partial_y v_1\right)(\nu_1^0 - \nu_1^h) - 2\mu\partial_y v_2(\nu_2^0 - \nu_2^h) - \lambda\left(\partial_x v_1 + \partial_y v_2\right)(\nu_2^0 - \nu_2^h)\\
&+ \mu\left(\partial_y v_2\sigma' y + \partial_y v_1(\sigma - 1)\right)\nu_1^h + 2\mu\partial_y v_2(\sigma - 1)\nu_2^h + \lambda\left(\partial_y v_1\sigma' y + \partial_y v_2(\sigma - 1)\right)\nu_2^h
\end{aligned}$$

Hence, using (143) we obtain (144) with

$$\hat{g}_1^v := 2\mu\partial_x v_1(\nu_1^0 - \nu_1^h) + \mu\partial_x v_2(\nu_2^0 - \nu_2^h) + \lambda\partial_x v_1(\nu_1^0 - \nu_1^h)$$

$$\hat{g}_2^v := \mu\partial_x v_2(\nu_1^0 - \nu_1^h) + \lambda\partial_x v_1(\nu_2^0 - \nu_2^h)$$

and

$$\begin{aligned}\check{g}_1^v &:= \mu\partial_y v_1(\nu_2^0 - \nu_2^h) + \lambda\partial_y v_2(\nu_1^0 - \nu_1^h) - 2\mu\partial_y v_1\sigma' h\nu_1^h \\ &\quad - \mu(\partial_y v_2\sigma' h + \partial_y v_1(\sigma-1))\nu_2^h - \lambda\left(\partial_y v_1\sigma' h + \partial_y v_2(\sigma-1)\right)\nu_1^h\,, \\ \check{g}_2^v &:= \mu\partial_y v_1(\nu_1^0 - \nu_1^h) + 2\mu\partial_y v_2(\nu_2^0 - \nu_2^h) + \lambda\partial_y v_2(\nu_2^0 - \nu_2^h) \\ &\quad - \mu\left(\partial_y v_2\sigma' h + \partial_y v_1(\sigma-1)\right)\nu_1^h - 2\mu\partial_y v_2(\sigma-1)\nu_2^h - \lambda\left(\partial_y v_1\sigma' h + \partial_y v_2(\sigma-1)\right)\nu_2^h\,.\end{aligned}$$

Using (142) we obtain (145), (146), (147), and (148). □

For technical reasons we need a precise estimate of the L^∞ norms of the functions ω_i defined in (142) and of their derivatives.

Lemma 11. *Under the assumptions (120)–(123), let r be as in (124), and let ω_i, $i = 1, \dots, 6$ be defined as in (142). Then there exists a constant $C = C(\eta_0, M) > 0$ such that*

$$\|\omega_i\|_{L^\infty(I_r)} \le Cr\|h''\|_{L^\infty(I_r)} + Cr^2(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}) \tag{149}$$

$$\begin{aligned}\|\omega_i'\|_{L^\infty(I_r)} \le C\|h''\|_{L^\infty(I_r)} &+ Cr\big(\|h''\|^2_{L^\infty(I_r)} + \|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big) \\ &+ Cr^3\big(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)}\big)\end{aligned} \tag{150}$$

for $i = 1, \dots, 6$.

Proof. Define

$$f_1(t) = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad f_2(t) = \frac{1}{\sqrt{1+t^2}} \quad \text{for } t \in \mathbb{R}\,. \tag{151}$$

Observe that $\omega_1(x) = f_1(h'(x)) - f_1(h_0')$ and $\omega_2(x) = f_2(h_0') - f_2(h'(x))$. Since

$$f_1'(t) = \frac{1}{(1+t^2)^{3/2}} \quad \text{and} \quad f_2'(t) = -\frac{t}{(1+t^2)^{3/2}}\,, \tag{152}$$

the functions f_1 and f_2 are Lipschitz continuous with Lipschitz constant one. Hence,

$$|\omega_i(x)| \le |h'(x) - h_0'| \quad \text{for } i = 1, 2\,,$$

and the estimate (149) follows by the mean value theorem. On the other hand,

$$|\omega_1'(x)| = |f_1'(h'(x))h''(x)| \le |h''(x)| \quad \text{and} \quad |\omega_2'(x)| = |f_2'(h'(x))h''(x)| \le |h''(x)|\,,$$

which gives (150) for $i = 1$ and 2.

By (121) and (122) we have $h(x) \le L_0 x$ for $x \in (\alpha, \beta)$, hence $|\omega_3(x)| \le L_0 r|\sigma'(x)|$ for $x \in I_r$, and (149) is a consequence of (133), while

$$|\omega_3'(x)| \le |\sigma''(x)h(x)f_1(h'(x))| + |\sigma'(x)h'(x)f_1(h'(x))| + |\sigma'(x)h(x)f_1'(h'(x))h''(x)|$$

$$\leq L_0 r|\sigma''(x)| + L_0|\sigma'(x)| + L_0 r|\sigma'(x)h''(x)|\,.$$

Using (133), (134), and the inequality

$$\begin{aligned}&2r^2\|h''\|_{L^\infty(I_r)}(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\\ &\leq 2r\|h''\|^2_{L^\infty(I_r)} + r^3(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\,,\end{aligned}$$

we obtain (150). The estimates for ω_5 are similar.

We have $|\omega_4(x)| \leq |(\sigma(x)-1)f_1(h'(x))| \leq |\sigma(x)-1|$, and so (149) follows from (132). On the other hand,

$$\begin{aligned}|\omega_4'(x)| &\leq |\sigma'(x)f_1(h'(x))| + |(\sigma(x)-1)f_1'(h'(x))h''(x)|\\ &\leq |\sigma'(x)| + |(\sigma(x)-1)h''(x)|\,,\end{aligned}$$

so that (150) is a consequence of (132) and (133). A similar estimate holds for ω_6. □

5 Regularity: Preliminary Results

In this section we study the regularity of solutions to the Lamé system in the triangle A_r^m introduced in (119). The exponent in the regularity theorem will depend on the complex solutions z of the equation

$$\sin^2(z\omega) = K_1 - Kz^2\sin^2\omega\,, \tag{153}$$

where $\omega = \arctan m$ is the angle of the triangle A_r^m at the vertex $(0,0)$, and

$$K := \frac{\lambda+\mu}{\lambda+3\mu} < 1 < K_1 := \frac{(\lambda+2\mu)^2}{(\lambda+\mu)(\lambda+3\mu)}\,, \tag{154}$$

where λ and μ are the Lamé coefficients. In particular, we will use the results of the following lemma.

Lemma 12. *There exists a constant* $\xi_0 = \xi_0(\lambda,\mu) \in (1/2,1)$ *such that for every* $0 < \omega \leq \pi/2$ *the equation* (153) *has no complex solutions* z *with* $\operatorname{Re} z \in (0,\xi_0)$.

Proof. We follow the proof of [40, Theorem 2.2]. If $\operatorname{Im} z \neq 0$, then using the fact that $0 < \omega \leq \frac{\pi}{2}$, the argument in the first case of that proof shows that there are no solutions z with $\operatorname{Re} z \in (0,1]$. If $\operatorname{Im} z = 0$, then (153) reduces to

$$\sin^2(\omega\operatorname{Re} z) = K_1 - K(\operatorname{Re} z)^2\sin^2\omega\,. \tag{155}$$

By (154) there exists $\varepsilon_0 > 0$ such that $K < 1 < K_1 - \varepsilon_0$. By a trigonometric computation we find that $\sin^2(\omega/2) \leq 1 - (\sin^2\omega)/4$, hence

$$\sin^2(\omega/2) < K_1 - \varepsilon_0 - K(\sin^2\omega)/4\,.$$

Since for every $0 < \omega \leq \frac{\pi}{2}$ the function $\xi \mapsto \sin^2(\omega\xi) + K\xi^2 \sin^2\omega$ is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $\pi + 2$, there exists a constant $\xi_0 \in (1/2, 1)$, depending only on ε_0, such that

$$\sin^2(\omega\xi_0) < K_1 - K\xi_0^2 \sin^2\omega$$

for every $0 < \omega \leq \frac{\pi}{2}$. Since $\xi \mapsto \sin^2(\omega\xi)$ is increasing and $\xi \mapsto K_1 - K\xi^2 \sin^2\omega$ is decreasing on $[0, \xi_0]$, it follows that

$$\sin^2(\omega \operatorname{Re} z) < K_1 - K(\operatorname{Re} z)^2 \sin^2\omega$$

for all z with $\operatorname{Re} z \in [0, \xi_0]$. This shows that (155) is impossible and concludes the proof. □

Let p_0 be such that $2 - \frac{2}{p_0} = \xi_0$. Then

$$\frac{4}{3} < p_0 = \frac{2}{2 - \xi_0} < 2. \tag{156}$$

We recall that for every $0 < m \leq L_0$ and $r > 0$ the triangle A_r^m is defined in (119). Since the conjugate exponent of p_0 satisfies $p_0' < 4$, by the Sobolev embedding theorem we have $H_0^1(A_r^m; \mathbb{R}^2) \hookrightarrow L^{p_0'}(A_r^m; \mathbb{R}^2)$. Hence, by duality $L^{p_0}(A_r^m; \mathbb{R}^2) \hookrightarrow H^{-1}(A_r^m; \mathbb{R}^2)$. Similarly, given $g \in W^{1,p_0}(A_r^m; \mathbb{R}^2)$, its trace on

$$\Gamma_r^m = \{(x, mx) : 0 < x < r\}, \tag{157}$$

still denoted by g, belongs to $W^{1-1/p_0,p_0}(\Gamma_r^m; \mathbb{R}^2)$ and, by the embedding theorem for fractional Sobolev spaces, we have $g \in L^2(\Gamma_r^m; \mathbb{R}^2) \hookrightarrow H^{-1/2}(\Gamma_r^m; \mathbb{R}^2)$. Therefore, given $f \in L^{p_0}(A_r^m; \mathbb{R}^2)$ and $g \in W^{1,p_0}(A_r^m; \mathbb{R}^2)$ there exists a unique weak solution $w \in H^1(A_r^m; \mathbb{R}^2)$ to the problem

$$\begin{cases} -\operatorname{div} \mathbb{C}Ew = f & \text{in } A_r^m, \\ (\mathbb{C}Ew)\nu^m = g & \text{on } \Gamma_r^m, \\ w = 0 & \text{on } \partial A_r^m \setminus \Gamma_r^m, \end{cases} \tag{158}$$

where $\nu^m := (-m, 1)/\sqrt{1 + m^2}$ is the outer unit normal to A_r^m on Γ_r^m.

In the next theorem we will use [40, Theorems 2.1 and 2.2] (see also [41, Theorem I]).

Theorem 13. *Let p_0 be as in (156), let $r > 0$, let $0 < \eta_0 \leq m \leq L_0$, let A_r^m and Γ_r^m be defined by (119) and (157), let $f \in L^{p_0}(A_r^m; \mathbb{R}^2)$, let $g \in W^{1,p_0}(A_r^m; \mathbb{R}^2)$, with $g = 0$ on one of the sides of the triangle A_r^m different from Γ_r^m, and let $w \in H^1(A_r^m; \mathbb{R}^2)$ be the unique weak solution to the problem (158). Then w belongs to $W^{2,p_0}(A_r^m; \mathbb{R}^2)$ and*

$$\|\nabla^2 w\|_{L^{p_0}(A_r^m)} \leq \kappa\big(\|f\|_{L^{p_0}(A_r^m)} + \|\nabla g\|_{L^{p_0}(A_r^m)}\big), \tag{159}$$

for a constant $\kappa > 0$ depending on λ, μ, η_0, and L_0, but independent of r, m, f, and g.

Proof. By a rescaling argument we see that it is enough to prove the result for $r = 1$.

Step 1: Assume $g = 0$. By [42, Theorem 3.1], Lemma 12, and (156), we have that $w \in W^{2,p_0}(A_1^m;\mathbb{R}^2)$. Let X^m be the space of functions $z \in W^{2,p_0}(A_1^m;\mathbb{R}^2)$ such that $z = 0$ on $\partial A_1^m \setminus \Gamma_1^m$ and $(\mathbb{C}Ew)\nu = 0$ on Γ_1^m endowed with the norm of $W^{2,p_0}(A_1^m;\mathbb{R}^2)$. Consider the continuous linear operator $\mathcal{L} : X^m \to L^{p_0}(A_1^m;\mathbb{R}^2)$ defined by $\mathcal{L}(z) := -\operatorname{div}\mathbb{C}Ez$. By what we just proved (see (158)), $\mathcal{L}$ is invertible, and so by the Closed Graph Theorem we obtain that there exists a constant $C > 0$, depending on λ, μ, and m, such that

$$\|w\|_{W^{2,p_0}(A_1^m)} \le C\|f\|_{L^{p_0}(A_1^m)} \tag{160}$$

for every solution w to (158) with $g = 0$.

Step 2: In the general case $g \in W^{1,p_0}(A_1^m;\mathbb{R}^2)$, with $g = 0$ on one of the sides of the triangle A_1^m different from Γ_1^m, recalling that $1 < p_0 < 2$, we can reason as in [23, Lemma 3.12] and using [43, Theorem 1.5.2.8] we can find $w^1 \in W^{2,p_0}(A_1^m;\mathbb{R}^2)$, with $(\mathbb{C}Ew^1)\nu^m = g$ on Γ_1^m and $w^1 = 0$ on ∂A_1^m, such that

$$\|w^1\|_{W^{2,p_0}(A_1^m)} \le C_1\|g\|_{W^{1-1/p_0,p_0}(\Gamma_1^m)} \le C_2\|\nabla g\|_{L^{p_0}(A_1^m)} \tag{161}$$

for some constants C_1, $C_2 > 0$ depending on λ, μ, and m, where in the last inequality we used the trace estimate and Poincaré's inequality. Then the function $v := w - w^1$ is a weak solution to (158) in A_1^m with f replaced by $f + \operatorname{div}\mathbb{C}Ew^1$ and g replaced by zero. Hence, by the previous step $v \in W^{2,p_0}(A_1^m;\mathbb{R}^2)$. Moreover, using (160) for v and (161) for w_1 we obtain a constant $\kappa_m > 0$, depending on λ, μ, and m, but independent of f and g, such that

$$\|\nabla^2 w\|_{L^{p_0}(A_1^m)} \le \kappa_m\big(\|f\|_{L^{p_0}(A_1^m)} + \|\nabla g\|_{L^{p_0}(A_1^m)}\big)\,. \tag{162}$$

Step 3: Let $m_0 \in [\eta_0, L_0]$. We want to prove that there exists $\varepsilon_0 > 0$ such that, if $m \in [\eta_0, L_0]$ and $|m - m_0| < \varepsilon_0$, then w satisfies the estimate

$$\|\nabla^2 w\|_{L^{p_0}(A_1^m)} \le 2\kappa_{m_0}\big(\|f\|_{L^{p_0}(A_1^m)} + \|\nabla g\|_{L^{p_0}(A_1^m)}\big)\,, \tag{163}$$

with the constant κ_{m_0} corresponding to m_0.

Since $w \in W^{2,p_0}(A_1^m;\mathbb{R}^2)$ satisfies (158), a direct computation (using Remark 1) shows that the function $z \in W^{2,p_0}(A_1^{m_0};\mathbb{R}^2)$ defined by $z(x,y) := w(x, \frac{m}{m_0}y)$ satisfies

$$-\operatorname{div}\mathbb{C}Ez = \tilde f + f^z \quad \text{in } A_1^{m_0}\,, \tag{164}$$

$$(\mathbb{C}Ez)\nu^{m_0} = \tilde g + \hat g^z + \check g^z \quad \text{on } \Gamma_1^{m_0}\,, \tag{165}$$

$$z = 0 \quad \text{on } \partial A_1^{m_0} \setminus \Gamma_1^{m_0}\,, \tag{166}$$

with $\tilde f, f^z \in L^{p_0}(A_1^{m_0};\mathbb{R}^2)$ and $\tilde g, \hat g^z, \check g^z \in W^{1,p_0}(A_1^{m_0};\mathbb{R}^2)$ defined by

$$\tilde f(x,y) := f(x, \tfrac{m}{m_0}y) \quad \text{and} \quad \tilde g(x,y) := g(x, \tfrac{m}{m_0}y)\,, \tag{167}$$

$$f_1^z := \mu\big((\tfrac{m_0}{m})^2 - 1\big)\partial^2_{yy}z_1 + (\lambda+\mu)(\tfrac{m_0}{m} - 1)\partial^2_{xy}z_2\,, \tag{168}$$

$$f_2^z := (\lambda+2\mu)\big((\tfrac{m_0}{m})^2 - 1\big)\partial^2_{yy}z_2 + (\lambda+\mu)(\tfrac{m_0}{m} - 1)\partial^2_{xy}z_1\,, \tag{169}$$

$$\hat{g}^z_1 := (2\mu+\lambda)\big(\tfrac{m}{\sqrt{1+m^2}} - \tfrac{m_0}{\sqrt{1+m_0^2}}\big)\partial_x z_1 - \mu\big(\tfrac{1}{\sqrt{1+m^2}} - \tfrac{1}{\sqrt{1+m_0^2}}\big)\partial_x z_2\,, \tag{170}$$

$$\begin{aligned}\check{g}^z_1 &:= -\mu\big(\tfrac{1}{\sqrt{1+m^2}} - \tfrac{1}{\sqrt{1+m_0^2}}\big)\partial_y z_1 + \lambda\big(\tfrac{m}{\sqrt{1+m^2}} - \tfrac{m_0}{\sqrt{1+m_0^2}}\big)\partial_y z_2 \\ &\quad - \tfrac{\mu}{\sqrt{1+m^2}}\big(\tfrac{m_0}{m}-1\big)\partial_y z_1 + \tfrac{\lambda m}{\sqrt{1+m^2}}\big(\tfrac{m_0}{m}-1\big)\partial_y z_2\,,\end{aligned} \tag{171}$$

$$\hat{g}^z_2 := \mu\big(\tfrac{m}{\sqrt{1+m^2}} - \tfrac{m_0}{\sqrt{1+m_0^2}}\big)\partial_x z_2 + \lambda\big(\tfrac{m}{\sqrt{1+m^2}} - \tfrac{m_0}{\sqrt{1+m_0^2}}\big)\partial_x z_1\,, \tag{172}$$

$$\begin{aligned}\check{g}^z_2 &:= \mu\big(\tfrac{m}{\sqrt{1+m^2}} - \tfrac{m_0}{\sqrt{1+m_0^2}}\big)\partial_y z_1 - 2\mu\big(\tfrac{1}{\sqrt{1+m^2}} - \tfrac{1}{\sqrt{1+m_0^2}}\big)\partial_y z_2 - \tfrac{2\mu}{\sqrt{1+m^2}}\big(\tfrac{m_0}{m}-1\big)\partial_y z_2 \\ &\quad - \lambda\big(\tfrac{1}{\sqrt{1+m^2}} - \tfrac{1}{\sqrt{1+m_0^2}}\big)\partial_y z_2 + \tfrac{\mu m}{\sqrt{1+m^2}}\big(\tfrac{m_0}{m}-1\big)\partial_y z_1 - \tfrac{\lambda}{\sqrt{1+m^2}}\big(\tfrac{m_0}{m}-1\big)\partial_y z_2\,.\end{aligned} \tag{173}$$

Since $z \in W^{2,p_0}(A^{m_0}_1;\mathbb{R}^2)$ and $z = 0$ on $\partial A^{m_0}_1 \setminus \Gamma^{m_0}_1$, we have that $\partial_x z = 0$ a.e. on $[0,1]\times\{0\}$ and $\partial_y z = 0$ a.e. on $\{1\}\times[0,m_0]$, hence $\hat{g}^z = 0$ a.e. on $[0,1]\times\{0\}$ and $\check{g}^z = 0$ a.e. on $\{1\}\times[0,m_0]$. Moreover $\tilde{g}$ vanishes on one of the sides of $A^{m_0}_1$ different from $\Gamma^{m_0}_1$. We may assume that $\tilde{g} = 0$ a.e. on $[0,1]\times\{0\}$, the other case being analogous.

Let $\hat{z}$ and $\check{z}$ be the unique solutions of the problems

$$\begin{cases} -\operatorname{div}\mathbb{C}E\hat{z} = \tilde{f} + f^z & \text{in } A^{m_0}_1\,,\\ (\mathbb{C}E\hat{z})\nu^{m_0} = \tilde{g} + \hat{g}^z & \text{on } \Gamma^{m_0}_1\,,\\ \hat{z} = 0 & \text{on } \partial A^{m_0}_1 \setminus \Gamma^{m_0}_1\,,\end{cases} \qquad \text{and} \qquad \begin{cases} -\operatorname{div}\mathbb{C}E\check{z} = 0 & \text{in } A^{m_0}_1\,,\\ (\mathbb{C}E\check{z})\nu^{m_0} = \check{g}^z & \text{on } \Gamma^{m_0}_1\,,\\ \check{z} = 0 & \text{on } \partial A^{m_0}_1 \setminus \Gamma^{m_0}_1\,.\end{cases}$$

By linearity we have $z = \hat{z} + \check{z}$. Since $\tilde{g} + \hat{g}^z$ and $\check{g}^z$ vanish on one of the sides of $A^{m_0}_1$ different from $\Gamma^{m_0}_1$, by (162) we have

$$\begin{aligned}&\|\nabla^2\hat{z}\|_{L^{p_0}(A^{m_0}_1)} \le \kappa_{m_0}\big(\|\tilde{f} + f^z\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\tilde{g} + \nabla\hat{g}^z\|_{L^{p_0}(A^{m_0}_1)}\big)\,,\\ &\|\nabla^2\check{z}\|_{L^{p_0}(A^{m_0}_1)} \le \kappa_{m_0}\|\nabla\check{g}^z\|_{L^{p_0}(A^{m_0}_1)}\,.\end{aligned}$$

hence

$$\|\nabla^2 z\|_{L^{p_0}(A^{m_0}_1)} \le \kappa_{m_0}\big(\|\tilde{f}+f^z\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\tilde{g}+\hat{g}^z\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\check{g}^z\|_{L^{p_0}(A^{m_0}_1)}\big)\,. \tag{174}$$

Let us fix $\omega > 0$. Since $z \in W^{2,p_0}(A^{m_0}_1;\mathbb{R}^2)$, by (167), (168), (169), (170), (171), (172), and (173) there exists $\varepsilon_\omega > 0$ such that, if $m \in [\eta_0, L_0]$ and $|m-m_0| < \varepsilon_\omega$, then

$$\|\nabla^2 w\|_{L^{p_0}(A^m_1)} \le (1+\omega)\|\nabla z^2\|_{L^{p_0}(A^{m_0}_1)}\,, \tag{175}$$

$$\|\tilde{f}\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\tilde{g}\|_{L^{p_0}(A^{m_0}_1)} \le (1+\omega)\big(\|f\|_{L^{p_0}(A^m_1)} + \|\nabla g\|_{L^{p_0}(A^m_1)}\big)\,, \tag{176}$$

$$\|f^z\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\hat{g}^z\|_{L^{p_0}(A^{m_0}_1)} + \|\nabla\check{g}^z\|_{L^{p_0}(A^{m_0}_1)} \le \omega\|\nabla^2 z\|_{L^{p_0}(A^{m_0}_1)}\,. \tag{177}$$

From (174), (176), and (177) we obtain

$$\|\nabla^2 z\|_{L^{p_0}(A^{m_0}_1)} \le (1+\omega)\kappa_{m_0}\big(\|f\|_{L^{p_0}(A^m_1)} + \|\nabla g\|_{L^{p_0}(A^m_1)}\big) + \omega\kappa_{m_0}\|\nabla^2 z\|_{L^{p_0}(A^{m_0}_1)}\,. \tag{178}$$

We now choose $\omega < 1/3$ such that $\omega\kappa_{m_0} < 1/9$. If $m \in [\eta_0, L_0]$ and $|m - m_0| < \varepsilon_\omega$, then (178) gives

$$\tfrac{8}{9}\|\nabla^2 z\|_{L^{p_0}(A_1^{m_0})} \le \tfrac{4}{3}\kappa_{m_0}\left(\|f\|_{L^{p_0}(A_1^m)} + \|\nabla g\|_{L^{p_0}(A_1^m)}\right).$$

Using (175) we obtain

$$\tfrac{2}{3}\|\nabla^2 w\|_{L^{p_0}(A_1^m)} \le \tfrac{4}{3}\kappa_{m_0}\left(\|f\|_{L^{p_0}(A_1^m)} + \|\nabla g\|_{L^{p_0}(A_1^m)}\right),$$

which implies (163).

Step 4: By compactness we can cover $[\eta_0, L_0]$ by a finite number of open intervals $(m_i - \varepsilon_i, m_i + \varepsilon_i)$, with $m_i \in [\eta_0, L_0]$, such that (163) holds with m_0 and ε_0 replaced by m_i and ε_i. Then (159) holds with $\kappa := 2\max_i \kappa_{m_i}$. □

6 Regularity at Corners

In this section we obtain a precise estimate on the W^{2,p_0} norm of the equilibrium solution u in a neighborhood of the corners.

Next we use a fixed point theorem to prove that u belongs to W^{2,p_0} near $(\alpha, 0)$.

Theorem 14. *Under the assumptions of Theorem 6, we have that $u \in W^{2,p_0}(\Omega_h)$.*

To prove this theorem we need some auxiliary results. We begin with Poincaré's inequality.

Lemma 15 (Poincaré's inequality)**.** *Let $0 < r < \beta - \alpha$ and let $p \ge 1$. Then*

$$\|v\|_{L^p(\Omega_h^{\alpha,\alpha+r})} \le L_0 r \|\nabla v\|_{L^p(\Omega_h^{\alpha,\alpha+r})} \tag{179}$$

for every $v \in W^{1,p}(\Omega_h^{\alpha,\alpha+r})$ such that $v(x,0) = 0$ for $x \in (\alpha, \alpha + r)$ (in the sense of traces).

Proof. By density we can assume that $v \in C^\infty(\mathbb{R}^2)$. For $(x,y) \in \Omega_h^{\alpha,\alpha+r}$, by the fundamental theorem of calculus and Hölder's inequality we have

$$|v(x,y)| \le (L_0 r)^{1/p'}\left(\int_0^{h(x)} |\partial_y v(x,t)|^p dt\right)^{1/p},$$

where we used the fact that $h(x) \le L_0 r$. Raising both sides to power p and integrating over $\Omega_h^{\alpha,\alpha+r}$ gives

$$\int_{\Omega_h^{\alpha,\alpha+r}} |v(x,y)|^p dxdy \le (L_0 r)^{1+p/p'} \int_{\Omega_h^{\alpha,\alpha+r}} |\partial_y v(x,y)|^p dxdy,$$

which yields (179) and concludes the proof. □

Remark 6. *Let $r > 0$, let $0 < \eta_0 \le m \le L_0$, and let A_r^m be the triangle defined in (119). With a proof similar to the one of Lemma 15, one can show that*

$$\|v\|_{L^p(A_r^m)} \le \max\{L_0, 1/\eta_0\} r \|\nabla v\|_{L^p(A_r^m)}$$

for every $v \in W^{1,p}(A_r^m)$ *such that* $v(x,0) = 0$ *for* $x \in (0,r)$ *or* $v(r,y) = 0$ *for* $y \in (0, mr)$ *(in the sense of traces).*

We turn to the proof of Theorem 14.

Proof of Theorem 14. In view of Theorem 4, it is enough to prove that there exists $r > 0$ sufficiently small such that

$$u \in W^{2,p_0}(\Omega_h^{\alpha,\alpha+r} \cup \Omega_h^{\beta-r,\beta}),$$

where $\Omega_h^{c,d}$ is defined in (35). We will only show that $u \in W^{2,p_0}(\Omega_h^{\alpha,\alpha+r})$, since the other endpoint can be treated in a similar way. By a translation, without loss of generality, we may assume that $\alpha = 0$. We modify u far from $(0,0)$ to obtain a new function $\tilde{u}$ vanishing away from $(0,0)$. We then use the transformation (126) to map $\Omega_h^{0,r}$ into the triangle A_r^m defined in (119) with $m = h'(0)$. A fixed point argument will allow us to show that the resulting system has a solution in $W^{2,p_0}(A_r^m)$, and due to uniqueness, this solution is $\tilde{u} \circ \Phi$ itself. To apply the Banach fixed point theorem, we will use Lemmas 8 and 11.

Step 1: Localization. Let

$$w^0(x,y) := (e_0 x, 0), \quad (x,y) \in \mathbb{R}^2. \tag{180}$$

For every $r > 0$ let $\varphi_r \in C^\infty(\mathbb{R}^2)$ be a function such that $0 \le \varphi_r \le 1$, $\varphi_r(x,y) = 1$ for $x \le 5r/8$, $\varphi_r(x,y) = 0$ for $x \ge 7r/8$, $\|\nabla\varphi_r\|_{L^\infty(\mathbb{R}^2)} \le C/r$ and $\|\nabla^2\varphi_r\|_{L^\infty(\mathbb{R}^2)} \le C/r^2$, where $C > 0$ is a constant independent of r. In Ω_h we define

$$\tilde{u} := (u - w^0)\varphi_r. \tag{181}$$

If $0 < r < \beta$ we have $\tilde{u}(x,0) = 0$ for $0 \le x \le r$ and $\tilde{u}(r,y) = 0$ for $0 \le y \le h(r)$. In other words $\tilde{u} = 0$ on $\partial\Omega_h^{0,r} \setminus \Gamma_h^{0,r}$, where $\Gamma_h^{0,r} = \{(x,h(x)) : 0 < x < r\}$. Moreover, we have

$$\nabla\tilde{u} = \varphi_r\nabla(u - w^0) + (u - w^0) \otimes \nabla\varphi_r.$$

By Poincaré's inequality (see Lemma 15) we obtain

$$\|\nabla\tilde{u}\|_{L^2(\Omega_h^{0,r})} \le C\|\nabla(u - w^0)\|_{L^2(\Omega_h^{0,r})}. \tag{182}$$

By direct computation, it follows from Remark 1 that

$$\begin{cases} -\operatorname{div}\mathbb{C}E\tilde{u} = -\mu\Delta\tilde{u} - (\lambda+\mu)\nabla\operatorname{div}\tilde{u} = f & \text{in } \Omega_h^{0,r}, \\ (\mathbb{C}E\tilde{u})\nu^h = 2\mu(E\tilde{u})\nu^h + \lambda(\operatorname{div}\tilde{u})\nu^h = g & \text{on } \Gamma_h^{0,r}, \\ \tilde{u} = 0 & \text{on } \partial\Omega_h^{0,r} \setminus \Gamma_h^{0,r}, \end{cases} \tag{183}$$

where $\nu^h(x) := (-h'(x), 1)/\sqrt{1 + (h'(x))^2}$ denotes the outer unit normal to Ω_h on Γ_h, and

$$f := -\mu\big(2(\nabla u - \nabla w^0)\nabla\varphi_r + (u - w^0)\Delta\varphi_r\big) \tag{184}$$

$$- (\lambda+\mu)\big((\operatorname{div} u - \operatorname{div} w^0)\nabla\varphi_r + (\nabla u - \nabla w^0)^T\nabla\varphi_r + \nabla^2\varphi_r(u-w^0)\big)\,,$$
$$g := \mu\big((u-w^0)\otimes\nabla\varphi_r + \nabla\varphi_r\otimes(u-w^0)\big)\nu^h + \lambda\,\mathrm{trace}((u-w^0)\otimes\nabla\varphi_r)\nu^h\,. \tag{185}$$

Note that $f \in L^2(\Omega_h^{0,r};\mathbb{R}^2)$. Since $h \in C^3([0,\beta])$ by Theorem 6, we can consider ν^h as a C^2 function in the strip $[0,\beta]\times\mathbb{R}$. This shows that $g \in H^1(\Omega_h^{0,r};\mathbb{R}^2)$. Since $u(x,0) - w^0(x,0) = 0$ for a.e. $x \in [0,r]$ and $\varphi_r(r,y) = 0$ for every $y \in \mathbb{R}$, we conclude that the trace of g vanishes on $\partial\Omega_h^{0,r} \setminus \Gamma_h^{0,r}$. We also observe that for every $f \in L^2(\Omega_h^{0,r};\mathbb{R}^2)$ and every $g \in H^1(\Omega_h^{0,r};\mathbb{R}^2)$ problem (183) has a unique weak solution in $H^1(\Omega_h^{0,r};\mathbb{R}^2)$.

Step 2: Straightening the boundary. Let r be as in (124) and let

$$v := \tilde{u}\circ\Phi \in H^1(A_r^m;\mathbb{R}^2)\,, \tag{186}$$

where A_r^m and Φ are defined in (119) and (126), with $m = h_0'$. Recalling Remark 1 and the fact that $\tilde{u}(x,y) = v(x,\sigma(x)y)$, it follows by direct computation and by Lemma 10 that v satisfies the boundary value problem

$$\begin{cases} -\operatorname{div}\mathbb{C}Ev = -\mu\Delta v - (\lambda+\mu)\nabla\operatorname{div} v = f\circ\Phi + f^v & \text{in } A_r^m\,,\\ (\mathbb{C}Ev)\nu^0 = 2\mu(Ev)\nu^0 + \lambda(\operatorname{div} v)\nu^0 = g\circ\Phi + \hat{g}^v + \check{g}^v & \text{on } \Gamma_r^m\,,\\ v = 0 & \text{on } \partial A_r^m\setminus\Gamma_r^m\,, \end{cases} \tag{187}$$

where $\nu^0 := (-h_0',1)/\sqrt{1+(h_0')^2}$ is the outer unit normal to A_r^m on $\Gamma_r^m := \{(x,h_0'x) : 0<x<r\}$, $f^v = (f_1^v,f_2^v) \in H^{-1}(A_r^m;\mathbb{R}^2)$ is defined by

$$\begin{aligned} f_1^v := \mu\bigg[\Big(\sigma^2 - 1 + y^2\frac{(\sigma')^2}{\sigma^2}\Big)\partial^2_{yy}v_1 + 2y\frac{\sigma'}{\sigma}\partial^2_{xy}v_1\bigg] \\ + (\lambda+\mu)\bigg[y^2\frac{(\sigma')^2}{\sigma^2}\partial^2_{yy}v_1 + 2y\frac{\sigma'}{\sigma}\partial^2_{xy}v_1 + (\sigma-1)\partial^2_{xy}v_2 + y\sigma'\partial^2_{yy}v_2\bigg]\,, \\ + \mu y\frac{\sigma''}{\sigma}\partial_y v_1 + (\lambda+\mu)\bigg[y\frac{\sigma''}{\sigma}\partial_y v_1 + \sigma'\partial_y v_2\bigg]\,, \end{aligned} \tag{188}$$

$$\begin{aligned} f_2^v := \mu\bigg[\Big(\sigma^2 - 1 + y^2\frac{(\sigma')^2}{\sigma^2}\Big)\partial^2_{yy}v_2 + 2y\frac{\sigma'}{\sigma}\partial^2_{xy}v_2\bigg] \\ + (\lambda+\mu)[(\sigma-1)\partial^2_{xy}v_1 + \sigma' y\partial^2_{yy}v_1 + (\sigma^2-1)\partial^2_{yy}v_2] \\ + \mu y\frac{\sigma''}{\sigma}\partial_y v_2 + (\lambda+\mu)\sigma'\partial_y v_1\,, \end{aligned} \tag{189}$$

while $\hat{g}^v = (\hat{g}_1^v,\hat{g}_2^v) \in W^{1,p_0}(A_r^m;\mathbb{R}^2)$ and $\check{g}^v = (\check{g}_1^v,\check{g}_2^v) \in W^{1,p_0}(A_r^m;\mathbb{R}^2)$ are defined by (145)–(148).

Step 3: Fixed Point Argument. To prove the W^{2,p_0} regularity of v we use a fixed point argument in the space

$$X_r^m := \{z \in W^{2,p_0}(A_r^m;\mathbb{R}^2) : z = 0 \text{ on } \partial A_r^m\setminus\Gamma_r^m\}\,,$$

endowed with the norm

$$\|z\|_{X_r^m} := \|\nabla^2 z\|_{L^{p_0}(A_r^m)}\,. \tag{190}$$

Let us first prove that $\|\cdot\|_{X_r^m}$ is a norm equivalent to $\|\cdot\|_{W^{2,p_0}(A_r^m,\mathbb{R}^2)}$. By Poincaré's inequality (see Remark 6) there exists a constant $C>0$, depending only on η_0 and L_0, such that

$$\|\varphi\|_{L^{p_0}(A_r^m)} \leq Cr\|\nabla\varphi\|_{L^{p_0}(A_r^m)} \tag{191}$$

for every $\varphi \in W^{1,p_0}(A_r^m,\mathbb{R}^2)$ such that $\varphi = 0$ on one of the sides of the triangle A_r^m different from Γ_r^m. If $z \in X_r^m$, then z, $\partial_x z$, and $\partial_y z$ satisfy this property, hence

$$\begin{aligned}
\|z\|_{L^{p_0}(A_r^m)} &\leq Cr\|\nabla z\|_{L^{p_0}(A_r^m)}\,,\\
\|\partial_x z\|_{L^{p_0}(A_r^m)} &\leq Cr\|\nabla \partial_x z\|_{L^{p_0}(A_r^m)}\,,\\
\|\partial_y z\|_{L^{p_0}(A_r^m)} &\leq Cr\|\nabla \partial_y z\|_{L^{p_0}(A_r^m)}\,.
\end{aligned}$$

These inequalities imply that there exists a constant $K_r>0$ such that

$$\|z\|_{W^{2,p_0}(A_r^m)} \leq K_r\|\nabla^2 z\|_{L^{p_0}(A_r^m)}$$

for every $z \in X_r^m$, concluding the proof of the equivalence between the norms $\|\cdot\|_{X_r^m}$ and $\|\cdot\|_{W^{2,p_0}(A_r^m,\mathbb{R}^2)}$. This shows, in particular, that X_r^m is a Banach space.

Let r be as in (124). Given $z \in W^{2,p_0}(A_r^m;\mathbb{R}^2)$, let $w := F_r^m(z)$ be the solution to the boundary value problem

$$\begin{cases}
-\operatorname{div}(\mathbb{C}Ew) = f\circ\Phi + f^z & \text{in } A_r^m\,,\\
(\mathbb{C}Ew)\nu^0 = g\circ\Phi + \hat{g}^z + \check{g}^z & \text{on } \Gamma_r^m\,,\\
w = 0 & \text{on } \partial A_r^m \setminus \Gamma_r^m\,,
\end{cases} \tag{192}$$

let w_1 be the solution to the boundary value problem

$$\begin{cases}
-\operatorname{div}(\mathbb{C}Ew_1) = f\circ\Phi & \text{in } A_r^m\,,\\
(\mathbb{C}Ew_1)\nu^0 = g\circ\Phi & \text{on } \Gamma_r^m\,,\\
w_1 = 0 & \text{on } \partial A_r^m \setminus \Gamma_r^m\,,
\end{cases}$$

and let $w_2 := G_r^m(z)$ be the solution to the boundary value problem

$$\begin{cases}
-\operatorname{div}(\mathbb{C}Ew_2) = f^z & \text{in } A_r^m\,,\\
(\mathbb{C}Ew_2)\nu^0 = \hat{g}^z + \check{g}^z & \text{on } \Gamma_r^m\,,\\
w_2 = 0 & \text{on } \partial A_r^m \setminus \Gamma_r^m\,.
\end{cases}$$

By linearity, we have $F_r^m(z) = w_1 + G_r^m(z)$.

We claim that for $r>0$ sufficiently small the linear map $G_r^m : X_r^m \to X_r^m$ is a contraction. By linearity it suffices to show that

$$\|G_r^m(z)\|_{X_r^m} \leq 1/2 \quad \text{for all } z\in X_r^m \text{ with } \|z\|_{X_r^m} \leq 1\,. \tag{193}$$

Fix $z \in X_r^m$ with $\|z\|_{X_r^m} \leq 1$. By linearity, using Theorem 13 we obtain that $w_2 = G_r^m(z) \in W^{2,p_0}(A_r^m;\mathbb{R}^2)$ with

$$\|\nabla^2 w_2\|_{L^{p_0}(A_r^m)} \leq \kappa\big(\|f^z\|_{L^{p_0}(A_r^m)} + \|\nabla \hat{g}^z\|_{L^{p_0}(A_r^m)} + \|\nabla \check{g}^z\|_{L^{p_0}(A_r^m)}\big)\,, \tag{194}$$

where κ depends only on λ, μ, η_0, and L_0; in particular it does not depend on r, h, f, g, and z. Therefore, (193) follows provided we can prove that

$$\kappa\big(\|f^z\|_{L^{p_0}(A_r^m)} + \|\nabla \hat{g}^z\|_{L^{p_0}(A_r^m)} + \|\nabla \check{g}^z\|_{L^{p_0}(A_r^m)}\big) \leq 1/2\,. \tag{195}$$

Step 4: Proof of (195). In this step we prove (195). Below C_h denotes a constant independent of $r \in (0, \eta_0^2/(4M))$ and $m \in [2\eta_0, L_0]$, but depending on h, whose value can change from formula to formula, while $C_{\lambda,\mu}$ denotes a constant depending only on the Lamé coefficients. We begin with $\|f^z\|_{L^{p_0}(A_r^m)}$. For $0 < r < \eta_0^2/(4M)$, by Lemma 8, Remark 3, (188), and (189), in A_r^m we have the pointwise estimate

$$\begin{aligned}|f^z| &\leq C_{\lambda,\mu}\Big(\|\sigma - 1\|_{L^\infty(I_r)} + \sup_{(x,y)\in A_r^m} |y\sigma'(x)| + \sup_{(x,y)\in A_r^m} |y\sigma'(x)|^2\Big)|\nabla^2 z| \\ &\quad + C_{\lambda,\mu}\Big(\|\sigma'\|_{L^\infty(I_r)} + \sup_{(x,y)\in A_r^m} |y\sigma''(x)|\Big)|\nabla z| \leq C_h r|\nabla^2 z| + C_h|\nabla z|\,,\end{aligned} \tag{196}$$

where in the last inequality we used (132), (135), and (138). In turn, using Poincaré's inequality given in Remark 6 together with the inequality $\|z\|_{X_r^m} \leq 1$, with norm defined by (190), we obtain

$$\|f^z\|_{L^{p_0}(A_r^m)} \leq C_h r\|\nabla^2 z\|_{L^{p_0}(A_r^m)} + C\|\nabla z\|_{L^{p_0}(A_r^m)} \leq C_h r\|\nabla^2 z\|_{L^{p_0}(A_r^m)} \leq C_h r\,. \tag{197}$$

We now estimate $\|\nabla \hat{g}^z\|_{L^{p_0}(A_r^m)}$. We observe that by (145) and (146) we can write each component of $\nabla \hat{g}^z$ as linear combinations of products of the functions ω_i and second order partial derivatives of the component of z, as well as products of the derivatives ω_i' and first order partial derivatives of the components of z. Therefore, we obtain that

$$\begin{aligned}\|\nabla \hat{g}^z\|_{L^{p_0}(A_r^m)} &\leq C_{\lambda,\mu}\Big(\sum_{i=1}^{6} \|\omega_i\|_{L^\infty(I_r)}\|\nabla^2 z\|_{L^{p_0}(A_r^m)} + \sum_{i=1}^{6} \|\omega_i'\|_{L^\infty(I_r)}\|\nabla z\|_{L^{p_0}(A_r^m)}\Big) \\ &\leq C_h r\,,\end{aligned} \tag{198}$$

where in the last inequality we used the fact that $\|\nabla^2 z\|_{L^{p_0}(A_r^m)} = \|z\|_{X_r^m} \leq 1$ and the estimates (149) and (150), together with Poincaré's inequality for $\partial_x z$ and $\partial_y z$, which vanish on one of the sides of A_r^m different from Γ_r^m (see Remark 6). Similarly, we prove that

$$\|\nabla \check{g}^z\|_{L^{p_0}(A_r^m)} \leq C_h r\,. \tag{199}$$

From (197), (198), and (199) it follows that

$$\kappa\big(\|f^z\|_{L^{p_0}(A_r^m)} + \|\nabla \hat{g}^z\|_{L^{p_0}(A_r^m)} + \|\nabla \check{g}^z\|_{L^{p_0}(A_r^m)}\big) \leq C_h r\,. \tag{200}$$

Therefore (195) is satisfied if $r \in (0, \eta_0^2/(4M))$ is sufficiently small. This shows that G_r^m is a contraction in the Banach space X_r^m. Consequently F_r^m is a contraction.

Step 5: Conclusion. By the Banach fixed point theorem applied to F_r^m, there exists $z_0 \in X_r^m$ such that $z_0 = F_r^m(z_0)$. By (192) the function z_0 solves (187). By (131) the function $u_0 := z_0 \circ \Psi$ belongs to $W^{2,p_0}(\Omega_h^{0,r};\mathbb{R}^2)$ and, by direct computation, it solves (183). Since $W^{2,p_0}(\Omega_h^{0,r};\mathbb{R}^2) \subset H^1(\Omega_h^{0,r};\mathbb{R}^2)$ and problem (183) has a unique weak solution in $H^1(\Omega_h^{0,r};\mathbb{R}^2)$, we conclude that $u_0 = \tilde{u}$, hence $\tilde{u} \in W^{2,p_0}(\Omega_h^{0,r};\mathbb{R}^2)$. Recalling that $\tilde{u} = u - w^0$ in $\Omega_h^{0,r/2}$, we obtain that $u \in W^{2,p_0}(\Omega_h^{0,r/2};\mathbb{R}^2)$, which concludes the proof. □

Theorem 16. *Under the hypotheses of Theorem 6, we have*

$$h''(\alpha) = h''(\beta) = 0\,, \tag{201}$$

and

$$\sigma_0 \frac{\alpha - \alpha^0}{\tau} = \frac{\gamma}{J(\alpha)} - \gamma_0 + \nu_0 \frac{h'(\alpha)}{J(\alpha)^2}\Big(\frac{h''}{J^3}\Big)'(\alpha)\,, \tag{202}$$

$$\sigma_0 \frac{\beta - \beta^0}{\tau} = -\frac{\gamma}{J(\beta)} + \gamma_0 - \nu_0 \frac{h'(\beta)}{J(\beta)^2}\Big(\frac{h''}{J^3}\Big)'(\beta)\,, \tag{203}$$

where J is defined in (42).

We begin with a preliminary lemma.

Lemma 17. *Under the hypotheses of Theorem 6, let $1 \le p < \infty$ and let $v \in W^{2,p}(\Omega_h;\mathbb{R}^2)$ be such that $v(x,0) = 0$ for a.e. $x \in (\alpha,\beta)$. Then there exists $\hat{v} \in W^{2,p}\mathbb{R}^2;\mathbb{R}^2)$ such that $\hat{v} = v$ in Ω_h and $\hat{v}(x,0) = 0$ for a.e. $x \in \mathbb{R}$.*

Proof. Since Ω_h is a domain with Lipschitz boundary, we can extend v to a function $\hat{v} \in W^{2,p_0}((\alpha,\beta)\times\mathbb{R};\mathbb{R}^2)$ (see [44, Theorem 13.17]). Let $\tilde{\alpha} := \alpha - \frac{\beta-\alpha}{2}$ and $\tilde{\beta} := \beta + \frac{\beta-\alpha}{2}$. For $x \in [\tilde{\alpha},\alpha]$ and $y \in \mathbb{R}$ we set

$$\hat{v}(x,y) := 3\hat{v}(2\alpha - x, y) - 2\hat{v}(3\alpha - 2x, y)\,.$$

Since $\hat{v}$, $\partial_x\hat{v}$, and $\partial_y\hat{v}$ have the same trace on both sides of the line $x = \alpha$, the function $\hat{v}$ belongs to $W^{2,p_0}((\tilde{\alpha},\beta)\times\mathbb{R};\mathbb{R}^2)$. Similarly, for $x \in [\beta,\tilde{\beta}]$ and $y \in \mathbb{R}$, we set

$$\hat{v}(x,y) := 3\hat{v}(2\beta - x, y) - 2\hat{v}(3\beta - 2x, y)$$

and we obtain that $\hat{v}$ belongs to $W^{2,p_0}((\tilde{\alpha},\tilde{\beta})\times\mathbb{R};\mathbb{R}^2)$. By construction, we have $\hat{v}(x,0) = 0$ for a.e. $x \in (\tilde{\alpha},\tilde{\beta})$. Using a suitable cut-off function, we can modify $\hat{v}$ near the lines $x = \tilde{\alpha}$ and $x = \tilde{\beta}$ so that the modified function vanishes near these lines. The conclusion can be obtained by setting $\hat{v} = 0$ outside the strip $(\tilde{\alpha},\tilde{\beta})\times\mathbb{R}$. □

Proof of Theorem 16. Since $u \in W^{2,p_0}(\Omega_h;\mathbb{R}^2)$ by Theorem 14, by Lemma 17 we can extend u to a function $\hat{u} \in W^{2,p_0}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ such that $u(x,0) = (e_0x, 0)$ for a.e. $x \in \mathbb{R}$. We take φ, α_ε, and h_ε as in (81), (83), and (87), respectively, and we define u_ε as restriction of $\hat{u}$ to Ω_{h_ε}. Due to our choice of the extension, we have $u_\varepsilon(x,0) = (e_0 x, 0)$ for a.e. $x \in (\alpha_\varepsilon, \beta)$.

Step 1: We claim that

$$\frac{d}{d\varepsilon}\mathcal{E}(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon)\Big|_{\varepsilon=0} = \int_\alpha^\beta W(Eu(x,h(x)))\varphi(x)\,dx\,. \tag{204}$$

By (102) there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned}\frac{\mathcal{E}(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon) - \mathcal{E}(\alpha, \beta, h, u)}{\varepsilon} &= -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\Big(\int_{h_\varepsilon(x)}^{h(x)} W(E\hat{u}(x,y))\,dy\Big)dx \\ &\quad - \frac{1}{\varepsilon}\int_\alpha^{\alpha_\varepsilon}\Big(\int_0^{h(x)} W(Eu(x,y))\,dy\Big)dx =: I_\varepsilon + II_\varepsilon\end{aligned}$$

for all $-\varepsilon_1 < \varepsilon < 0$. Since $\hat{u} \in W^{2,p_0}(\mathbb{R}^2;\mathbb{R}^2)$, with $p_0 > \frac{4}{3}$, by the Sobolev–Gagliardo–Nirenberg embedding theorem, we have that $\nabla\hat{u} \in L^{p_0^*}(\mathbb{R}^2;\mathbb{R}^{2\times 2})$, where $p_0^* > 4$. Hence, by Hölder's inequality and (17),

$$\begin{aligned}\int_\alpha^{\alpha_\varepsilon}\int_0^{h(x)} W(Eu(x,y))\,dydx &\le C_W\int_\alpha^{\alpha_\varepsilon}\int_0^{h(x)}|\nabla u(x,y)|^2 dydx \\ &\le C_W\Big(\int_\alpha^{\alpha_\varepsilon}\int_0^{h(x)}|\nabla u(x,y)|^4 dydx\Big)^{1/2}\Big(\int_\alpha^{\alpha_\varepsilon} h(x)\,dx\Big)^{1/2} \\ &\le C\Big(\int_\alpha^{\alpha_\varepsilon}\int_0^{h(x)}|\nabla u(x,y)|^4 dydx\Big)^{1/2}|\varepsilon|\end{aligned}$$

where we used the fact that $|h(x)| \le L_0|\alpha_\varepsilon - \alpha|$ and (85). This shows that $II_\varepsilon \to 0$ as $\varepsilon \to 0^-$.

Since the function $\zeta := E\hat{u}$ belongs to $W^{1,p_0}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, we have that $W \circ \zeta \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$. Indeed, using the fact that $W(\xi) = \mu|\xi|^2 + (\lambda+\mu)(\operatorname{tr}\xi)^2$, for every bounded Lebesgue measurable set $E \subset \mathbb{R}^2$, by Hölder's inequality, we have

$$\int_E |\nabla(W\circ\zeta)|\,dxdy \le C\int_E |\zeta||\nabla\zeta|\,dxdy \le C\|\zeta\|_{L^{p_0'}(E)}\|\nabla\zeta\|_{L^{p_0}(E)}. \tag{205}$$

Since $p_0 > \frac{4}{3}$, it follows that $p_0' = \frac{p_0}{p_0-1} < p_0^* = \frac{2p_0}{2-p_0}$, and thus the right-hand side of (205) is finite.

By considering the representative of ζ that is locally absolutely continuous on a.e. line parallel to the axes, we have that

$$W(\zeta(x,y)) = W(\zeta(x,h(x))) - \int_y^{h(x)} \partial_y(W\circ\zeta)(x,s)\,ds$$

for a.e. $x \in (\alpha_\varepsilon, \alpha + 2\delta_0)$. Hence,

$$I_\varepsilon = -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)} W(\zeta(x,y))\,dydx = -\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)} W(\zeta(x,h(x)))\,dydx$$
$$+\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)}\int_{y}^{h(x)} \partial_y(W\circ\zeta)(x,s)\,dsdydx =: A_\varepsilon + B_\varepsilon.$$

By (87) and (89), we have

$$A_\varepsilon = \int_{\alpha_\varepsilon}^{\alpha+2\delta_0} W(\zeta(x,h(x)))\left(\varphi(x) + \frac{\omega_\varepsilon}{\varepsilon}\psi(x)\right)\,dx \to \int_{\alpha}^{\alpha+2\delta_0} W(\zeta(x,h(x)))\varphi(x)\,dx\,. \tag{206}$$

On the other hand, by Fubini's theorem

$$B_\varepsilon = \frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)} (s - h_\varepsilon(x))\partial_y(W\circ\zeta)(x,s)\,dsdx\,,$$

and so,

$$|B_\varepsilon| \le C\left\|\varphi + \frac{\omega_\varepsilon}{\varepsilon}\psi\right\|_\infty \int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)} |\partial_y(W\circ\zeta)(x,s)|\,dsdx \tag{207}$$
$$\le C\left\|\varphi + \frac{\omega_\varepsilon}{\varepsilon}\psi\right\|_\infty \|\zeta\|_{L^{p_0'}((\alpha,\beta)\times(0,L_0(\beta-\alpha)))}\left(\int_{\alpha_\varepsilon}^{\alpha+2\delta_0}\int_{h_\varepsilon(x)}^{h(x)} |\partial_y\zeta(x,s)|^{p_0}\,dsdx\right)^{1/p_0} \to 0$$

as $\varepsilon \to 0^-$, where in the last inequality we used (205).

It follows from (206) and (207) that

$$I_\varepsilon \to \int_\alpha^\beta W(Eu(x,h(x)))\varphi(x)\,dx\,,$$

which proves (204) when $\varepsilon \to 0^-$.

On the other hand, if $0 < \varepsilon < \varepsilon_1$, then by (83) and (86) we have $\alpha - \delta_0 < \alpha_\varepsilon < \alpha$, and since $\operatorname{supp}\psi \subset (\alpha + \delta_0, \alpha + 2\delta_0)$ we write

$$\frac{\mathcal{E}(\alpha_\varepsilon,\beta,h_\varepsilon,u_\varepsilon) - \mathcal{E}(\alpha,\beta,h,u)}{\varepsilon} = \frac{1}{\varepsilon}\int_{\alpha}^{\alpha+2\delta_0}\Big(\int_{h(x)}^{h_\varepsilon(x)} W(E\hat{u}(x,y))\,dy\Big)dx$$
$$+\frac{1}{\varepsilon}\int_{\alpha_\varepsilon}^{\alpha}\Big(\int_{0}^{h(x)+\varepsilon\varphi(x)} W(E\hat{u}(x,y))\,dy\Big)dx$$
$$=: III_\varepsilon + IV_\varepsilon\,.$$

The proof that $IV_\varepsilon \to 0$ as $\varepsilon \to 0^+$ can be done as in the proof of II_ε. The term III_ε can be treated similarly to I_ε.

Step 2: By (99), (118), and (204), we have that the derivative $\frac{d}{d\varepsilon}\mathcal{F}^0(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon)$ exists at $\varepsilon = 0$, and by minimality,

$$\frac{d}{d\varepsilon}\mathcal{F}^0(\alpha_\varepsilon, \beta, h_\varepsilon, u_\varepsilon)\bigg|_{\varepsilon=0} = 0\,. \tag{208}$$

Hence, using also (100), we get

$$\begin{aligned}\gamma\int_\alpha^\beta \frac{h'\varphi'}{J}dx + \gamma\frac{J(\alpha)}{h'(\alpha)} - \gamma_0\frac{1}{h'(\alpha)} + \nu_0\int_\alpha^\beta \frac{h''\varphi''}{J^5}dx - \frac{5}{2}\nu_0\int_\alpha^\beta \frac{h'(h'')^2\varphi'}{J^7}dx \\ + \frac{\nu_0}{2}\frac{(h''(\alpha))^2}{J(\alpha)^5}\frac{1}{h'(\alpha)} + \int_\alpha^\beta \overline{W}\varphi\,dx - \frac{1}{\tau}\int_\alpha^\beta \overline{H}\varphi dx - \sigma_0\frac{\alpha-\alpha^0}{\tau}\frac{1}{h'(\alpha)} = 0\,,\end{aligned}$$

where $\overline{W}$, $\overline{H}$, and J are defined in (52) and (42). We integrate by parts once the integrals containing φ' and twice the integral containing φ'' to obtain

$$\begin{aligned}-\gamma\int_\alpha^\beta \Big(\frac{h'}{J}\Big)'\varphi dx + \nu_0\int_\alpha^\beta \Big(\frac{h''}{J^5}\Big)''\varphi\,dx + \frac{5}{2}\nu_0\int_\alpha^\beta \Big(\frac{h'(h'')^2}{J^7}\Big)'\varphi dx \\ + \int_\alpha^\beta \overline{W}\varphi\,dx - \frac{1}{\tau}\int_\alpha^\beta \overline{H}\varphi dx - \gamma\frac{h'(\alpha)}{J(\alpha)} + \gamma\frac{J(\alpha)}{h'(\alpha)} - \gamma_0\frac{1}{h'(\alpha)} - \nu_0\frac{h''(\alpha)}{J(\alpha)^5}\varphi'(\alpha) \\ + \nu_0\Big(\frac{h''}{J^5}\Big)'(\alpha) + \frac{5}{2}\nu_0\frac{h'(\alpha)(h''(\alpha))^2}{J(\alpha)^7} + \frac{\nu_0}{2}\frac{(h''(\alpha))^2}{J(\alpha)^5}\frac{1}{h'(\alpha)} - \sigma_0\frac{\alpha-\alpha^0}{\tau}\frac{1}{h'(\alpha)}\,,\end{aligned}$$

where we used the facts that $\operatorname{supp}\varphi \subset (\alpha - \delta_0, \alpha + \delta_0)$ and $\varphi(\alpha) = 1$. By (53), and the fact that $m\int_\alpha^\beta \varphi\,dx = 0$, we have

$$\begin{aligned}-\gamma\frac{h'(\alpha)}{J(\alpha)} + \gamma\frac{J(\alpha)}{h'(\alpha)} - \gamma_0\frac{1}{h'(\alpha)} - \nu_0\frac{h''(\alpha)}{J(\alpha)^5}\varphi'(\alpha) + \nu_0\Big(\frac{h''}{J^5}\Big)'(\alpha) \\ + \frac{5}{2}\nu_0\frac{h'(\alpha)(h''(\alpha))^2}{J(\alpha)^7} + \frac{\nu_0}{2}\frac{(h''(\alpha))^2}{J(\alpha)^5}\frac{1}{h'(\alpha)} - \sigma_0\frac{\alpha-\alpha^0}{\tau}\frac{1}{h'(\alpha)} = 0\,.\end{aligned}$$

Since $\varphi_0'(0)$ can be chosen arbitrarily, by (81) so can $\varphi'(\alpha)$. Hence, by dividing the previous equation by $\varphi'(\alpha) > 0$ and letting $\varphi'(\alpha) \to \infty$ we get

$$h''(\alpha) = 0. \tag{209}$$

Using (209) and multiplying the previous equation by $h'(\alpha)$ we obtain

$$\begin{aligned}\sigma_0\frac{\alpha-\alpha^0}{\tau} &= \gamma\frac{(J(\alpha))^2 - (h'(\alpha))^2}{J(\alpha)} - \gamma_0 + \nu_0 h'(\alpha)\Big(\frac{h''}{J^5}\Big)'(\alpha) \\ &= \frac{\gamma}{J(\alpha)} - \gamma_0 + \nu_0\frac{h'(\alpha)}{J(\alpha)^2}\Big(\frac{h''}{J^3}\Big)'(\alpha)\,.\end{aligned}$$

In a similar way we can prove that $h''(\beta) = 0$ and that

$$\sigma_0 \frac{\beta - \beta^0}{\tau} = -\frac{\gamma}{J(\beta)} + \gamma_0 - \nu_0 h'(\beta) \Big(\frac{h''}{J^5}\Big)'(\beta) = -\frac{\gamma}{J(\beta)} + \gamma_0 - \nu_0 \frac{h'(\beta)}{J(\beta)^2} \Big(\frac{h''}{J^3}\Big)'(\beta)\,.$$

We omit the details. This concludes the proof. □

We now estimate the L^{p_0} norm of $\nabla^2 u$ near α. To simplify the exposition we assume that $\alpha = 0$. Given $r > 0$, we set $I_r := (0, r)$, $\Omega_h^{0,r} := \Omega_h \cap (I_r \times \mathbb{R})$, and $\Gamma_h^{0,r} := \Gamma_h \cap (I_r \times \mathbb{R}) = \{(x, h(x)) : 0 < x < r\}$.

Theorem 18. *Under the assumptions of Theorem 6, let $0 < r < \delta_0$. Then there exist two constants $0 < c_1 = c_1(\eta_0, \eta_1, M) < 1$ and $c_2 = c_2(\eta_0, \eta_1, M) > 0$, independent of r, such that if*

$$r^2 (\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}) \le c_1\,, \tag{210}$$

then

$$\|\nabla^2 u\|_{L^{p_0}(\Omega_h^{0,r/2})} \le c_2 + \frac{c_2}{r} \|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})}\,, \tag{211}$$

$$\|\nabla u\|_{L^{p_0/(2-p_0)}(\Gamma_h^{0,r/2})} \le c_2 + \frac{c_2}{r} \|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})}\,. \tag{212}$$

We begin with a preliminary lemma.

Lemma 19. *Let $0 < \eta_0 \le m \le L_0$, $1 \le p < \infty$, $\mathbb{R}^2_+ := \mathbb{R} \times (0, +\infty)$,*

$$A^m_\infty := \{(x, y) \in \mathbb{R}^2 : \, x > 0\,,\, 0 < y < mx\} \tag{213}$$

and let $v \in W^{2,p}(A^m_\infty)$ be such that $v(x, 0) = 0$ for $x > 0$. Then the function

$$\hat{v}(x, y) := \begin{cases} 3v(2y/m - x, y) - 2v(3y/m - 2x, y) & \text{if } (x, y) \in \mathbb{R}^2_+ \setminus A^m_\infty\,, \\ v(x, y) & \text{if } (x, y) \in A^m_\infty\,, \end{cases}$$

belongs to $W^{2,p}(\mathbb{R}^2_+)$ with $\hat{v}(x, 0) = 0$ for all $x \in \mathbb{R}$. Moreover,

$$\|\nabla^2 \hat{v}\|_{L^p(\mathbb{R}^2_+)} \le C \|\nabla^2 v\|_{L^p(A^m_\infty)} \tag{214}$$

for some constant $C > 0$ depending on η_0, L_0, and p.

Proof. Since $\hat{v}$, $\partial_x \hat{v}$, and $\partial_y \hat{v}$ have the same trace on both sides of the line $y = mx$, the function $\hat{v}$ belongs to $W^{2,p_0}(\mathbb{R}^2_+; \mathbb{R}^2)$. By direct computation we check that $\hat{v}(x, 0) = 0$ for a.e. $x < 0$. The estimate (214) can be obtained by computing the second derivatives of $\hat{v}$. □

Proof of Theorem 18. **Step 1:** We proceed as in the proof of Theorem 14. We recall that we are taking $\alpha = 0$. Let $m = h'(0)$, let r be as in (124), let A^m_r be the triangle defined in (119), and let v be defined as in (186). In view of Theorem 14, we have that $v \in W^{2,p_0}(A_r^m;\mathbb{R}^2)$, and so, by (187) and Theorem 13, we have that

$$\begin{aligned}\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \le \kappa\big(&\|f\circ\Phi\|_{L^{p_0}(A_r^m)} + \|f^v\|_{L^{p_0}(A_r^m)} + \|\nabla(g\circ\Phi)\|_{L^{p_0}(A_r^m)}\\ &+ \|\hat{g}^v\|_{L^{p_0}(A_r^m)} + \|\check{g}^v\|_{L^{p_0}(A_r^m)}\big)\,.\end{aligned}$$

Since $v=0$ near the line $x=r$ by (181), we can extend it to A_∞^m by setting $v=0$ in $A_\infty^m\setminus A_r^m$, and the extended function belongs to $W^{2,p_0}(A_\infty^m;\mathbb{R}^2)$ and satisfies the estimate

$$\begin{aligned}\|\nabla^2 v\|_{L^{p_0}(A_\infty^m)} \le \kappa\big(&\|f\circ\Phi\|_{L^{p_0}(A_r^m)} + \|f^v\|_{L^{p_0}(A_r^m)}\\ &+ \|\nabla(g\circ\Phi)\|_{L^{p_0}(A_r^m)} + \|\hat{g}^v\|_{L^{p_0}(A_r^m)} + \|\check{g}^v\|_{L^{p_0}(A_r^m)}\big)\,. \end{aligned}\tag{215}$$

We underline that κ is independent of $r\in(0,\delta_0)$, $m\in[2\eta_0,L_0]$, and h satisfying (120)–(123). Since $h''(\alpha)=0$ by Theorem 16, it follows by the mean value theorem that

$$\|h''\|_{L^\infty(I_r)} \le r\|h'''\|_{L^\infty(I_r)}\,. \tag{216}$$

Hence, also by (196), (132), (133), (137), and (138) we have

$$\begin{aligned}\|f^v\|_{L^{p_0}(A_r^m)} &\le Cr^2(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\\ &+ Cr^4(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\\ &+ Cr(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\|\nabla v\|_{L^{p_0}(A_r^m)}\\ &+ Cr^3(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla v\|_{L^{p_0}(A_r^m)}\,.\end{aligned}$$

Since $\partial_x v$ and $\partial_y v$ vanish on one of the sides of A_r^m different from Γ_r^m, by Poincaré's inequality (see Remark 6) we obtain

$$\begin{aligned}\|f^v\|_{L^{p_0}(A_r^m)} &\le Cr^2(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\\ &+ Cr^4(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\,.\end{aligned}\tag{217}$$

By (198), (149), and (150) we have

$$\begin{aligned}\|\nabla\hat{g}^v\|_{L^{p_0}(A_r^m)} \le Cr^2\big(&\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big)\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\\ &+ Cr(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\|\nabla v\|_{L^{p_0}(A_r^m)}\\ &+ Cr^3(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla v\|_{L^{p_0}(A_r^m)}\,.\end{aligned}$$

Using again Poincaré's inequality (see Remark 6) we obtain

$$\begin{aligned}\|\nabla\hat{g}^v\|_{L^{p_0}(A_r^m)} \le Cr^2\big(&\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big)\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\\ &+ Cr^4(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\end{aligned}\tag{218}$$

In the same way we prove that

$$\begin{aligned}\|\nabla \check{g}^v\|_{L^{p_0}(A_r^m)} &\leq Cr^2\big(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)}\big)\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \\ &\quad + Cr^4(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\end{aligned} \tag{219}$$

By (184), it follows from Lemma 9 that

$$\begin{aligned}\|f\circ\Phi\|_{L^{p_0}(A_r^m)} &\leq C\|f\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\leq C\|\nabla^2\varphi_r\|_{C^0(\mathbb{R}^2)}\|u-w^0\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\quad + C\|\nabla\varphi_r\|_{C^0(\mathbb{R}^2)}\|\nabla u-\nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\leq \frac{C}{r}\|\nabla u-\nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})}\,,\end{aligned} \tag{220}$$

where we used the Poincaré inequality (see Lemma 15) and the inequalities $\|\nabla\varphi_r\|_{C^0(\mathbb{R}^2)} \leq C/r$, and $\|\nabla^2\varphi_r\|_{C^0(\mathbb{R}^2)} \leq C/r^2$.

By (185), it follows from Lemmas 8 and 9 that

$$\begin{aligned}\|\nabla(g\circ\Phi)\|_{L^{p_0}(A_r^m)} &\leq C\big(1+\sup_{(x,y)\in A_r^m} y|\sigma'(x)|\big)\|\nabla g\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\leq C\big(1+\sup_{(x,y)\in A_r^m} y|\sigma'(x)|\big)\big[\|\nabla^2\varphi_r\|_{C^0(\mathbb{R}^2)}\|u-w^0\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\quad + \|\nabla\varphi_r\|_{C^0(\mathbb{R}^2)}\|\nabla u-\nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})} \\ &\quad + \|\nabla\varphi_r\|_{C^0(\mathbb{R}^2)}\|u-w^0\|_{L^{p_0}(\Omega_h^{0,r})}\|h''\|_{L^\infty(I_r)}\big] \\ &\leq \frac{C}{r}\big[1+r^2(\|h'''\|_{L^\infty(I_r)}+\|h^{(\mathrm{iv})}\|_{L^1(I_r)}))+r^4(\|h'''\|^2_{L^\infty(I_r)} \\ &\quad + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})^2\big]\|\nabla u-\nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})}\,,\end{aligned} \tag{221}$$

where we used the Poincaré inequality (see Lemma 15), the inequalities $\|\nabla\varphi_r\|_{C^0(\mathbb{R}^2)} \leq C/r$ and $\|\nabla^2\varphi_r\|_{C^0(\mathbb{R}^2)} \leq C/r^2$, and (216).

Combining inequalities (215), and (217)–(221), we finally obtain

$$\begin{aligned}\|\nabla^2 v\|_{L^{p_0}(A_r^m)} &\leq Cr^2(\|h'''\|_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \\ &+ Cr^4(\|h'''\|^2_{L^\infty(I_r)} + \|h^{(\mathrm{iv})}\|^2_{L^1(I_r)})\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \\ &+ \frac{C}{r}\big[1+r^2(\|h'''\|_{L^\infty(I_r)}+\|h^{(\mathrm{iv})}\|_{L^1(I_r)})) \\ &+ r^4(\|h'''\|^2_{L^\infty(I_r)}+\|h^{(\mathrm{iv})}\|_{L^1(I_r)})^2\big]\|\nabla u-\nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})}\,.\end{aligned} \tag{222}$$

Let us fix $c_1 = c_1(\eta_0,\eta_1,M) > 0$ such that $c_1 < 1$ and $Cc_1 < 1/4$, where $C = C(\eta_0,\eta_1,M)$ is the constant in (222). Suppose that (210) holds. Then $r^4\|h'''\|^2_{L^\infty(I_r)} \leq r^2\|h'''\|_{L^\infty(I_r)}$ and $r^4\|h^{(\mathrm{iv})}\|^2_{L^1(I_r)} \leq r^2\|h^{(\mathrm{iv})}\|_{L^1(I_r)}$, hence (222) and the inequality $|\Omega_h^{0,r}| \leq L_0 r^2/2$ give

$$\frac{1}{2}\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \leq \frac{C}{r}\|\nabla u - \nabla w^0\|_{L^{p_0}(\Omega_h^{0,r})} \leq \frac{C}{r}\|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})} + \frac{C}{r} r^{2/p_0}\,,$$

with new constants C depending only on η_0, η_1, and M. Since $p_0 < 2$ by (156), we obtain

$$\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \leq \frac{C}{r}\|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})} + C \tag{223}$$

In turn, by Remark 4, (133), (135), (138), (186), (210), (216), and Remark 6 applied to $\partial_x v$ and $\partial_y v$, the previous inequality gives

$$\|\nabla^2 \tilde{u}\|_{L^p(\Omega_h^{0,r})} \leq C\|\nabla^2 v\|_{L^p(A_r^m)} \leq \frac{C}{r}\|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})} + C\,.$$

Since $\tilde{u} = u - w^0$ in $\Omega_h^{0,r/2}$ and w^0 is linear, inequality (211) follows.

Step 2: It follows from (181) and (186) that the function $v \in W^{2,p_0}(A_r^m;\mathbb{R}^2)$ is zero outside $A_{7r/8}^m$. Hence, by extending v to be zero, we can assume that $v \in W^{2,p_0}(A_\infty^m;\mathbb{R}^2)$, where A_∞^m is defined in (213) with $m = h'(0)$. By Lemma 19 we have that the function

$$\hat{v}(x,y) := \begin{cases} 3v(2y/m - x, y) - 2v(3y/m - 2x, y) & \text{if } (x,y) \in \mathbb{R}^2_+ \setminus A_\infty^m\,, \\ v(x,y) & \text{if } (x,y) \in A_\infty^m\,, \end{cases}$$

belongs to $W^{2,p_0}(\mathbb{R}^2_+)$ with $\hat{v}(x,0) = 0$ for all $x \in \mathbb{R}$. Moreover,

$$\|\nabla^2 \hat{v}\|_{L^{p_0}(\mathbb{R}^2_+)} \leq C\|\nabla^2 v\|_{L^{p_0}(A_\infty^m)}\,.$$

Next we extend $\hat{v}$ across the x-axis by setting

$$\bar{v}(x,y) := \begin{cases} 3\hat{v}(x,-y) - 2\hat{v}(x,-2y) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+, \\ \hat{v}(x,y) & \text{if } (x,y) \in \mathbb{R}^2_+\,. \end{cases}$$

Then $\bar{v} \in W^{2,p_0}(\mathbb{R}^2)$ with

$$\|\nabla^2 \bar{v}\|_{L^{p_0}(\mathbb{R}^2)} \leq C\|\nabla^2 \hat{v}\|_{L^{p_0}(\mathbb{R}^2_+)} \leq C\|\nabla^2 v\|_{L^{p_0}(A_r^m)}.$$

It follows from [44, Theorem 18.24] that the trace of $\nabla \bar{v}$ along the half- line $\Gamma_\infty^m := \{(x, mx) : x \geq 0\}$ belongs to $L^{p_0/(2-p_0)}(\Gamma_\infty^m;\mathbb{R}^{2\times 2})$ with

$$\|\nabla \bar{v}\|_{L^{p_0/(2-p_0)}(\Gamma_\infty^m)} \leq C\|\nabla^2 \bar{v}\|_{L^{p_0}(\mathbb{R}^2)} \leq C\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\,. \tag{224}$$

Since $\varphi_r = 1$ on $A_{r/2}^m$, it follows from (181) and (186) that $\bar{v} = (u - w^0) \circ \Phi$ in $A_{r/2}^m$. Hence, $u - w^0 = \bar{v} \circ \Psi$ in $\Omega_h^{0,r/2}$. By (126) and the chain rule,

$$|\nabla(u - w^0)(x,y)| \leq C(1 + |\sigma'(x)y|)|\nabla \bar{v}(x, \sigma(x)y)|\,.$$

Taking $y = h(x)$ and using (125) gives

$$|\nabla(u - w^0)(x, h(x))| \leq C \sup_{(x,y)\in\Omega_h^{0,r/2}} (1 + |\sigma'(x)y|)|\nabla\bar{v}(x, mx)|$$

for a.e. $0 < x < r/2$. Hence, by (224),

$$\begin{aligned}\|\nabla(u - w^0)\|_{L^{p_0/(2-p_0)}(\Gamma_h^{0,r/2})} &\leq C \sup_{(x,y)\in\Omega_h^{0,r/2}} (1 + |\sigma'(x)y|)\|\nabla\bar{v}\|_{L^{p_0/(2-p_0)}(\Gamma_\infty^m)} \\ &\leq C \sup_{(x,y)\in\Omega_h^{0,r}} (1 + |\sigma'(x)y|)\|\nabla^2 v\|_{L^{p_0}(A_r^m)}\,.\end{aligned}$$

Using (135), (201), (210), and (216),

$$\|\nabla(u - w^0)\|_{L^{p_0/(2-p_0)}(\Gamma_h^{0,r/2})} \leq C\|\nabla^2 v\|_{L^{p_0}(A_r^m)} \leq C + \frac{C}{r}\|\nabla u\|_{L^{p_0}(\Omega_h^{0,r})}\,,$$

where the last inequality follows from (223). Since by (180),

$$\|\nabla w^0\|_{L^{p_0/(2-p_0)}(\Gamma_h^{0,r/2})} \leq |e_0|((1 + L_0^2)^{1/2} r/8)^{(2-p_0)/p_0} \leq C\,,$$

inequality (212) follows. □

Corollary 20. *Under the hypotheses of Theorem 6, let* $1 \leq p \leq p_0/(2 - p_0)$ *and let*

$$r_1 := \min\left\{\frac{c_1^{1/2}}{2c_0}\frac{1}{B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)}, \frac{\delta_0}{2}, \frac{\eta_1}{2}\right\}, \tag{225}$$

where $B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)$ *is defined in (61),* $c_0 > 1$ *is the constant in the statement of Theorem 6, and* $0 < c_1 < 1$ *is the constant in (210). Then there exist two constants* $c_3 = c_3(\eta_0, \eta_1, M) > 0$ *and* $c_4 = c_4(\eta_0, \eta_1, M, p) > 0$ *such that*

$$\|\nabla^2 u\|_{L^{p_0}(\Omega_h^{\alpha,\alpha+r_1})} \leq c_3 B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{2-2/p_0}\,, \tag{226}$$

$$\|\nabla u\|_{L^p(\Gamma_h^{\alpha,\alpha+r_1})} \leq c_4 B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{1-1/p}\,. \tag{227}$$

Proof. By Hölder's inequality with exponent $q = 2/p_0$, and Lemma 3, for every $0 < r \leq \beta - \alpha$,

$$\|\nabla u\|_{L^{p_0}(\Omega_h^{\alpha,\alpha+r})} \leq C(r^2)^{1/(q'p_0)}\|\nabla u\|_{L^2(\Omega_h^{\alpha,\beta})} \leq Cr^{2/p_0-1}\,, \tag{228}$$

where we used the fact that $q' = 2/(2 - p_0)$.

By (68), (70), and (225), recalling that $2c_0 > 1$, we have

$$r_1^2(\|h'''\|_{L^\infty((\alpha,\beta))} + \|h^{(\mathrm{iv})}\|_{L^1((\alpha,\beta))}) \leq 2c_0 r_1^2 B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^2 \leq c_1\,.$$

Hence, we can apply Theorem 18 to obtain (211) and (212). In turn, by (228),

$$\|\nabla^2 u\|_{L^{p_0}(\Omega_h^{\alpha,\alpha+r_1})} \le c_2 + \frac{c_2}{r_1}\|\nabla u\|_{L^{p_0}(\Omega_h^{\alpha,\alpha+2r_1})}$$
$$\le c_2 + c_2 C\frac{1}{r_1^{2-2/p_0}} \le c_3 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^{2-2/p_0},$$

where in the last inequality we used (225) and the inequality $1 \le B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)$.

If $p < p_0/(2-p_0)$, we use Hölder's inequality with exponent $p_1 = p_0/[p(2-p_0)]$, (212), (228), and the fact that $r_1 \le 1$ to get

$$\|\nabla u\|_{L^p(\Gamma_h^{\alpha,\alpha+r_1})} \le C r_1^{1/(pp_1')}\|\nabla u\|_{L^{p_0/(2-p_0)}(\Gamma_h^{\alpha,\alpha+r_1})}$$
$$= C r_1^{1+1/p-2/p_0}\|\nabla u\|_{L^{p_0/(2-p_0)}(\Gamma_h^{\alpha,\alpha+r_1})}$$
$$\le C + C r_1^{1/p-2/p_0}\|\nabla u\|_{L^{p_0}(\Omega_h^{\alpha,\alpha+2r_1})} \le C + C r_1^{1/p-1}$$
$$\le C B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^{1-1/p},$$

where in the last inequality we used (225) and the inequality $1 \le B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)$. The same inequality holds if $p = p_0/(2-p_0)$. □

7 Global Regularity

In this section we obtain global estimates for $\nabla^2 u$ in the entire domain Ω_h. Given $r > 0$ and $x_0 \in \mathbb{R}$, we set $I_r(x_0) := (x_0 - r, x_0 + r)$. Under the assumptions of Theorem 6, let us fix r and x_0 such that

$$0 < 2r \le \min\{\delta_0, \eta_1\} \quad \text{and} \quad \alpha + 8r/\eta_0 \le x_0 \le \beta - 8r/\eta_0\,. \tag{229}$$

If $x \in I_{4r}(x_0)$, the inequality $\eta_0 < 1$ implies that

$$\alpha + 4r/\eta_0 < \alpha + 8r/\eta_0 - 4r < x < \beta - 8r/\eta_0 + 4r < \beta - 4r/\eta_0\,. \tag{230}$$

Using the fact that $2r \le \delta_0 < \eta_0$, by (79) we have $h(x) \ge \eta_0(x-\alpha) \ge 4r$ for $\alpha + 4r/\eta_0 \le x \le \alpha + \delta_0$; in the same way we prove that $h(x) \ge 4r$ for $\beta - \delta_0 \le x \le \beta - 4r/\eta_0$. Using (63) we have also $h(x) \ge 2\eta_1 \ge 4r$ for $\alpha + \delta_0 \le x \le \beta - \delta_0$. Hence, by (230), we have

$$h(x) \ge 4r \tag{231}$$

for every $x \in I_{4r}(x_0)$ and r and x_0 as in (229).

We recall that

$$\Omega_{-h} := \{(x,y) \in \mathbb{R}^2 : \ \alpha < x < \beta\,,\ -h(x) < y < 0\}\,,$$
$$\Omega_{h,r}^{a,b} := \{(x,y) \in \mathbb{R}^2 : \ a < x < b\,,\ h(x) - r < y < h(x)\}\,,$$

$$\Gamma_h^{a,b} := \{(x, h(x)) : a < x < b\} .$$

Theorem 21. *Under the assumptions of Theorem 6, let r and x_0 be as in (229). Then for every $2 \le q < \infty$ there exist constants $c_5 = c_5(\eta_0, \eta_1, M) > 0$, and $c_6 = c_6(\eta_0, \eta_1, M, q) > 0$, independent of h, r and x_0, such that if*

$$\|h''\|_{L^\infty(I_{4r}(x_0))} \le \frac{1}{r} , \tag{232}$$

then $u \in H^2(\Omega_{h,2r}^{x_0-2r,x_0+2r})$ and we have

$$\|\nabla^2 u\|_{L^2(\Omega_{h,2r}^{x_0-2r,x_0+2r})} \le \frac{c_5}{r} \|\nabla u\|_{L^2(\Omega_{h,4r}^{x_0-4r,x_0+4r})} , \tag{233}$$

$$\|\nabla u\|_{L^q(\Gamma_h^{x_0-2r,x_0+2r})} \le \frac{c_6}{r^{1-1/q}} \|\nabla u\|_{L^2(\Omega_{h,4r}^{x_0-4r,x_0+4r})} . \tag{234}$$

Proof. Without loss of generality we assume that $x_0 = 0$. For every $\rho > 0$ define $R_\rho := (-\rho, \rho) \times (-\rho, 0)$ and $J_\rho := (-\rho, \rho) \times \{0\}$.

Step 1: Let

$$v(x, y) := u(x, y + h(x)) . \tag{235}$$

In view of (231) we have $R_{4r} \subset \Omega_{-h}$. Hence, (58) gives

$$\begin{cases} \operatorname{div}(\mathbb{A}\nabla v) = 0 \text{ in } R_{4r}, \\ (\mathbb{A}\nabla v)e_2 = 0 \quad \text{on } J_{4r} . \end{cases}$$

Define $v_r(x, y) := v(rx, ry)$, $(x, y) \in R_4$. Then v_r satisfies the boundary value problem

$$\begin{cases} \operatorname{div}(\mathbb{A}_r\nabla v_r) = 0 \text{ in } R_4, \\ (\mathbb{A}_r\nabla v_r)e_2 = 0 \quad \text{on } J_4 , \end{cases}$$

where $\mathbb{A}_r(x) := \mathbb{A}(rx)$. Using using (59) and (232) and the fact that $\operatorname{Lip} h \le L_0$, we have

$$\|\mathbb{A}_r\|_{C^1(\overline{R}_2)} \le C .$$

Standard elliptic regularity ([45, Proof of Theorem 20.4]) gives

$$\|\nabla^2 v_r\|_{L^2(R_2)} \le C\|\nabla v_r\|_{L^2(R_4)} .$$

In turn, we obtain

$$\|\nabla^2 v\|_{L^2(R_{2r})} \le \frac{C}{r}\|\nabla v\|_{L^2(R_{4r})} . \tag{236}$$

Since $u(x, y) = v(x, y - h(x))$ and

$$\begin{aligned} &\partial_x u = \partial_x v - \partial_y v h' , \quad \partial_y u = \partial_y v \\ &\partial^2_{xx} u = \partial^2_x v - 2\partial^2_{xy} v h' + \partial^2_{yy} v (h')^2 - \partial_y v h'' , \quad \partial^2_{xy} u = \partial_{xy} v - \partial^2_{yy} v h' , \end{aligned} \tag{237}$$

$$\partial^2_{yy} u = \partial^2_{yy} v \,,$$

we have

$$\begin{aligned}\|\nabla^2 u\|_{L^2(\Omega_{h,2r}^{-2r,2r})} &\le C\big(\|\nabla^2 v\|_{L^2(R_{2r})} + \|h''\|_{L^\infty(I_{4r})}\|\nabla v\|_{L^2(R_{2r})}\big)\\ &\le \frac{C}{r}\|\nabla v\|_{L^2(R_{4r})} \le \frac{C}{r}\|\nabla u\|_{L^2(\Omega_{h,4r}^{-4r,4r})}\,,\end{aligned}$$

which proves (233).

Step 2: In this step we prove (234). We begin by proving

$$\|\nabla v\|_{L^2(J_{2r})} \le \frac{C}{r^{1/2}}\|\nabla v\|_{L^2(R_{4r})}\,, \tag{238}$$

$$|\nabla v|_{H^{1/2}(J_{2r})} \le \frac{C}{r}\|\nabla v\|_{L^2(R_{4r})}\,. \tag{239}$$

Define $z = \nabla v$. Then by standard trace theory [46, Theorem 2.5.3] and a rescaling argument, we have

$$\begin{aligned}\|z\|^2_{L^2(J_{2r})} &\le \frac{C}{r}\|z\|^2_{L^2(R_{2r})} + Cr\|\nabla z\|^2_{L^2(R_{2r})}\,,\\ |z|^2_{H^{1/2}(J_{2r})} &\le C\|\nabla z\|^2_{L^2(R_{2r})}\,.\end{aligned}$$

Combining the previous inequalities with (236) gives (238) and (239).

Next, fix $2 < q < \infty$. We claim that

$$\|\nabla v\|_{L^q(J_{2r})} \le \frac{C}{r^{1-1/q}}\|\nabla v\|_{L^2(R_{4r})}\,. \tag{240}$$

Let $s = \frac{q-2}{2q} < \frac{1}{2}$. Then s is subcritical so, by [47, Corollary 2.3],

$$\|\nabla v\|_{L^q(J_{2r})} = \|\nabla v\|_{L^{2^*_s}(J_{2r})} \le \frac{C}{r^{1-1/q}}\|\nabla v\|_{L^1(J_{2r})} + C|\nabla v|_{H^s(J_{2r})}\,,$$

where $2^*_s = \frac{2}{1-2s} = q$ is Sobolev critical exponent. On the other hand, by [47, Lemma 2.6],

$$|\nabla v|_{H^s(J_{2r})} \le Cr^{1/2-s}|\nabla v|_{H^{1/2}(J_{2r})} = Cr^{1/q}|\nabla v|_{H^{1/2}(J_{2r})}\,.$$

Combining the last two inequalities and using Hölder's inequality and (238) and (239), we deduce

$$\begin{aligned}\|\nabla v\|_{L^q(J_{2r})} &\le \frac{C}{r^{1/2-1/q}}\|\nabla v\|_{L^2(J_{2r})} + Cr^{1/q}|\nabla v|_{H^{1/2}(J_{2r})}\\ &\le \frac{C}{r^{1-1/q}}\|\nabla v\|_{L^2(R_{4r})}\,.\end{aligned}$$

By changing variables using (237), we obtain (234). □

Corollary 22. *Under the assumptions of Theorem 6, let r be such that*

$$0 < 2r \leq \min\{\delta_0, \eta_1\}\,, \tag{241}$$

and let

$$\alpha_r := \alpha + 9r/\eta_0 \quad \textit{and} \quad \beta_r := \beta - 9r/\eta_0\,. \tag{242}$$

Then there exists a constant $c_7 = c_7(\eta_0, \eta_1, M) > 0$ such that, if

$$\|h''\|_{L^\infty(I_{4r}(x_0))} \leq \frac{1}{r} \tag{243}$$

for every $x_0 \in [\alpha_r, \beta_r]$, then $u \in H^2(\Omega_h^{\alpha_r,\beta_r})$ and

$$\|\nabla^2 u\|_{L^2(\Omega_h^{\alpha_r,\beta_r})} \leq \frac{c_7}{r}\|\nabla u\|_{L^2(\Omega_h)}\,. \tag{244}$$

Proof. Assume (243) and let

$$\begin{aligned}
\Omega_1 &:= \{(x,y) : \alpha_r < x < \beta_r,\ 0 < y < 2r\}\,,\\
\Omega_2 &:= \{(x,y) : \alpha_r < x < \beta_r,\ r < y < h(x) - r\}\,,\\
\Omega_3 &:= \{(x,y) : \alpha_r < x < \beta_r,\ h(x) - 2r < y < h(x)\}\,.
\end{aligned}$$

Since $\Omega_h^{\alpha_r,\beta_r} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, it is enough to prove that $u \in H^2(\Omega_i)$ and that

$$\|\nabla^2 u\|_{L^2(\Omega_i)} \leq \frac{C}{r}\|\nabla u\|_{L^2(\Omega_h)}\,. \tag{245}$$

For every (x_0, y_0) in $\mathbb{R}^2$ and every $\rho > 0$ let $Q_\rho(x_0, y_0)$ be the cube with center (x_0, y_0) and sides with length 2ρ parallel to the coordinate axes. We set $Q_\rho^+(x_0, y_0) := \{(x, y) \in Q_\rho(x_0, y_0) : y > 0\}$. We take for granted that every solution of the Lamé system in a cube of the form $Q_{2\rho}(x_0, y_0)$ belongs to $H^2(Q_\rho(x_0, y_0))$ and satisfies the inequality

$$\int_{Q_\rho(x_0,y_0)} |\nabla^2 u|^2\, dxdy \leq \frac{C}{\rho^2}\int_{Q_{2\rho}(x_0,y_0)} |\nabla u|^2\, dxdy\,, \tag{246}$$

(see [45, Theorem 20.1]). Moreover, every solution in a rectangle of the form $Q_{2\rho}^+(x_0, 0)$ with homogeneous Dirichlet boundary condition on $I_{2\rho}(x_0) \times \{0\}$ belongs to $H^2(Q_\rho^+(x_0, 0))$ and satisfies the inequality

$$\int_{Q_\rho^+(x_0,0)} |\nabla^2 u|^2\, dxdy \leq \frac{C}{\rho^2}\int_{Q_{2\rho}^+(x_0,0)} |\nabla u|^2\, dxdy\,. \tag{247}$$

In both cases the dependence of the estimate on ρ follows from a standard dimensional argument.

To prove the estimate for Ω_1, for every integer i we define $x_i := 2ir$ and we consider the set Z_1 of integers i such that $\alpha_r < x_i < \beta_r$. Since x_i satisfies (229), by (231) we have $h(x) \geq 4r$ for every $i \in Z_1$ and $x \in I_{4r}(x_i)$. It follows that $Q^+_{4r}(x_i, 0) \subset \Omega_h$. Therefore $u \in H^2(Q^+_{2r}(x_i, 0))$ and by (247),

$$\int_{Q^+_{2r}(x_i,0)} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{4r^2} \int_{Q^+_{4r}(x_i,0)} |\nabla u|^2 \, dxdy \, .$$

Using the inclusion $\Omega_1 \subset \bigcup_{i \in Z_1} Q^+_{2r}(x_i, 0)$, we obtain $u \in H^2(\Omega_1)$ and

$$\int_{\Omega_1} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{4r^2} \sum_{i \in Z_1} \int_{Q^+_{4r}(x_i,0)} |\nabla u|^2 \, dxdy \, .$$

Since each rectangle $Q^+_{4r}(x_i, 0)$ intersects at most 7 rectangles of the form $Q^+_{4r}(x_{\hat{\imath}}, 0)$, from the previous inequality and from the inclusion $Q_{4r}(x_{2i}, 0)^+ \subset \Omega_h$ we obtain

$$\int_{\Omega_1} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{4r^2} \int_{\Omega_h} |\nabla u|^2 \, dxdy \, . \tag{248}$$

To prove the estimate for Ω_2, we set $\rho := r/(3+3L_0)$ and we consider the set Z_2 of all pairs of integers (i, j) such that $Q_\rho(i\rho, j\rho) \cap \Omega_2 \neq \emptyset$. We claim that $Q_{2\rho}(i\rho, j\rho) \subset \Omega_h$ for every $(i, j) \in Z_2$. Indeed, if $(i, j) \in Z_2$, then there exists $(x_0, y_0) \in Q_\rho(i\rho, j\rho) \cap \Omega_2$. Hence $\alpha_r < x_0 < \beta_r$ and $r < y_0 < h(x_0) - r$. If $(x, y) \in Q_{2\rho}(i\rho, j\rho)$, we have $|x - x_0| < 3\rho$ and $|y - y_0| < 3\rho$. Recalling the Lipschitz estimate for h, this implies that $0 < r - 3\rho < y_0 - 3\rho < y < y_0 + 3\rho < h(x_0) - r + 3\rho \leq h(x) + L_0|x - x_0| - r + 3\rho < h(x) + 3L_0\rho - r + 3\rho = h(x)$, which gives $(x, y) \in \Omega_h$ and concludes the proof of the inclusion $Q_{2\rho}(i\rho, j\rho) \subset \Omega_h$ for $(i, j) \in Z_2$. Therefore $u \in H^2(Q_\rho(i\rho, j\rho))$ and by (246),

$$\int_{Q_\rho(i\rho,j\rho)} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{r^2} \int_{Q_{2\rho}(i\rho,j\rho)} |\nabla u|^2 \, dxdy \, .$$

Using the inclusion $\Omega_2 \subset \bigcup_{(i,j) \in Z_2} Q_\rho(i\rho, j\rho)$ we obtain $u \in H^2(\Omega_2)$ and

$$\int_{\Omega_2} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{r^2} \sum_{(i,j) \in Z_2} \int_{Q_{2\rho}(i\rho,j\rho)} |\nabla u|^2 \, dxdy \, .$$

Since each rectangle $Q_{2\rho}(i\rho, j\rho)$ intersects at most 49 rectangles of the form $Q_{2\rho}(\hat{\imath}\rho, \hat{\jmath}\rho)$, from the previous inequality and from the inclusion $Q_{2\rho}(i\rho, j\rho) \subset \Omega_h$ we obtain

$$\int_{\Omega_2} |\nabla^2 u|^2 \, dxdy \leq \frac{C}{r^2} \int_{\Omega_h} |\nabla u|^2 \, dxdy \, . \tag{249}$$

To prove the estimate for Ω_3, we consider x_i and Z_1 as for Ω_1. By Theorem 21 we obtain that $u \in H^2(\Omega^{x_i - 2r, x_i + 2r}_{h,2r})$ and that

$$\int_{\Omega^{x_i - 2r, x_{2i} + 2r}_{h,2r}} |\nabla^2 u|^2 \, dxdy \leq \frac{c_5^2}{r^2} \int_{\Omega^{x_i - 4r, x_i + 4r}_{h,4r}} |\nabla u|^2 \, dxdy \, .$$

Using the inclusion $\Omega_3 \subset \bigcup_{i\in Z_1} \Omega_{h,2r}^{x_i-2r,x_i+2r}$ we obtain $u \in H^2(\Omega_3)$ and

$$\int_{\Omega_3} |\nabla^2 u|^2 \, dxdy \le \frac{c_5^2}{r^2} \sum_{i\in Z_1} \int_{\Omega_{h,4r}^{x_i-4r,x_i+4r}} |\nabla u|^2 \, dxdy \, . \tag{250}$$

Since each set $\Omega_{h,4r}^{x_i-4r,x_i+4r}$ intersects at most 7 sets of the form $\Omega_{h,4r}^{x_i-4r,x_i+4r}$, from the previous inequality and from the inclusion $\Omega_{h,4r}^{x_i-4r,x_i+4r} \subset \Omega_h$, we obtain

$$\int_{\Omega_3} |\nabla^2 u|^2 \, dxdy \le \frac{7c_5^2}{r^2} \int_{\Omega_h} |\nabla u|^2 \, dxdy \, , \tag{251}$$

which concludes the proof. □

Theorem 23. *Under the hypotheses of Theorem 6, let* $2 \le p \le p_0/(2-p_0)$*. Then there exists a constant* $c_8 = c_8(\eta_0, \eta_1, M, p) > 0$ *such that*

$$\|\nabla u\|_{L^p(\Gamma_h)} \le c_8 B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{1-1/p}.$$

Proof. By (227),

$$\int_\alpha^{\alpha+r_1} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2} dx \le c_4^p B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{p-1}. \tag{252}$$

A similar estimate holds in $(\beta - r_1, \beta)$. It remains to estimate ∇u over $(\alpha + r_1, \beta - r_1)$. Let $r := \eta_0 r_1/4$ and for every integer i we define $x_i := ir$. Let Z be the set of integers i such that $\alpha + r_1 < x_i < \beta - r_1$. Since $(\alpha + r_1, \beta - r_1) \subset \bigcup_{i\in Z} I_r(x_i)$, we have

$$\int_{\alpha+r_1}^{\beta-r_1} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2} dx \le \sum_{i\in Z} \int_{I_r(x_i)} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2} dx \, . \tag{253}$$

Recalling that $r < r_1 < \delta_0$, we see that (229) is satisfied with r replaced by $r/2$. Moreover, from (67) and the definitions of r_1 and r, we obtain

$$r\|h''\|_{L^\infty((\alpha,\beta))} \le c_0 r B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0) \le c_1^{1/2} \eta_0/8 < 1 \, .$$

Hence we can apply Theorem 21, with r replaced by $r/2$, to obtain (234) in each interval $I_r(x_i)$.

By (234),

$$\|\nabla u\|_{L^p(\Gamma_h^{x_i-r,x_i+r})} \le \frac{c_6}{r^{1-1/p}} \|\nabla u\|_{L^2(\Omega_{h,2r}^{x_i-2r,x_i+2r})} \, .$$

Combining this inequality with (253) gives

$$\int_{\alpha+r_1}^{\beta-r_1} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2} dx \le \frac{C}{r^{p-1}} \sum_{i\in Z} \|\nabla u\|^p_{L^2(\Omega_{h,2r}^{x_i-2r,x_i+2r})}$$

$$\leq \frac{C}{r^{p-1}} \left(\sum_{i \in Z} \|\nabla u\|^2_{L^2(\Omega^{x_i-2r, x_i+2r}_{h,2r})} \right)^{p/2} ,$$

where we used the fact that $p \geq 2$. Since each set $\Omega^{x_i-2r,x_i+2r}_{h,2r}$ intersects at most 7 sets of the form $\Omega^{x_i-2r,x_i+2r2}_{h,2r}$, from the previous inequality and from the inclusion $\Omega^{x_i-2r,x_i+2r}_{h,2r} \subset \Omega_h$ we obtain

$$\int_{\alpha+r_1}^{\beta-r_1} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2}dx \leq \frac{C}{r^{p-1}} \|\nabla u\|^p_{L^2(\Omega_h)} \leq \frac{C}{r_1^{p-1}} , \tag{254}$$

where in the last inequality we used (22) and the equality $r := \eta_0 r_1/4$. By (225) we have

$$\begin{aligned} \frac{1}{r_1} &= \max\Big\{ \frac{2c_0}{c_1^{1/2}} B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0), \frac{2}{\delta_0}, \frac{2}{\eta_1} \Big\} \\ &\leq \max\Big\{ \frac{2c_0}{c_1^{1/2}}, \frac{2}{\delta_0}, \frac{2}{\eta_1} \Big\} B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0) , \end{aligned}$$

where in the last inequality we used the fact that $1 \leq B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)$. Therefore, (254) yields

$$\int_{\alpha+r_1}^{\beta-r_1} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2}dx \ \leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^{p-1}.$$

Summing this inequality to (252) and to the corresponding inequality in $(\beta - r_1, \beta)$ gives

$$\int_{\alpha}^{\beta} |\nabla u(x,h(x))|^p \sqrt{1+(h'(x))^2}dx \ \leq CB_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^{p-1} ,$$

which concludes the proof. □

Theorem 24. *Under the hypotheses of Theorem 6, let* $1 < p \leq \frac{p_0}{4-2p_0}$. *Then we have* $h^{(\mathrm{iv})} \in L^{p_0/(4-2p_0)}((\alpha,\beta))$ *and there exists a constant* $c_9 = c_9(\eta_0,\eta_1,M,p) > 0$ *such that*

$$\|h^{(\mathrm{iv})}\|^p_{L^p((\alpha,\beta))} \leq c_9 B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)^q, \tag{255}$$

where $B_\tau(H,H^0,\alpha,\alpha^0,\beta,\beta^0)$ *is defined in* (61) *and*

$$q := \max\{p, 5p/2-1, 3p-2, 2p-1\} .$$

Proof. By (53),

$$-\gamma\Big(\frac{h'}{J}\Big)' + \nu_0\Big(\frac{h''}{J^5}\Big)'' + \frac{5}{2}\nu_0\Big(\frac{h'(h'')^2}{J^7}\Big)' + \overline{W} - \frac{1}{\tau}\bar{H} = m$$

in (α,β), where we recall that $\overline{W}(x) := W(Eu(x,h(x)))$ and

$$\overline{H}(x) := \int_{-\infty}^{x} (H(s) - H^0(s))\sqrt{1 + ((\check{h}^0)'(s))^2}\, ds\,.$$

Hence,

$$|h^{(\mathrm{iv})}|^p \le C|h''|^p + C|h'''|^p|h''|^p + C|h''|^{3p} + C\overline{W}^p + \frac{C}{\tau^p}|\overline{H}|^p + C|m|^p \tag{256}$$

for some constant $C > 0$ depending on L_0.

By (67) and (105) we have

$$\int_\alpha^\beta |h''|^p dx \le (\beta - \alpha)\|h''\|^p_{L^\infty(\alpha,\beta)} \le CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^p. \tag{257}$$

If $1 < p < 2$, by Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$,

$$\begin{aligned}\int_\alpha^\beta |h'''|^p|h''|^p dx &\le \Big(\int_\alpha^\beta |h''|^2 dx\Big)^{p/2}\Big(\int_\alpha^\beta |h'''|^{2p/(2-p)}dx\Big)^{(2-p)/2}\\ &\le M^{p/2}\|h'''\|^{(3p-2)/2}_{L^\infty((\alpha,\beta))}\|h'''\|^{(2-p)/2}_{L^1((\alpha,\beta))},\end{aligned}$$

where in the last inequality we used (64) and the fact that $\big(2p/(2-p) - 1\big)(2-p)/2 = (3p-2)/2$. Using (68) and (69), it follows from the previous inequality that

$$\int_\alpha^\beta |h'''|^p|h''|^p dx \le CB_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{5p/2-1}, \tag{258}$$

where we used the fact that $3p - 2 + (2-p)/2 = 5p/2 - 1$. If $p \ge 2$, by (64), (67), and (68) we have

$$\begin{aligned}\int_\alpha^\beta |h'''|^p|h''|^p dx &\le \|h'''\|^p_{L^\infty((\alpha,\beta))}\|h''\|^{p-2}_{L^\infty((\alpha,\beta))}\int_\alpha^\beta |h''|^2 dx\\ &\le C(B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0))^{3p-2}.\end{aligned} \tag{259}$$

On the other hand, by (64) and (67),

$$\int_\alpha^\beta |h''|^{3p}dx \le \|h''\|^{3p-2}_{L^\infty((\alpha,\beta))}\int_\alpha^\beta |h''|^2 dx \le C(B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0))^{3p-2}M\,. \tag{260}$$

By the previous theorem

$$\int_\alpha^\beta \overline{W}^p dx \le \int_\alpha^\beta (W(Eu(x,h(x))))^p\sqrt{1 + (h'(x))^2}dx$$

$$\leq C \int_{\alpha}^{\beta} |\nabla u(x, h(x))|^{2p} \sqrt{1 + (h'(x))^2} dx \leq C B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^{2p-1}.$$

Since $\frac{1}{\tau}|\overline{H}| \leq C B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)$ and $|m| \leq C B_\tau(H, H^0, \alpha, \alpha_0, \beta, \beta^0)$ by (61) and (112), respectively, combining these two inequalities with (256)– (260) yields (255). □

Remark 7. *Note that* $5p/2 - 1 \leq 2$ *for* $p \leq \frac{6}{5}$, $3p - 2 \leq 2$ *for* $p \leq \frac{4}{3}$, *and* $2p - 1 \leq 2$ *for* $p \leq 3/2$. *Hence, by taking* $1 < p \leq \min\{6/5, p_0/(4 - 2p_0)\}$ *we have that* $q \leq 2$. *Moreover, as in the proof of* (20), *we have that* $H - H^0 = 0$ *for* $x \notin (\min\{\alpha, \alpha^0\}, \max\{\beta, \beta^0\})$. *Hence, by Hölder's inequality*

$$\left(\int_{\mathbb{R}} |H - H^0|\, dx\right)^2 = \left(\int_{\min\{\alpha,\alpha^0\}}^{\max\{\beta,\beta^0\}} |H - H^0|\, dx\right)^2$$
$$\leq (\max\{\beta, \beta^0\} - \min\{\alpha, \alpha^0\}) \int_{\min\{\alpha,\alpha^0\}}^{\max\{\beta,\beta^0\}} |H - H^0|^2\, dx\,.$$

In turn, by (61) *for* $q \leq 2$,

$$B_\tau(H, H^0, \alpha, \alpha^0, \beta, \beta^0)^q \leq 4 + 4\frac{(\max\{\beta, \beta^0\} - \min\{\alpha, \alpha^0\})}{\tau^2} \int_{\mathbb{R}} |H - H^0|^2 dx \quad (261)$$
$$+ \frac{4}{\tau^2}|\alpha - \alpha^0|^2 + \frac{4}{\tau^2}|\beta - \beta^0|^2.$$

8 Discrete Time Approximation

For a given function $f = f(x, t)$ we set

$$\dot{f}(x, t) := \frac{\partial f}{\partial t}(x, t) \quad \text{and} \quad f'(x, t) := \frac{\partial f}{\partial x}(x, t)\,.$$

Fix $(\alpha_0, \beta_0, h_0, u_0) \in \mathcal{A}$ with $\operatorname{Lip} h_0 < L_0$ and define

$$H_0(x) := \int_{-\infty}^{x} \check{h}_0(\rho)\, d\rho\,, \quad (262)$$

where $\check{h}_0$ is the extension of h_0 by zero outside the interval (α_0, β_0). For every $k \in \mathbb{N}$ we set $\tau_k := \frac{1}{k}$ and $t_k^i := i\tau_k$, $i \in \mathbb{N} \cup \{0\}$. We define $(\alpha_k^i, \beta_k^i, h_k^i, u_k^i) \in \mathcal{A}$ inductively with respect to i as follows:

$$(\alpha_k^0, \beta_k^0, h_k^0, u_k^0) := (\alpha_0, \beta_0, h_0, u_0)\,, \quad (263)$$

and given $(\alpha_k^{i-1}, \beta_k^{i-1}, h_k^{i-1}, u_k^{i-1}) \in \mathcal{A}$ we let $(\alpha_k^i, \beta_k^i, h_k^i, u_k^i) \in \mathcal{A}$ be a minimizer of the functional

$$\mathcal{F}_k^{i-1}(\alpha, \beta, h, u) := \mathcal{S}(\alpha, \beta, h) + \mathcal{E}(\alpha, \beta, h, u) + \mathcal{T}_{\tau_k}(\alpha, \beta, h; \alpha_k^{i-1}, \beta_k^{i-1}, h_k^{i-1})\,, \quad (264)$$

whose existence is guaranteed by Theorem 2. We introduce the linear interpolations (in time) for α, β, h given by

$$\alpha_k(t) := \alpha_k^{i-1} + \frac{t - t_k^{i-1}}{\tau_k}(\alpha_k^i - \alpha_k^{i-1}) \,, \tag{265}$$

$$\beta_k(t) := \beta_k^{i-1} + \frac{t - t_k^{i-1}}{\tau_k}(\beta_k^i - \beta_k^{i-1}) \,, \tag{266}$$

$$h_k(t, x) := \check{h}_k^{i-1}(x) + \frac{t - t_k^{i-1}}{\tau_k}(\check{h}_k^i(x) - \check{h}_k^{i-1}(x)) \,, \tag{267}$$

for $t \in [t_k^{i-1}, t_k^i]$, $i \in \mathbb{N}$, and $x \in \mathbb{R}$, where $\check{h}_k^i$ is the extension of h_k^i by zero outside the interval $[\alpha_k^i, \beta_k^i]$.

We also introduce the piecewise constant interpolations (in time) for α, β, h given by

$$\hat{\alpha}_k(t) := \alpha_k^i \,, \quad \hat{\beta}_k(t) := \beta_k^i \,, \quad \hat{h}_k(t, x) := \check{h}_k^i(x) \tag{268}$$

for $t \in (t_k^{i-1}, t_k^i]$, $i \in \mathbb{N}$, and $x \in \mathbb{R}$. For $t = 0$ we set $\hat{\alpha}_k(0) := \alpha_k^0 = \alpha_0$, $\hat{\beta}_k(0) := \beta_k^0 = \beta_0$, and $\hat{h}_k(0, x) := h_k^0(x) = h_0(x)$. Observe that, since $(\alpha_k^i, \beta_k^i, h_k^i) \in \mathcal{A}_s$ for every i and k, we have $\hat{h}_k(t, \cdot) \in H^2((\hat{\alpha}_k(t), \hat{\beta}_k(t)))$ for every $t \geq 0$ and, by Lemma 1,

$$\hat{\beta}_k(t) - \hat{\alpha}_k(t) \geq \sqrt{\frac{2A_0}{L_0}}. \tag{269}$$

Note that since $\operatorname{Lip} \check{h}_k^{i-1} \leq L_0$ and $\operatorname{Lip} \check{h}_k^i \leq L_0$, we have $\operatorname{Lip} h_k(t, \cdot) \leq L_0$ and $\operatorname{Lip} \hat{h}_k(t, \cdot) \leq L_0$ for every $t \in [0, \infty)$. Moreover by (6),

$$\int_{\mathbb{R}} h_k(t, x) \, dx = \int_{\mathbb{R}} \hat{h}_k(t, x) \, dx = A_0 \tag{270}$$

for every $t \in [0, \infty)$ and all k.

Lemma 25. *There exists a constant $M_0 > 0$ such that*

$$\mathcal{S}(\alpha_k^i, \beta_k^i, h_k^i) + \mathcal{E}(\alpha_k^i, \beta_k^i, h_k^i, u_k^i) + \sum_{j=1}^{i} \mathcal{T}_{\tau_k}(\alpha_k^j, \beta_k^j, h_k^j; \alpha_k^{j-1}, \beta_k^{j-1}, h_k^{j-1}) \leq M_0$$

for every i and k.

Proof. Fix $i \in \mathbb{N}$ and let $1 \leq j \leq i$. Since $(\alpha_k^j, \beta_k^j, h_k^j, u_k^j) \in \mathcal{A}$ is a minimizer of the functional $\mathcal{F}_k^{j-1}$ defined in (264), we have

$$\begin{aligned} &\mathcal{S}(\alpha_k^j, \beta_k^j, h_k^j) + \mathcal{E}(\alpha_k^j, \beta_k^j, h_k^j, u_k^j) + \mathcal{T}_{\tau_k}(\alpha_k^j, \beta_k^j, h_k^j; \alpha_k^{j-1}, \beta_k^{j-1}, h_k^{j-1}) \\ &\quad \leq \mathcal{S}(\alpha_k^{j-1}, \beta_k^{j-1}, h_k^{j-1}) + \mathcal{E}(\alpha_k^{j-1}, \beta_k^{j-1}, h_k^{j-1}, u_k^{j-1}) \,. \end{aligned}$$

Summing both sides of this inequality over $j = 1, \ldots, i$, we obtain

$$\mathcal{S}(\alpha_k^i, \beta_k^i, h_k^i) + \mathcal{E}(\alpha_k^i, \beta_k^i, h_k^i, u_k^i) + \sum_{j=1}^{i} \mathcal{T}_{\tau_k}(\alpha_k^j, \beta_k^j, h_k^j; \alpha_k^{j-1}, \beta_k^{j-1}, h_k^{j-1})$$
$$\leq \mathcal{S}(\alpha_0, \beta_0, h_0) + \mathcal{E}(\alpha_0, \beta_0, h_0, u_0) =: M_0\,,$$

and this concludes the proof. □

Proposition 26. *There exists a constant $M_1 > 0$ such that*

$$\int_0^\infty (\dot\alpha_k(t))^2 dt \leq M_1 \quad \text{and} \quad \int_0^\infty (\dot\beta_k(t))^2 dt \leq M_1 \tag{271}$$

for every k. In particular,

$$|\alpha_k(t_2) - \alpha_k(t_1)| \leq M_1^{1/2} |t_2 - t_1|^{1/2}, \quad |\beta_k(t_2) - \beta_k(t_1)| \leq M_1^{1/2} |t_2 - t_1|^{1/2} \tag{272}$$

for every t_1,$t_2 \in [0, \infty)$, and

$$\alpha^0 - (tM_1)^{1/2} \leq \alpha_k(t) \leq \beta_k(t) \leq \beta^0 + (tM_1)^{1/2} \tag{273}$$

for every $t \in [0, \infty)$.

Proof. By (14), (18), and Lemma 25 we obtain

$$\frac{1}{\tau_k} \sum_{j=1}^{\infty} (\alpha_k^j - \alpha_k^{j-1})^2 \leq \frac{2M_0}{\sigma_0} =: M_1\,.$$

By (265) the previous inequality can be written as

$$\int_0^\infty (\dot\alpha_k(t))^2 dt = \sum_{j=1}^{\infty} \int_{t_k^{j-1}}^{t_k^j} (\dot\alpha_k(t))^2 dt \leq M_1\,.$$

The first inequality (272) now follows from the fundamental theorem of calculus and Hölder's inequality. Since $\alpha_k(0) = \alpha^0$ the first inequality (273) is a direct consequence of (272). Similar estimates hold for β_k. □

Define

$$H_k^i(x) := \int_{-\infty}^{x} \check h_k^i(\rho)\, d\rho\,, \quad H_k(t, x) := \int_{-\infty}^{x} h_k(t, \rho)\, d\rho\,. \tag{274}$$

$$\hat H_k(t, x) := \int_{-\infty}^{x} \hat h_k(t, \rho)\, d\rho\,. \tag{275}$$

Observe that by (267),

$$H_k(t,x) = H_k^{i-1}(x) + \frac{t - t_k^{i-1}}{\tau_k}(H_k^i(x) - H_k^{i-1}(x)) \quad \text{for } t \in [t_k^{i-1}, t_k^i]\,, \tag{276}$$

$$\hat{H}_k(t,x) = H_k^i(x) \quad \text{for } (t_k^{i-1}, t_k^i]\,, \tag{277}$$

for $i \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proposition 27. *For every k,*

$$\int_0^\infty \|\dot{H}_k(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \le 2M_0\,, \tag{278}$$

where M_0 is the constant in Lemma 25. In particular,

$$\|H_k(t_2,\cdot) - H_k(t_1,\cdot)\|_{L^2(\mathbb{R})} \le (2M_0)^{1/2}|t_2 - t_1|^{1/2} \tag{279}$$

$$\|H_k(t,\cdot)\|_{L^2(\mathbb{R})} \le (2M_0)^{1/2} t^{1/2} + \|H^0\|_{L^2(\mathbb{R})} \tag{280}$$

for every $t, t_1, t_2 \in [0,\infty)$ and for every k, where H^0 is defined in (262).

Proof. By (14), (18), and Lemma 25,

$$\frac{1}{2\tau_k}\sum_{j=1}^\infty \int_{\mathbb{R}} (H_k^j(x) - H_k^{j-1}(x))^2 dx \le M_0\,.$$

Using (276) we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} (\dot{H}_k(t,x))^2 dx dt &= \sum_{j=1}^\infty \int_{t_k^{j-1}}^{t_k^j} \int_{\mathbb{R}} (\dot{H}_k(t,x))^2 dx dt \\ &= \frac{1}{\tau_k}\sum_{j=1}^\infty \int_{\mathbb{R}} (H_k^j(x) - H_k^{j-1}(x))^2 dx \le 2M_0\,,\end{aligned}$$

which gives (278). The estimate (279) follows from the fundamental theorem of calculus and Hölder's inequality. From (262) and (279) we obtain (280). □

Proposition 28. *There exists a constant $M_2 > 0$ such that*

$$\|h_k(t,\cdot)\|_{L^2(\mathbb{R})} \le M_2(t^{1/2} + (\beta_0 - \alpha_0))\,, \tag{281}$$

$$\|h_k(t_2,\cdot) - h_k(t_1,\cdot)\|_{L^2(\mathbb{R})} \le M_2|t_2 - t_1|^{3/10}, \tag{282}$$

for every $t, t_1, t_2 \in [0,\infty)$ and for every k.

Proof. By (267) and (273) we have that $h_k(t,x) = 0$ for every $t \in [0,\infty)$ and for $x \notin [\alpha^0 - (tM_1)^{1/2}, \beta^0 + (tM_1)^{1/2}]$. Since $\operatorname{Lip} h_k(t,\cdot) \le L_0$ for every $t \in [0,\infty)$, we obtain (281). To prove (282), fix $t_1, t_2 \in [0,\infty)$ and k and apply Theorem 7.41 in [44] to the function $v(x) := H_k(t_2, x) - H_k(t_1, x)$, $x \in \mathbb{R}$, to get

$$\begin{aligned}
\|h_k(t_2,\cdot) - h_k(t_1,\cdot)\|_{L^2(\mathbb{R})} &= \|H_k'(t_2,\cdot) - H_k'(t_1,\cdot)\|_{L^2(\mathbb{R})} \\
&\le C_{GN}\|H_k(t_2,\cdot) - H_k(t_1,\cdot)\|_{L^2(\mathbb{R})}^{3/5}\|H_k''(t_2,\cdot) - H_k''(t_1,\cdot)\|_{L^\infty(\mathbb{R})}^{2/5} \\
&\le C_{GN}(2M_0)^{3/10}|t_2 - t_1|^{3/10}\|h_k'(t_2,\cdot) - h_k'(t_1,\cdot)\|_{L^\infty(\mathbb{R})}^{2/5} \\
&\le C_{GN}(2M_0)^{3/10}(2L_0)^{2/5}|t_2 - t_1|^{3/10}
\end{aligned}$$

where in the last inequalities we used (279) and the fact that $\operatorname{Lip} h_k(t,\cdot) \le L_0$ for every $t \in [0,\infty)$. □

Proposition 29. *There exists a constant $M_3 > 1$ such that for every i, k we have*

$$\int_{\alpha_k^i}^{\beta_k^i} |(h_k^i)''(x)|^2 dx \le M_3 \,. \tag{283}$$

In particular,

$$|(h_k^i)'(x_2) - (h_k^i)'(x_1)| \le M_3^{1/2}(x_2 - x_1)^{1/2} \tag{284}$$

for all $\alpha_k^i \le x_1 \le x_2 \le \beta_k^i$.

Proof. It follows from (13), (14), Lemma 25, and the fact that $\operatorname{Lip} h_k^i \le L_0$ that

$$\int_{\alpha_k^i}^{\beta_k^i} ((h_k^i)''(x))^2 dx \le (1 + L_0^2)^{5/2}\int_{\alpha_k^i}^{\beta_k^i} \frac{((h_k^i)''(x))^2}{(1 + ((h_k^i)'(x))^2)^{5/2}} dx \le 2\nu_0^{-1}(1 + L_0^2)^{5/2} M_0 \,.$$

By the fundamental theorem of calculus and Hölder's inequality this implies (284). □

9 Convergence to the Evolution Problem

Throughout this section we assume that

$$(\alpha_0, \beta_0, h_0) \in \mathcal{A}_s \,, \tag{285}$$

$$h_0(x) > 0 \quad \text{for every } x \in (\alpha_0, \beta_0) \,, \tag{286}$$

$$h_0'(\alpha_0) > 0 \quad \text{and} \quad h_0'(\beta_0) < 0 \,, \tag{287}$$

$$\operatorname{Lip} h_0 < L_0 \,. \tag{288}$$

Proposition 30. *Let α_k, β_k, $\hat\alpha_k$, and $\hat\beta_k$ be defined as in (265), (266), and (268). Then there exist functions α, $\beta : (0,\infty) \to \mathbb{R}$ such that $\alpha, \beta \in H^1((0,T))$ for every $T > 0$ and, up to subsequences (not relabeled),*

$$\alpha_k \rightharpoonup \alpha \quad \textit{and} \quad \beta_k \rightharpoonup \beta \quad \textit{weakly in } H^1((0,T)) \,, \tag{289}$$

$$\alpha_k \to \alpha \quad \textit{and} \quad \beta_k \to \beta \quad \textit{uniformly in } [0,T] \,, \tag{290}$$

$$\hat{\alpha}_k \to \alpha \quad \textit{and} \quad \hat{\beta}_k \to \beta \quad \textit{uniformly in } [0,T]\,, \tag{291}$$

for every $T > 0$. *Moreover,*

$$\alpha(0) = \alpha_0\,, \quad \beta(0) = \beta_0\,, \quad \textit{and} \quad \beta(t) - \alpha(t) \geq \sqrt{\frac{2A_0}{L_0}} \tag{292}$$

for every $t \geq 0$.

Proof. Proposition 26, together with the initial conditions $\alpha_k(0) = \alpha_0$ and $\beta_k(0) = \beta_0$, implies (289), which in turn yields (290) in view of the compact embedding. By (265) and (268), for every k and $t \in [0,T]$, there exists a $\hat{t} \in [0,T]$ with $t \leq \hat{t} < t + \tau_k$ such that $\hat{\alpha}_k(t) = \alpha_k(\hat{t})$. By (272), $|\hat{\alpha}_k(t) - \alpha_k(t)| \leq M_1^{1/2}\tau_k$. This implies that $\hat{\alpha}_k - \alpha_k \to 0$ uniformly. Similarly, $\hat{\beta}_k - \beta_k \to 0$ uniformly. Hence, (291) follows from (290). The equalities in (292) follow from (290), since $\alpha_k(0) = \alpha_0$ and $\beta_k(0) = \beta_0$. The inequality in (292) follows from (269) and (291). □

Proposition 31. *Let* h_k *be defined as in* (267). *Then there exist a subsequence (not relabeled) and a nonnegative function* $h_* \in C^{0,3/10}([0,\infty); L^2(\mathbb{R}))$ *such that for every* $t \in [0,\infty)$,

$$\operatorname{Lip} h_*(t,\cdot) \leq L_0 \quad \textit{and} \quad h_*(t,x) = 0 \quad \textit{for } x \notin (\alpha(t),\beta(t)) \tag{293}$$

$$h_k(t,\cdot) \to h_*(t,\cdot) \quad \textit{uniformly in } \mathbb{R}\,, \tag{294}$$

$$h_k'(t,\cdot) \overset{*}{\rightharpoonup} h_*'(t,\cdot) \quad \textit{weakly star in } L^\infty(\mathbb{R})\,. \tag{295}$$

Moreover, if we let $h(t,\cdot)$ *be the restriction of* $h^*(t,\cdot)$ *to* $(\alpha(t),\beta(t))$, *then* $h(t,\cdot) \in H_0^1((\alpha(t),\beta(t))) \cap H^2((\alpha(t),\beta(t)))$ *and*

$$\int_{\alpha(t)}^{\beta(t)} h(t,x)\,dx = A_0\,, \tag{296}$$

$$\int_{\alpha(t)}^{\beta(t)} |h''(t,x)|^2 dx \leq M_3\,, \tag{297}$$

$$|h'(t,x_2) - h'(t,x_1)| \leq M_3^{1/2}(x_2 - x_1)^{1/2} \quad \textit{for all } x_1, x_2 \textit{ with } \alpha(t) \leq x_1 \leq x_2 \leq \beta(t) \tag{298}$$

for every $t \in [0,\infty)$.

Remark 8. *It follows from the previous proposition that* $(\alpha(t),\beta(t),h(t,\cdot)) \in \mathcal{A}_s$ *for every* $t \in [0,\infty)$.

Proof of Proposition 31. By Proposition 28 and the Ascoli–Arzelà theorem there exist a subsequence (not relabeled) and a nonnegative function $h_* \in C^{0,3/18}([0,\infty); L^2(\mathbb{R}))$ such that for every $t \in [0,\infty)$,

$$h_k(t,\cdot) \rightharpoonup h_*(t,\cdot) \quad \text{weakly in } L^2(\mathbb{R})\,. \tag{299}$$

Since $\operatorname{Lip} h_k(t,\cdot) \le L_0$ for every $t \in [0,\infty)$ and every k, we obtain that $\operatorname{Lip} h_*(t,\cdot) \le L_0$ and that (295) holds.

Fix $t \in [0,\infty)$ and k and find i such $t_k^{i-1} \le t \le t_k^i$. Since $\alpha_k(t_k^{i-1}) = \alpha_k^{i-1}$ and $\alpha_k(t_k^i) = \alpha_k^i$, (see (265)), by (272) we have

$$\alpha_k(t) - (M_1\tau_k)^{1/2} \le \min\{\alpha_k^{i-1}, \alpha_k^i\} \le \max\{\alpha_k^{i-1}, \alpha_k^i\} \le \alpha_k(t) + (M_1\tau_k)^{1/2}, \quad (300)$$

where we used the fact that $t_k^i - t_k^{i-1} = \tau_k$. Similarly we can show that

$$\beta_k(t) - (M_1\tau_k)^{1/2} \le \min\{\beta_k^{i-1}, \beta_k^i\} \le \max\{\beta_k^{i-1}, \beta_k^i\} \le \beta_k(t) + (M_1\tau_k)^{1/2}. \quad (301)$$

Therefore by (267) we have

$$h_k(t,x) = 0 \quad \text{for } x \notin (\alpha_k(t) - (M_1\tau_k)^{1/2}, \beta_k(t) + (M_1\tau_k)^{1/2}). \quad (302)$$

Hence, by (290) and (299) we obtain that $h_*(t,x) = 0$ for $x \notin (\alpha(t), \beta(t))$. In turn, since $\operatorname{Lip} h_k(t,\cdot) \le L_0$, we can apply the Ascoli–Arzelà theorem in the x variable and we obtain (294). Properties (270), (290), (294), and (302) imply (296).

To prove (297) observe that by (267), (300), and (301), for every $x \in (\alpha_k(t) + (M_1\tau_k)^{1/2}, \beta_k(t) - (M_1\tau_k)^{1/2})$ we have

$$h_k(t,x) = h_k^{i-1}(x) + \frac{t - t_k^{i-1}}{\tau_k}(h_k^i(x) - h_k^{i-1}(x)).$$

Fix $a < b$ such that $\alpha(t) < a < b < \beta(t)$. By (290) we have that $\alpha_k(t) + (M_1\tau_k)^{1/2} < a < b < \beta_k(t) - (M_1\tau_k)^{1/2}$ for all k sufficiently large, and so $h_k''(t,\cdot) \in L^2((a,b))$ and by (283), we have that

$$\int_b^a |h_k''(t,x)|^2 dx \le \frac{t - t_k^{i-1}}{\tau_k} \int_b^a |(h_k^i)''(x)|^2 dx + \frac{t_k^i - t}{\tau_k} \int_b^a |(h_k^{i-1})''(x)|^2 dx \le M_3.$$

It follows from (294) that $h''(t,\cdot) \in L^2((a,b))$ and

$$\int_a^b |h''(t,x)|^2 dx \le M_3. \quad (303)$$

Taking the limit as $a \to \alpha(t)$ and $b \to \beta(t)$ we conclude that $h(t,\cdot) \in H^2((\alpha(t),\beta(t)))$ and that (297) holds. In turn, by the fundamental theorem of calculus and Hölder's inequality, we obtain (298). □

Proposition 32. *Let $\{h_k\}_k$ and h_* be the subsequence and the function given in Proposition 31, and let $\{H_k\}_k$ be defined by (274). Then up to a further subsequence (not relabeled),*

$$H_k \rightharpoonup H \quad \textit{weakly in } H^1((0,T); L^2(\mathbb{R})) \textit{ for every } T > 0, \quad (304)$$

with

$$H(t,x) := \int_{-\infty}^{x} h_*(t,\rho)\, d\rho\,. \tag{305}$$

Proof. By (278) and (280) there exist a subsequence (not relabeled) and a function $H\colon (0,\infty) \to L^2(\mathbb{R})$ such that $H \in H^1((0,T);L^2(\mathbb{R}))$ for every $T>0$ and (304) holds. It suffices to prove that H equals the right-hand side of (305).

Let $\varphi \in C_c^\infty((0,\infty))$ and $\psi \in C_c^\infty(\mathbb{R})$. Then by (304),

$$\int_0^\infty \int_{\mathbb{R}} \varphi(t)\psi(x)H_k(t,x)\, dxdt \to \int_0^\infty \int_{\mathbb{R}} \varphi(t)\psi(x)H(t,x)\, dxdt$$

as $k\to\infty$. On the other hand,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \varphi(t)\psi(x)H_k(t,x)\, dxdt &= \int_0^\infty \int_{\mathbb{R}} \int_{-\infty}^{x} \varphi(t)\psi(x)h_k(t,\rho)\, d\rho dxdt \\ &\to \int_0^\infty \int_{\mathbb{R}} \int_{-\infty}^{x} \varphi(t)\psi(x)h(t,\rho)\, d\rho dxdt\end{aligned}$$

as $k\to\infty$, where we used the Lebesgue dominated convergence and (281), (290), (294), and (302). Given the arbitrariness of φ and ψ, we obtain (305). □

Next we study the convergence of the piecewise constant interpolations $\hat{H}_k$.

Proposition 33. *Let $\{H_k\}_k$ and H be the subsequence and the function given in Proposition 32, and let $\{\hat{H}_k\}_k$ be defined by (275). Then*

$$\hat{H}_k(t,\cdot) \rightharpoonup H(t,\cdot) \quad \textit{weakly in } L^2(\mathbb{R}) \textit{ for every } t\geq 0\,. \tag{306}$$

Proof. Let $\varphi\in L^2(\mathbb{R})$ and define

$$\psi_k(t) := \int_{\mathbb{R}} (H_k(t,x) - H(t,x))\varphi(x)\, dx\,.$$

By Proposition 32, for every $T>0$ the function $\psi_k \in H^1((0,T))$ and $\psi_k \rightharpoonup 0$ weakly in $H^1((0,T))$. This implies that $\psi_k(t)\to 0$ for every $t\in(0,T)$. By the arbitrariness of φ and T, we deduce that

$$H_k(t,\cdot) \rightharpoonup H(t,\cdot) \quad \text{weakly in } L^2(\mathbb{R}) \text{ for every } t\geq 0\,. \tag{307}$$

By (276) and (279), for every $t\in(t_k^{i-1},t_k^i]$,

$$\|\hat{H}_k(t,\cdot) - H_k(t,\cdot)\|_{L^2(\mathbb{R})} = \|H_k(t_k^i,\cdot) - H_k(t,\cdot)\|_{L^2(\mathbb{R})} \leq (2M_0)^{1/2}\tau_k^{1/2} \to 0$$

as $k\to\infty$. Together with (307), this implies (306). □

Corollary 34. *Let h_* be as in Proposition 31 and let $\{H_k\}_k$ and H be the subsequence and the function given in Proposition 32. Fix $t \geq 0$. Then the corresponding subsequence $\{\hat{h}_k\}_k$ satisfies*

$$\hat{h}_k(t, \cdot) \to h_*(t, \cdot) \quad \textit{uniformly in } \mathbb{R}\,, \tag{308}$$

$$\hat{h}'_k(t, \cdot) \overset{*}{\rightharpoonup} h'_*(t, \cdot) \quad \textit{weakly star in } L^\infty(\mathbb{R})\,. \tag{309}$$

Proof. Since $\hat{h}_k(t, \cdot) = \hat{H}'_k(t, \cdot) \rightharpoonup H'(t, \cdot) = h_*(t, \cdot)$ weakly in $H^{-1}(\mathbb{R})$ in view of the previous proposition, and $\{\hat{h}_k(t, \cdot)\}_k$ is bounded in $W^{1,\infty}(\mathbb{R})$, the conclusion follows. □

In the remaining of this section we always assume that the sequences $\{\alpha_k\}_k$, $\{\beta_k\}_k$, $\{h_k\}_k$, $\{H_k\}_k$ satisfy (289), (290), (294), (295), and (304), (306), (308), and (309), and that $h(t, \cdot)$ is the restriction of $h_*(t, \cdot)$ to $(\alpha(t), \beta(t))$.

Lemma 35. *Let $\{t_k\}_k$ be a sequence of nonnegative numbers converging to some $t_0 \geq 0$. Then*

$$h_k(t_k, \cdot) \to h_*(t_0, \cdot)\,, \quad \hat{h}_k(t_k, \cdot) \to h_*(t_0, \cdot) \quad \textit{uniformly in } \mathbb{R}\,. \tag{310}$$

Moreover, if $\alpha(t_0) < a < b < \beta(t_0)$, then

$$h'_k(t_k, \cdot) \to h'(t_0, \cdot)\,, \quad \hat{h}'_k(t_k, \cdot) \to h'(t_0, \cdot) \quad \textit{uniformly on } [a, b]\,. \tag{311}$$

Finally, if $\{x_k\}_k$ is a sequence in $\mathbb{R}$ converging to some $x_0 \in \mathbb{R}$ such that $\hat{\alpha}_k(t_k) \leq x_k \leq \hat{\beta}_k(t_k)$, then $\alpha(t_0) \leq x_0 \leq \beta(t_0)$ and

$$\hat{h}'_k(t_k, x_k) \to h'(t_0, x_0) \tag{312}$$

Proof. Since $\operatorname{Lip} h_k(t_k, \cdot) \leq L_0$ for every k, by (302) we can apply the Ascoli–Arzelà theorem to obtain that up to a subsequence (not relabeled)

$$h_k(t_k, \cdot) \to g(\cdot) \quad \text{uniformly in } \mathbb{R}$$

for some Lipschitz continuous function g. On the other hand, by (279),

$$\|H_k(t_k, \cdot) - H_k(t_0, \cdot)\|_{L^2(\mathbb{R})} \leq (2M_0)^{1/2} |t_k - t_0|^{1/2}$$

and in view of (304) we have that

$$H_k(t_0, \cdot) \rightharpoonup H(t_0, \cdot) \quad \text{weakly in } L^2(\mathbb{R})\,.$$

The last two properties imply that

$$H_k(t_k, \cdot) \rightharpoonup H(t_0, \cdot) \quad \text{weakly in } L^2(\mathbb{R})\,.$$

Since $H_k'(t_k, \cdot) = h_k(t_k, \cdot)$, it follows that $g(\cdot) = H'(t_0, \cdot) = h_*(t_0, \cdot)$. Since the limit does not depend on the subsequence, this concludes the proof of (310).

By (283),

$$\int_a^b |h_k''(t_k, x)|^2 dx \le M_3$$

for all k sufficiently large. By Hölder's inequality this bound implies that $h_k'(t_k, \cdot)$ are Hölder continuous of exponent $\frac{1}{2}$ on $[a, b]$ uniformly with respect to k. By the Ascoli–Arzelà theorem again and by (310) we obtain (311). In particular, if $\alpha(t_0) < x < \beta(t_0)$, then

$$h_k'(t_k, x) \to h'(t_0, x) \,. \tag{313}$$

If $\alpha(t_0) < x_0 < \beta(t_0)$, then we can choose a and b with $\alpha(t_0) < a < x_0 < b < \beta(t_0)$ and (312) follows from (311). It remains to consider the cases $x_0 = \alpha(t_0)$ or $x_0 = \beta(t_0)$. We consider only the case $x_0 = \alpha(t_0)$. Fix $0 < \varepsilon < \beta(t_0) - \alpha(t_0)$ and observe that $\hat{\alpha}_k(t_k) < x_k + \varepsilon < \hat{\beta}_k(t_k)$ for all k sufficiently large. By what we just proved

$$\hat{h}_k'(t_k, x_k + \varepsilon) \to h'(t_0, x_0 + \varepsilon) \,. \tag{314}$$

By the fundamental theorem of calculus, Hölder's inequality, and (283),

$$|\hat{h}_k'(t_k, x_k + \varepsilon) - \hat{h}_k'(t_k, x_k)| \le \varepsilon^{1/2} \left(\int_{x_k}^{x_k+\varepsilon} |\hat{h}_k''(t_k, x)|^2 \right)^{1/2} \le \varepsilon^{1/2} M_3^{1/2} \,.$$

Similarly,

$$|h'(t_0, x_0 + \varepsilon) - h'(t_0, x_0)| \le \varepsilon^{1/2} M_3^{1/2} \,.$$

Therefore, by (314),

$$\limsup_{k\to\infty} |\hat{h}_k'(t_k, x_k) - h'(t_0, x_0)| \le 2\varepsilon^{1/2} M_3^{1/2}.$$

Letting $\varepsilon \to 0^+$, we obtain (312). □

In what follows $C_b(\mathbb{R})$ is the space of bounded continuous functions in $\mathbb{R}$ with the supremum norm.

Lemma 36. *The function $t \mapsto h_*(t, \cdot)$ from $[0, \infty)$ into $C_b(\mathbb{R})$ is continuous.*

Proof. Let $\{t_n\}_n$ be a sequence in $[0, \infty)$ converging to t_0. Since $h_* \in C^{3/10}([0, \infty); L^2(\mathbb{R}))$ we have $h_*(t_n, \cdot) \to h_*(t_0, \cdot)$ in $L^2(\mathbb{R})$. On the other hand, $\operatorname{Lip} h_*(t_n, \cdot) \le L_0$ for every n and by (293) the supports of the functions $h_*(t_n, \cdot)$ are contained in a compact set independent of n. Hence, the Ascoli–Arzelà theorem implies $h_*(t_n, \cdot) \to h_*(t_0, \cdot)$ in $C_b(\mathbb{R})$. □

Lemma 37. *Fix a bounded interval $I \subset [0, \infty)$ and let $a < b$ be such that*

$$\alpha(t) \le a < b \le \beta(t)$$

for all $t \in I$. Then the function $t \mapsto h'(t, \cdot)$ from I into $C^0([a, b])$ is continuous

Proof. Let $\{t_n\}_n$ be a sequence in I converging to $t_0 \in I$. By Lemma 36, $h(t_n, \cdot) \to h(t_0, \cdot)$ in $C^0([a, b])$. On the other hand, by (297), the sequence $\{h(t_n, \cdot)\}_n$ is bounded in $H^2((a, b))$ and it converges weakly to $h(t_0, \cdot)$ in $H^2((a, b))$ and so $h'(t_n, \cdot) \to h'(t_0, \cdot)$ in $C^0([a, b])$. □

Since $h(t, \cdot)$ is defined only in $[\alpha(t), \beta(t)]$, its space derivatives at the endpoints are one-sided.

Lemma 38. *The functions $t \mapsto h'(t, \alpha(t))$ and $t \mapsto h'(t, \beta(t))$ are continuous on $[0, \infty)$.*

Proof. It is enough to give the proof for $t \mapsto h'(t, \alpha(t))$. Let $\{t_n\}_n$ be a sequence in $[0, \infty)$ converging to t_0 and let $x \in (\alpha(t_0), \beta(t_0))$. By continuity of α and β there exist an open bounded interval $I \subset [0, \infty)$ and $a < b$ such that

$$\alpha(t) \le a < x < b \le \beta(t)$$

for all $t \in I$. By Lemma 37,

$$h'(t_n, x) \to h'(t_0, x)$$

as $n \to \infty$. By (298),

$$\begin{aligned} |h'(t_n, \alpha(t_n)) - h'(t_0, \alpha(t_0))| &\le |h'(t_n, \alpha(t_n)) - h'(t_n, x)| + |h'(t_n, x) - h'(t_0, x)| \\ &\quad + |h'(t_0, x) - h'(t_0, \alpha(t_0))| \\ &\le M_3^{1/2} |\alpha(t_n) - x|^{1/2} + |h'(t_n, x) - h'(t_0, x)| + M_3^{1/2} |\alpha(t_0) - x|^{1/2}. \end{aligned}$$

Letting $n \to \infty$ gives

$$\limsup_{n \to \infty} |h'(t_n, \alpha(t_n)) - h'(t_0, \alpha(t_0))| \le 2M_3^{1/2} |\alpha(t_0) - x|^{1/2}.$$

Taking the limit as $x \to \alpha(t_0)^+$ we conclude the proof. □

Theorem 39. *Under the assumptions (285)–(288), there exists $T_0 > 0$ such that for all $t \in [0, T_0]$,*

$$h(t, x) > 0 \quad \textit{for all } x \in (\alpha(t), \beta(t))\,, \tag{315}$$

$$h'(t, \alpha(t)) > 0 \quad \textit{and} \quad h'(t, \beta(t)) < 0\,. \tag{316}$$

Proof. Fix $0 < \varepsilon < \min\{h_0'(\alpha_0), -h_0'(\beta_0)\}$. By Lemma 38 there exists $T_1 > 0$ such that

$$h'(t, \alpha(t)) \ge \varepsilon \quad \text{and} \quad h'(t, \beta(t)) \le -\varepsilon$$

for all $t \in [0, T_1]$. Fix $\delta > 0$ such that $M_3^{1/2} \delta^{1/2} < \varepsilon$. By (298),

$$h'(t, x) \ge h'(t, \alpha(t)) - M_3^{1/2} |x - \alpha(t)|^{1/2} \ge \varepsilon - M_3^{1/2} \delta^{1/2} > 0$$

for all $\alpha(t) \le x \le \alpha(t)+\delta$. Since $h(t,\alpha(t)) = 0$ by the previous inequality we have that

$$h(t,x) > 0 \quad \text{for all } \alpha(t){<}x \le \alpha(t)+\delta \text{ and for all } 0 \le t \le T_1\,. \tag{317}$$

Moreover,

$$h(t,x) > 0 \quad \text{for all } \beta(t)-\delta \le x{<}\beta(t) \text{ and for all } 0 \le t \le T_1\,. \tag{318}$$

Fix $a < b$ such that $\alpha_0 < a < \alpha_0+\delta$ and $\beta_0-\delta < b < \beta_0$. Then there exists $0 < T_2 \le T_1$ such that

$$\alpha(t) < a < \alpha(t)+\delta \quad \text{and} \quad \beta(t)-\delta < b < \beta(t) \quad \text{for every } 0 \le t \le T_2\,. \tag{319}$$

Let $\eta := \min_{[a,b]} h_0 > 0$. By Lemma 36 there exists $0 < T_3 \le T_2$ such that

$$h(t,x) \ge \frac{\eta}{2} \quad \text{for all } a \le x \le b \text{ and for all } 0 \le t \le T_3\,. \tag{320}$$

Combining (317)–(320), we obtain $h(t,x) > 0$ for all $x \in (\alpha(t),\beta(t))$ and for all $0 \le t \le T_3$. □

Proposition 40. *Under the assumptions of Theorem 39, there exist $k_0 \in \mathbb{N}$ and $0 < \eta_0 < 1$ such that*

$$(h_k^i)'(\alpha_k^i) > 2\eta_0 \quad \textit{and} \quad (h_k^i)'(\beta_k^i) < -2\eta_0 \tag{321}$$

for all $k \ge k_0$, and all $0 \le i \le kT_0$.

Proof. Since the function $t \mapsto h'(t,\alpha(t))$ is continuous by Lemma 38, Theorem 39 implies that there exists $0 < \eta_0 < 1$ such that

$$h'(t,\alpha(t)) > 2\eta_0 \quad \text{for every } t \in [0,T_0]\,. \tag{322}$$

We claim that there exists $k_0 \in \mathbb{N}$ such that

$$(h_k^i)'(\alpha_k^i) > 2\eta_0 \quad \text{for all } k \ge k_0 \text{ and } 0 \le i \le kT_0\,. \tag{323}$$

If not, then for every $n \in \mathbb{N}$ there exist $k_n \ge n$ and $0 \le i_n \le k_n T_0$ such that

$$(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) \le 2\eta_0\,. \tag{324}$$

Define $t_n := t_{k_n}^{i_n} = i_n/k_n$. Since $0 \le t_n \le T_0$, up to a subsequence, $t_n \to t_0$ for some $t_0 \in [0,T_0]$, and by (290),

$$\alpha_{k_n}^{i_n} = \alpha_{k_n}(t_n) \to \alpha(t_0)\,. \tag{325}$$

By (267), we have $h_{k_n}^{i_n}(x) = h_{k_n}(t_n, x)$ for every $x \in [\alpha_{k_n}^{i_n}, \beta_{k_n}^{i_n}]$, and so $(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) = h'_{k_n}(t_n, \alpha_{k_n}^{i_n})$, where $h'_{k_n}(t_n, \alpha_{k_n}^{i_n})$ is the right derivative of $h'_{k_n}(t_n, \cdot)$ at $\alpha_{k_n}^{i_n}$. Fix $\alpha(t_0) < x < \beta(t_0)$ and let $a, b \in \mathbb{R}$ be such that $\alpha(t_0) < a < x < b < \beta(t_0)$. By (311),

$$h'_{k_n}(t_n, x) \to h'(t_0, x)\,. \tag{326}$$

By (284) and (298),

$$\begin{aligned}|(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) - h'(t_0, \alpha(t_0))| &\leq |(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) - (h_{k_n}^{i_n})'(x)| \\ &\quad + |h'_{k_n}(t_n, x) - h'(t_0, x)| + |h'(t_0, x) - h'(t_0, \alpha(t_0))| \\ &\leq M_3^{1/2}|\alpha_{k_n}(t_n) - x|^{1/2} + |h'_{k_n}(t_n, x) - h'(t_0, x)| + M_3^{1/2}|\alpha(t_0) - x|^{1/2}.\end{aligned}$$

Letting $n \to \infty$ and using (325) and (326), we get

$$\limsup_{n\to\infty} |(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) - h'(t_0, \alpha(t_0))| \leq 2M_3^{1/2}|\alpha(t_0) - x|^{1/2}.$$

Taking the limit as $x \to \alpha(t_0)^+$ we obtain that

$$(h_{k_n}^{i_n})'(\alpha_{k_n}^{i_n}) \to h'(t_0, \alpha(t_0))\,,$$

contradicting (322) and (324). This proves the claim. A similar argument holds for β_k^i and so (321) is satisfied. □

Proposition 41. *Under the assumptions of Theorem 39, let η_0 be as in Proposition 40, and let*

$$0 < \delta < \frac{1}{2} \min_{t\in[0,T_0]} (\beta(t) - \alpha(t))\,. \tag{327}$$

Then there exist $k_1 \geq k_0$ and $0 < \eta_1 < 1$ such that

$$h_k^i(x) \geq 2\eta_1 \quad \text{for all } x \in [\alpha_k^i + \delta, \beta_k^i - \delta]\,, \tag{328}$$

$$h_k^i(x) > 0 \quad \text{for all } x \in (\alpha_k^i, \beta_k^i)\,, \tag{329}$$

for all $k \geq k_1$ and $0 \leq i \leq kT_0$.

Proof. To prove (328), we argue by contradiction and assume that every $n \in \mathbb{N}$ there exist $k_n \geq n$, $0 \leq i_n \leq k_n T_0$, and $x_n \in [\alpha_{k_n}^{i_n} + \delta, \beta_{k_n}^{i_n} - \delta]$ such that $h_{k_n}^{i_n}(x_n) \leq 1/n$. Define $t_n := t_{k_n}^{i_n}$. Since $0 \leq t_n \leq T_0$, up to a subsequence (not relabeled), $t_n \to t_0$ for some $0 \leq t_0 \leq T_0$. By (290),

$$\alpha_{k_n}^{i_n} = \alpha_{k_n}(t_n) \to \alpha(t_0)\,, \quad \beta_{k_n}^{i_n} = \beta_{k_n}(t_n) \to \beta(t_0)\,.$$

Extracting a further subsequence (not relabeled), we have that $x_n \to x_0$ for some $x_0 \in [\alpha(t_0) + \delta, \beta(t_0) - \delta]$. Since

$$1/n \geq h_{k_n}^{i_n}(x_n) = h_{k_n}(t_n, x_n) \to h_*(t_0, x_0)$$

by (310), we obtain a contradiction by Theorem 39.

To show (329), fix $0 < \varepsilon < \frac{1}{2}\min_{t\in[0,T_0]}(\beta(t) - \alpha(t))$ such that $M_3^{1/2}\varepsilon^{1/2} < 2\eta_0$. By (284) and (323),

$$(h_k^i)'(x) \geq (h_k^i)'(\alpha_k^i) - M_3^{1/2}|x - \alpha_k^i|^{1/2} \geq 2\eta_0 - M_3^{1/2}\varepsilon^{1/2} > 0$$

for all $\alpha_k^i \leq x \leq \alpha_k^i + \varepsilon$ and $k \geq k_0$. Hence,

$$h_k^i(x) > 0 \quad \text{for } \alpha_k^i < x < \alpha_k^i + \varepsilon \text{ and } k \geq k_0\,, \tag{330}$$

and in the same way we can show that

$$h_k^i(x) > 0 \quad \text{for } \beta_k^i - \varepsilon < x < \beta_k^i \text{ and } k \geq k_0\,. \tag{331}$$

The positivity of $h_k^i(x)$ for $x \in [\alpha_k^i + \varepsilon, \beta_k^i - \varepsilon]$ is a consequence of (328) with $\delta = \varepsilon$. □

Theorem 42. *Under the assumptions (285)–(288), there exists $0 < T_1 \leq T_0$ such that*

$$\operatorname{Lip} h_*(t, \cdot) < L_0 \tag{332}$$

for every $t \in [0, T_1]$.

Proof. Fix L_1 with $\operatorname{Lip} h_0 < L_1 < L_0$. By Lemma 38 the function $t \mapsto h'(t, \alpha(t))$ is continuous and since $h'(0, \alpha(0)) = h_0'(\alpha_0) < L_1$, there exist $T_1 > 0$ such that

$$h'(t, \alpha(t)) < L_1 \quad \text{for all } t \in [0, T_1]\,.$$

Fix $\delta > 0$ such that $M_3^{1/2}\delta^{1/2} < L_0 - L_1$. By (298),

$$|h'(t, x)| \leq h'(t, \alpha(t)) + M_3^{1/2}\delta^{1/2} < L_1 + M_3^{1/2}\delta^{1/2} < L_0$$

for every $t \in [0, T_1]$ and every $\alpha(t) \leq x \leq \alpha(t) + \delta$. Similarly, taking T_1 smaller, if needed, we obtain

$$|h'(t, x)| < L_0 \quad \text{for every } t \in [0, T_1] \text{ and every } \beta(t) - \delta \leq x \leq \beta(t)\,.$$

It remains to prove that

$$\max_{x\in[\alpha(t)+\delta, \beta(t)-\delta]} |h'(t, x)| < L_0\,. \tag{333}$$

Let $g(t)$ denote the left-hand side of (333). To prove that g is continuous, fix $0 \leq t_0 \leq T_1$ and a, b with $\alpha(t_0) < a < \alpha(t_0) + \delta$ and $\beta(t_0) - \delta < b < \beta(t_0)$. By continuity of α and β there exists an open interval I containing t_0 such that $a < \alpha(t) + \delta$ and $\beta(t) - \delta < b$ for all $t \in I$. By Lemma 37, $t \mapsto h'(t, \cdot)$ from I into $C^0([a, b])$ is continuous. Since α and β are continuous, it follows that g is continuous in I. By the arbitrariness of t_0, we conclude that g is continuous in $[0, T_1]$. Using the fact that $g(0) \leq \operatorname{Lip} h_0 < L_0$, taking T_1 even smaller, if needed, we obtain that $g(t) < L_0$ for all $t \in [0, T_1]$. This concludes the proof. □

Proposition 43. *Let T_1 be as in Theorem 42 and k_1 be as in Proposition 41. Then there exists $k_2 \geq k_1$ such that*

$$\operatorname{Lip} \hat{h}_k(t, \cdot) < L_0 \tag{334}$$

for all $k \geq k_2$, and all $t \in [0, T_1]$.

Proof. Assume by contradiction that (334) does not hold. Recalling that $\operatorname{Lip} \hat{h}_k(t, \cdot) \leq L_0$ and that $\hat{h}_k(t, \cdot) \in C^{1,1/2}((\hat{\alpha}_k(t), \hat{\beta}_k(t)))$ for every t, for every $n \in \mathbb{N}$ there exist $k_n \geq n$, $t_n \in [0, T_1]$ and $x_n \in [\hat{\alpha}_{k_n}(t_n), \hat{\beta}_{k_n}(t_n)]$ such that

$$|\hat{h}'_{k_n}(t_n, x_n)| = L_0 \,. \tag{335}$$

Up to a subsequence, $t_n \to t_0$ for some $t_0 \in [0, T_1]$, and by (291),

$$\hat{\alpha}_{k_n}(t_n) \to \alpha(t_0) \quad \text{and} \quad \hat{\beta}_{k_n}(t_n) \to \beta(t_0) \,. \tag{336}$$

Hence, up to a subsequence (not relabeled), $x_n \to x_0$ for some $x_0 \in [\alpha(t_0), \beta(t_0)]$. By (312) we have that $\hat{h}'_{k_n}(t_n, x_n) \to h'(t_0, x_0)$. By (335) and Theorem 42 we obtain a contradiction. □

Let T_1 be as in Theorem 42. For every $t \in [0, T_1]$, let $u(t, \cdot, \cdot)$ be the unique minimizer of the problem

$$\min \left\{ \int_{\Omega_{h(t,\cdot)}} W(Ev(x, y)) \, dxdy : \, v \in \mathcal{A}_e(\alpha(t), \beta(t), h(t, \cdot)) \right\} , \tag{337}$$

where $\mathcal{A}_e$ is defined in (9).

Proposition 44. *Let T_1 be as in Theorem 42 and let $\{t_k\}_k$ be a sequence in $[0, T_1]$ converging to some t_0. Assume that for every k there exists $i_k \in \mathbb{N} \cup \{0\}$ such that $t_k = t_k^{i_k}$. Then $\{u_k^{i_k}\}_k$ converges to $u(t_0, \cdot, \cdot)$ weakly in $H^1(\tilde{\Omega}; \mathbb{R}^2)$ for every open set $\tilde{\Omega} \subset \Omega_{h(t_0,\cdot)}$ with $\operatorname{dist}(\tilde{\Omega}, \operatorname{graph}(h(t_0, \cdot)) > 0$.*

Proof. Let $u_k := u_k^{i_k}$. By minimality,

$$\int_{\Omega_{h_k(t_k,\cdot)}} W(Eu_k(x, y)) \, dxdy \leq \int_{\Omega_{h_k(t_k,\cdot)}} W(e_0 I) \, dxdy = W(e_0 I) A_0 \,, \tag{338}$$

where I is the 2×2 identity matrix and we used (6). Let $\alpha(t_0) < a < b < \beta(t_0)$ and let $\tilde{\Omega}$ be an open set with boundary of class C^∞ such that $[a, b] \times \{0\} \subset \partial\tilde{\Omega}$ and $\operatorname{dist}(\tilde{\Omega}, \operatorname{graph}(h(t_0, \cdot))) > 0$. By (290) and (310) we have $\operatorname{dist}(\tilde{\Omega}, \operatorname{graph}(h_k(t_k, \cdot))) > 0$ and $u_k(x, 0) = (e_0 x, 0)$ for all $x \in [a, b]$ and all k sufficiently large. By Korn's inequality (see Lemma 3) we have that $\{u_k\}_k$ is bounded in $H^1(\tilde{\Omega}; \mathbb{R}^2)$. Then there exist a subsequence (not relabeled) and a function $v \in H^1(\tilde{\Omega};\mathbb{R}^2)$ such that $u_k \rightharpoonup v$ weakly in $H^1(\tilde{\Omega};\mathbb{R}^2)$.

Take an increasing sequence of domains $\{\tilde{\Omega}_n\}_n$ as above such that their union is $\Omega_{h(t_0,\cdot)}$. By a diagonal argument, we can extract a further subsequence (not relabeled) and construct a function $v \in H^1_{\mathrm{loc}}(\Omega_{h(t_0,\cdot)};\mathbb{R}^2)$ such that $u_k \rightharpoonup v$ weakly in $H^1(\tilde{\Omega}_n;\mathbb{R}^2)$ for every $\tilde{\Omega}_n$. By (17), (338), and a lower semicontinuity argument we have that

$$\int_{\tilde{\Omega}_n} |Ev(x,y)|^2 dxdy \leq C_W W(e_0 I) A_0$$

for every n. By letting $n \to \infty$ we obtain

$$\int_{\Omega_{h(t_0,\cdot)}} |Ev(x,y)|^2 dxdy \leq C_W W(e_0 I) A_0 \,.$$

In view of (315), (316), and Theorem 42, the set $\Omega_{h(t_0,\cdot)}$ has Lipschitz continuous boundary and since, by construction, $v(x,0) = (e_0 x, 0)$ for all $x \in (a_n, b_n)$ for all n, we can apply Korn's inequality (see Lemma 3) to conclude that $v \in H^1(\Omega_{h(t_0,\cdot)};\mathbb{R}^2)$.

It remains to show that $v = u(t_0,\cdot,\cdot)$. Let $w \in \mathcal{A}_e(\alpha(t_0),\beta(t_0),h(t_0,\cdot))$. Since $h(t_0,\cdot) \in C^1([\alpha(t_0),\beta(t_0)])$ by Proposition 31 and (316) holds we can argue as at the end of Step 1 in the proof of Theorem 6 to extend w to a function $\tilde{w} \in H^1(U;\mathbb{R}^2)$, where $U := (\alpha(t_0)-1, \beta(t_0)+1) \times (0,\infty)$ and $\tilde{w}(x,0) = (e_0 x, 0)$ for all $x \in (\alpha(t_0)-1, \beta(t_0)+1)$. By the minimality of u_k in $\Omega_{h_k(t_k,\cdot)}$ we have

$$\int_{\Omega_{h_k(t_k,\cdot)}} W(Eu_k(x,y))\, dxdy \leq \int_{\Omega_{h_k(t_k,\cdot)}} W(E\tilde{w}(x,y))\, dxdy \,.$$

Letting $k \to \infty$ and using (310) we obtain

$$\lim_{k\to\infty} \int_{\Omega_{h_k(t_k,\cdot)}} W(E\tilde{w}(x,y))\, dxdy = \int_{\Omega_{h(t_0,\cdot)}} W(Ew(x,y))\, dxdy \,.$$

On the other, by lower semicontinuity

$$\begin{aligned}\int_{\tilde{\Omega}_n} W(Ev(x,y))\, dxdy &\leq \liminf_{k\to\infty} \int_{\tilde{\Omega}_n} W(Eu_k(x,y))\, dxdy \\ &\leq \liminf_{k\to\infty} \int_{\Omega_{h_k(t_k,\cdot)}} W(Eu_k(x,y))\, dxdy\end{aligned}$$

for every n. Taking the limit as $n \to \infty$ we obtain

$$\int_{\Omega_{h(t_0,\cdot)}} W(Ev(x,y))\, dxdy \leq \int_{\Omega_{h(t_0,\cdot)}} W(Ew(x,y))\, dxdy \,,$$

which proves the minimality of v. By uniqueness, $v = u(t_0,\cdot,\cdot)$, and since the limit is independent of the subsequence, the entire sequence $\{u_k\}_k$ converges to $u(t_0,\cdot,\cdot)$ as in the statement. □

In what follows

$$p_1 := \min\{6/5, p_0/(4-2p_0)\} \quad \text{and} \quad q_1 := 4p_1/(2+p_1)\,, \tag{339}$$

where p_0 is defined in (156). We observe that $1 < p_1 \le 6/5$ and $4/3 < q_1 \le 3/2$.

Theorem 45. . *Under the assumptions (285)–(288), let T_1 be as in Theorem 42 and let k_2 be as in Proposition 43. Then there exists $M_4 > 0$ such that*

$$\int_0^{T_1} \|\hat{h}_k'''(t,\cdot)\|^{p_1}_{L^{q_1}((\hat\alpha_k(t),\hat\beta_k(t)))} dt \le M_4 \Big(\int_0^{T_1} (\hat\beta_k(t) - \hat\alpha_k(t))^{(2-5p_1)/(4-2p_1)} \Big)^{1-p_1/2} + M_4\,, \tag{340}$$

$$\int_0^{T_1} \int_{\hat\alpha_k(t)}^{\hat\beta_k(t)} |\hat{h}_k^{(\mathrm{iv})}(t,x)|^{p_1} dx dt \le M_4\,, \tag{341}$$

for all $k \ge k_2$.

Proof. **Step 1.** In this proof C denotes a constant, independent of k and i, whose value can change from formula to formula. Let $\hat q := \max\{p_1, 5p_1/2-1, 3p_1-2, 2p_1-1\}$. Since $p_1 \le 6/5$, by Remark 7 we have $\hat q \le 2$, hence, by (261) and (273),

$$B_{\tau_k}(H_k^i, H_k^{i-1}, \alpha_k^i, \alpha_k^{i-1}, \beta_k^i, \beta_k^{i-1})^{\hat q} \le 4 + C\|\dot H_k(t,\cdot)\|^2_{L^2(\mathbb{R})} + 4|\dot\alpha_k(t)|^2 + 4|\dot\beta_k(t)|^2\,.$$

By Proposition 29, $(\alpha_k^i, \beta_k^i, h_k^i)$ satisfies (64) with $M = M_3 > 1$. By Proposition 40 there exists $0 < \eta_0 < 1$ such that $(\alpha_k^i, \beta_k^i, h_k^i)$ satisfies (62) for all $k \ge k_0$ and all $0 \le i \le kT_0$. By (303) and (322) for $t \in [0, T_0]$ we can apply Lemma 7 to $(\alpha(t), \beta(t), h(t,\cdot))$ obtaining $\beta(t) - \alpha(t) \ge 16\eta_0^2/M_3$, hence (327) is satisfied by $\delta_0 = \eta_0^2/(4M_3)$. Therefore, by Proposition 41 there exists $0 < \eta_1 < 1$ such that $(\alpha_k^i, \beta_k^i, h_k^i)$ satisfies (63) and the second inequality in (37) for all $k \ge k_1$ and all $0 \le i \le kT_0$. Moreover, Proposition 43 implies that h_k^i satisfies the first inequality in (37) for all $k \ge k_2$ and all $0 \le i \le kT_1$. We conclude that all assumptions of Theorem 6 are satisfied for all $k \ge k_2$ and all $0 \le i \le kT_1$.

Hence, by Theorem 24, we have

$$\begin{aligned}\int_{\alpha_k^i}^{\beta_k^i} |(h_k^i)^{(\mathrm{iv})}(x)|^{p_1} dx &\le c_9 B_{\tau_k}(H_k^i, H_k^{i-1}, \alpha_k^i, \alpha_k^{i-1}, \beta_k^i, \beta_k^{i-1})^{\hat q} \\ &\le C + C\|\dot H_k(t,\cdot)\|^2_{L^2(\mathbb{R})} + C|\dot\alpha_k(t)|^2 + C|\dot\beta_k(t)|^2\,.\end{aligned}$$

Integrating in time over $[t_k^{i-1}, t_k^i]$ and summing over all $1 \le i \le kT_1$ we obtain

$$\int_0^{T_1} \int_{\hat\alpha_k(t)}^{\hat\beta_k(t)} |\hat{h}_k^{(\mathrm{iv})}(t,x)|^{p_1} dx dt \le CT_1 + C\int_0^{T_1} \|\dot H_k(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \tag{342}$$

$$+ C\int_0^{T_1} |\dot{\alpha}_k(t)|^2 dt + C\int_0^{T_1} |\dot{\beta}_k(t)|^2 dt \le C\,,$$

where in the last inequality we used (271) and (278). This proves (341).

Step 2: By standard interpolation results ([44, Theorem 7.41]) for every $t \in [0, T_1]$,

$$\begin{aligned}\|\hat{h}_k'''(t,\cdot)\|_{L^{q_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} &\le C(\hat{\beta}_k(t)) - \hat{\alpha}_k(t))^{1/q_1-3/2}\|\hat{h}_k''(t,\cdot)\|_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} \\ &\quad + C\|\hat{h}_k''(t,\cdot)\|_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} + C\|\hat{h}_k^{(\mathrm{iv})}(t,\cdot)\|_{L^{p_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))}\,.\end{aligned}$$

In turn,

$$\begin{aligned}&\int_0^{T_1} \|\hat{h}_k'''(t,\cdot)\|^{p_1}_{L^{q_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt \\ &\quad\le C\int_0^{T_1} (\hat{\beta}_k(t)) - \hat{\alpha}_k(t))^{1/2-5p_1/4}\|\hat{h}_k''(t,\cdot)\|^{p_1}_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt \\ &\qquad + C\int_0^{T_1} \|\hat{h}_k''(t,\cdot)\|^{p_1}_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt + C\int_0^{T_1} \|\hat{h}_k^{(\mathrm{iv})}(t,\cdot)\|^{p_1}_{L^{p_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt\,.\end{aligned}$$

By Hölder's inequality, (283), and (341),

$$\begin{aligned}&\int_0^{T_1} \|\hat{h}_k'''(t,\cdot)\|^{p_1}_{L^{q_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt \\ &\le C\left(\int_0^{T_1} (\hat{\beta}_k(t)) - \hat{\alpha}_k(t))^{(2-5p_1)/(4-2p_1)}\right)^{1-p_1/2}\left(\int_0^{T_1} \|\hat{h}_k''(t,\cdot)\|^2_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt\right)^{p_1/2} \\ &\quad + CT_1^{1-p_1/2}\left(\int_0^{T_1} \|\hat{h}_k''(t,\cdot)\|^2_{L^2((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt\right)^{p_1/2} \\ &\quad + C\int_0^{T_1} \|\hat{h}_k^{(\mathrm{iv})}(t,\cdot)\|^{p_1}_{L^{p_1}((\hat{\alpha}_k(t),\hat{\beta}_k(t)))} dt \\ &\le C(T_1M_3)^{p_1/2}\left(\int_0^{T_1} (\hat{\beta}_k(t)) - \hat{\alpha}_k(t))^{(2-5p_1)/(4-2p_1)}\right)^{1-p_1/2} + CT_1M_3^{p_1/2} + M_4\,.\end{aligned}$$

This proves (340). □

Theorem 46. *Under the assumptions* (285)–(288)*, let* T_1 *be as in Theorem 42. Then for a.e.* $t \in (0, T_1)$ *we have*

$$\sigma_0\dot{\alpha}(t) = \frac{\gamma}{J(t,\alpha(t))} - \gamma_0 + \nu_0\frac{h'(t,\alpha(t))}{(J(t,\alpha(t)))^2}\left(\frac{h''(t,\cdot)}{(J(t,\cdot))^3}\right)'(\alpha(t))\,, \tag{343}$$

$$\sigma_0\dot{\beta}(t) = -\frac{\gamma}{J(t,\beta(t))} + \gamma_0 - \nu_0\frac{h'(t,\beta(t))}{J(t,\beta(t))^2}\left(\frac{h''(t,\cdot)}{J(t,\cdot)^3}\right)'(\beta(t))\,, \tag{344}$$

where $J(t,\cdot)$ is defined in (42) using $h(t,\cdot)$.

Remark 9. *To express (343) and (344) in an intrinsic way, for every $t \in [0,T_1]$ we define $s(t,x)$ for $\alpha(t) \le x \le \beta(t)$ by*

$$s(t,x) := \int_{\alpha(t)}^{x} \sqrt{1+(h'(t,\rho))^2}d\rho \tag{345}$$

The inverse of $s(t,\cdot)$, defined for $0 = s(t,\alpha(t)) \le s \le s(t,\beta(t))$, is denoted by $x(t,\cdot)$. Let $\kappa(t,\cdot) : \big(s(t,\alpha(t)), s(t,\beta(t))\big) \to \mathbb{R}$ be the signed curvature of the graph of $h(t,\cdot)$, considered as a function of arclength. To be precise, we have

$$\kappa(t,s) := \frac{h''(t,x(t,s))}{\Big(1+\big(h'(t,x(t,s))\big)^2\Big)^{3/2}}\,, \tag{346}$$

hence

$$\kappa(t,s(t,x)) = \frac{h''(t,x)}{J(t,x)^3}$$

Since

$$\Big(\frac{h''(t,\cdot)}{(J(t,\cdot))^3}\Big)'(x) = \partial_s\kappa\big(t,s(t,x)\big)J(t,x)\,,$$

we can rewrite (343) and (344) as

$$\sigma_0\dot\alpha(t) = \gamma\cos\theta_\alpha(t) - \gamma_0 + \nu_0\partial_s\kappa\big(t,s(t,\alpha(t))\big)\sin\theta_\alpha(t)\,,$$
$$\sigma_0\dot\beta(t) = -\gamma\cos\theta_\beta(t) + \gamma_0 - \nu_0\partial_s\kappa\big(t,s(t,\beta(t))\big)\sin\theta_\beta(t)\,,$$

where

$$\theta_\alpha(t) := \arcsin\frac{h'(t,\alpha(t))}{\sqrt{1+(h'(t,\alpha(t)))^2}} \quad \text{and} \quad \theta_\beta(t) := \arcsin\frac{h'(t,\beta(t))}{\sqrt{1+(h'(t,\beta(t)))^2}}\,, \tag{347}$$

are the oriented angles between the oriented x-axis and the tangent to the graph of $h(t,\cdot)$ at $(\alpha(t),0)$ and $(\beta(t),0)$, respectively.

Proof of Theorem 46. Fix $t \in [0,T_1]$ and $k \ge k_2$, where k_2 is as in Proposition 43, and find i such $t_k^{i-1} \le t \le t_k^i$. By (202),

$$\sigma_0\frac{\alpha_k^i - \alpha_k^{i-1}}{\tau_k} = \frac{\gamma}{J_k^i(\alpha_k^i)} - \gamma_0 + \nu_0\frac{(h_k^i)'(\alpha_k^i)}{(J_k^i(\alpha_k^i))^2}\Big(\frac{(h_k^i)''}{(J_k^i)^3}\Big)'(\alpha_k^i)\,,$$

where $J_k^i(x) := (1+((h_k^i)'(x))^2)^{1/2}$. Introducing $\hat J_k(t,x) := (1+(\hat h_k'(t,x))^2)^{1/2}$, for every $0 \le t \le T_1$ we have

$$\sigma_0\dot\alpha_k(t-) = \frac{\gamma}{\hat J_k(t,\hat\alpha_k(t))} - \gamma_0 + \nu_0\frac{\hat h_k'(t,\hat\alpha_k(t))}{(\hat J_k(t,\hat\alpha_k(t)))^2}\Big(\frac{\hat h_k''(t,\cdot)}{(\hat J_k(t,\cdot))^3}\Big)'(\hat\alpha_k(t))\,, \tag{348}$$

where $\dot{\alpha}_k(t-)$ is the left derivative. We claim that

$$\frac{\gamma}{\hat{J}_k(\cdot, \hat{\alpha}_k(\cdot))} \to \frac{\gamma}{(1 + (h'(\cdot, \alpha(\cdot)))^2)^{1/2}} \quad \text{strongly in } L^2((0, T_1)) \,. \tag{349}$$

Since the function $s \mapsto 1/(1 + s^2)^{1/2}$ is 1-Lipschitz continuous, by (312) we have

$$\left| \frac{1}{\hat{J}_k(t, \hat{\alpha}_k(t))} - \frac{1}{(1 + (h'(t, \alpha(t)))^2)^{1/2}} \right| \le |\hat{h}'_k(t, \hat{\alpha}_k(t)) - h'(t, \alpha(t))| \to 0 \,,$$

which gives (349) by the dominated convergence theorem.

To study the last term in (348) we observe that

$$\begin{aligned} \frac{\hat{h}'_k(t, \hat{\alpha}_k(t))}{\hat{J}_k(t, \hat{\alpha}_k(t))^2} \Big(\frac{\hat{h}''_k(t, \cdot)}{(\hat{J}_k(t, \cdot))^3} \Big)'(\hat{\alpha}_k(t)) &= \frac{\hat{h}'_k(t, \hat{\alpha}_k(t))}{(\hat{J}_k(t, \hat{\alpha}_k(t)))^5} \hat{h}'''_k(t, \hat{\alpha}_k(t)) \\ &- 3 \frac{(\hat{h}'_k(t, \hat{\alpha}_k(t)))^2}{(\hat{J}_k(t, \hat{\alpha}_k(t)))^7} (\hat{h}''_k(t, \hat{\alpha}_k(t)))^2 = \frac{\hat{h}'_k(t, \hat{\alpha}_k(t))}{(\hat{J}_k(t, \hat{\alpha}_k(t)))^5} \hat{h}'''_k(t, \hat{\alpha}_k(t)) \,, \end{aligned}$$

where we used (201). We claim that

$$\hat{h}'''_k(\cdot, \hat{\alpha}_k(\cdot)) \rightharpoonup h'''(\cdot, \alpha(\cdot)) \quad \text{weakly in } L^{p_1}((0, T_1)) \,. \tag{350}$$

By the fundamental theorem of calculus,

$$\hat{h}'''_k(t, \hat{\alpha}_k(t)) = \hat{h}'''_k(t, x) - \int_{\hat{\alpha}_k(t)}^{x} \hat{h}^{(\mathrm{iv})}_k(t, \rho) \, d\rho \,,$$

where $\hat{\alpha}_k(t) < x < \hat{\beta}_k(t)$. By Lemma 1 and the uniform continuity of α and β (see (289)), we can subdivide $[0, T_1]$ into a finite number of intervals I such that for each of them there exist $a, b \in \mathbb{R}$ such that $\alpha(t) < a < b < \beta(t)$ for all $t \in I$. To prove (350) it suffices to prove weak convergence in $L^{p_1}(I)$ for every such I. Fix $I = [t_1, t_2]$ and the corresponding a, b. By Hölder's inequality

$$|\hat{h}'''_k(t, \hat{\alpha}_k(t))|^{p_1} \le C |\hat{h}'''_k(t, x)|^{p_1} + C \int_{\hat{\alpha}_k(t)}^{\hat{\beta}_k(t)} |\hat{h}^{(\mathrm{iv})}_k(t, \rho)|^{p_1} d\rho \,.$$

Averaging in x over (a, b) and integrating in t over (t_1, t_2) gives

$$\int_{t_1}^{t_2} |\hat{h}'''_k(t, \hat{\alpha}_k(t))|^{p_1} dt \le \frac{C}{b - a} \int_{t_1}^{t_2} \int_a^b |\hat{h}'''_k(t, x)|^{p_1} dx dt + C \int_{t_1}^{t_2} \int_{\hat{\alpha}_k(t)}^{\hat{\beta}_k(t)} |\hat{h}^{(\mathrm{iv})}_k(t, \rho)|^{p_1} d\rho dt \,.$$

Note that the first integral on the right-hand side is bounded because $p_1 \le q_1$ and in view of (340), while the second integral is bounded by (341).

Let $\phi \in C_c^\infty((a,b))$ with $\int_a^b \phi(x)\,dx \neq 0$ and $\psi \in C_c^\infty((t_1,t_2))$. Then

$$\int_a^b \phi(x)\,dx \int_{t_1}^{t_2} \hat{h}_k'''(t,\hat{\alpha}_k(t))\psi(t)\,dt = \int_{t_1}^{t_2}\int_b^a \hat{h}_k'''(t,x)\phi(x)\psi(t)\,dxdt$$
$$- \int_{t_1}^{t_2}\int_b^a \int_{\mathbb{R}} \hat{h}_k^{(\mathrm{iv})}(t,\rho)\chi_{(\hat{\alpha}_k(t),x)}(\rho)\phi(x)\psi(t)\,d\rho dxdt,$$

where $\hat{h}_k^{(\mathrm{iv})}(t,\rho)\chi_{(\hat{\alpha}_k(t),x)}(\rho)$ is interpreted to be zero for $\rho \notin (\hat{\alpha}_k(t),x)$. By (291), (308), and (341), we have

$$\hat{h}_k^{(\mathrm{iv})}(t,\rho)\chi_{(\hat{\alpha}_k(t),x)}(\rho)\phi(x) \rightharpoonup h^{(\mathrm{iv})}(t,\rho)\chi_{(\alpha(t),x)}(\rho)\phi(x) \quad \text{weakly in } L^{p_1}((t_1,t_2)\times\mathbb{R})$$

for every fixed x. On the other hand, by (340),

$$\hat{h}_k'''(t,x) \rightharpoonup h'''(t,x) \quad \text{weakly in } L^{p_1}((t_1,t_2);L^{q_1}((a,b))\,.$$

Therefore,

$$\int_a^b \phi(x)\,dx \int_{t_1}^{t_2} \hat{h}_k'''(t,\hat{\alpha}_k(t))\psi(t)\,dt \to \int_{t_1}^{t_2}\int_b^a h'''(t,x)\phi(x)\psi(t)\,dxdt$$
$$- \int_{t_1}^{t_2}\int_b^a\int_{\alpha(t)}^x h^{(\mathrm{iv})}(t,\rho)(\rho)\phi(x)\psi(t)\,d\rho dxdt$$
$$= \int_a^b \phi(x)dx \int_{t_1}^{t_2} h'''(t,\alpha(t))\psi(t)\,dt\,.$$

Dividing by $\int_a^b \phi(x)\,dx$, we get

$$\int_{t_1}^{t_2} \hat{h}_k'''(t,\hat{\alpha}_k(t))\psi(t)\,dt \to \int_{t_1}^{t_2} h'''(t,\alpha(t))\psi(t)\,dt\,.$$

By the arbitrariness of ψ we obtain the weak convergence in I, which suffices to prove (350).

Arguing as in the first part of the proof

$$\frac{\hat{h}_k'(t,\hat{\alpha}_k(t))}{(\hat{J}_k(t,\hat{\alpha}_k(t)))^5} \to \frac{h'(t,\alpha(t))}{(J(t,\alpha(t)))^5}$$

pointwise and is uniformly bounded. Therefore

$$\frac{\hat{h}_k'(\cdot,\hat{\alpha}_k(\cdot))}{(\hat{J}_k(\cdot,\hat{\alpha}_k(\cdot)))^5}\hat{h}_k'''(\cdot,\hat{\alpha}_k(\cdot)) \rightharpoonup \frac{\hat{h}_k'(t,\hat{\alpha}_k(t))}{(\hat{J}_k(t,\hat{\alpha}_k(t)))^5}h'''(\cdot,\alpha(\cdot)) \quad \text{weakly in } L^{p_1}((0,T_1))\,. \tag{351}$$

Combining (348), (349), and (351), from (289) we obtain (343). □

Next we introduce the time derivative of h.

Proposition 47. *For a.e. $t \in [0, +\infty)$ there exists an element of $H^{-1}((\alpha(t), \beta(t)))$, denoted by $\dot{h}(t, \cdot)$, such that*

$$\frac{h(s, \cdot) - h(t, \cdot)}{s - t} \to \dot{h}(t, \cdot) \quad \textit{strongly in } H^{-1}((a, b)) \tag{352}$$

for every $\alpha(t) < a < b < \beta(t)$.

Proof. Let us fix $T > 0$. By Proposition 32, $H \in H^1((0, T); L^2(\mathbb{R}))$ and $H' = h_*$. It follows that

$$h_* \in H^1((0, T); H^{-1}(\mathbb{R}))\,, \tag{353}$$

and so for a.e. $t \in (0, T)$,

$$\frac{h_*(s, \cdot) - h_*(t, \cdot)}{s - t} \to \dot{h}_*(t, \cdot) \quad \text{strongly in } H^{-1}(\mathbb{R})\,. \tag{354}$$

In particular, if $t \in [0, T]$ satisfies (354) and $\alpha(t) < a < b < \beta(t)$, by the continuity of α and β (see Proposition 30), we have $\alpha(s) < a < b < \beta(s)$, for all s close to t. Therefore (354) implies (352). □

Theorem 48. *Under the assumptions (285)–(288), let T_1 be as in Theorem 42. Then for a.e. $t \in (0, T_1)$ we have*

$$\dot{h} = \Big[-\gamma \frac{1}{J}\Big(\frac{h'}{J}\Big)'' + \nu_0 \frac{1}{J}\Big(\frac{h''}{J^5}\Big)''' + \frac{5}{2}\nu_0 \frac{1}{J}\Big(\frac{h'(h'')^2}{J^7}\Big)'' + \frac{1}{J}\overline{W}'\Big]' \tag{355}$$

in $\mathcal{D}'((\alpha(t), \beta(t)))$, where $\overline{W}$ is defined in (42).

Remark 10. *To express (355) in an intrinsic way, besides the functions $s(t, \cdot)$, $x(t, \cdot)$, and $\kappa(t, \cdot)$ considered in Remark 9, for $0 = s(t, \alpha(t)) \le s \le s(t, \beta(t))$ we introduce the normal velocity $\widetilde{V}(t, s)$ of the time dependent curve $\Gamma_{h(t,\cdot)}$ at the point corresponding to the arclength parameter s, given by*

$$\widetilde{V}(t, s) := \frac{\dot{h}(t, x(t, s))}{\sqrt{1 + \big(h'(t, x(t, s))\big)^2}}\,.$$

Moreover, we introduce the chemical potential $\zeta(t, \cdot)\colon \big(s(t, \alpha(t)), s(t, \beta(t))\big) \to \mathbb{R}$ given by

$$\zeta(t, s) := -\gamma\kappa(t, s) + \nu_0\Big(\partial_{ss}\kappa(t, s) + \frac{\kappa(t, s)^3}{2}\Big) + \widetilde{W}(t, s)\,,$$

where

$$\widetilde{W}(t, s) := W\big(Eu(t, x(t, s), h(t, x(t, s))\big)\,.$$

By direct computations (see [10, Remark 3.2 and Lemma 6.7]) we obtain that (355) is equivalent to

$$\widetilde{V}(t, \cdot) = \partial_{ss}\zeta(t, \cdot)$$

in $\mathcal{D}\big((s(t, \alpha(t)), s(t, \beta(t))\big)$.

Proof. By the continuity of α and β (see Proposition 30), it suffices to prove that, given $a < b$ and a time interval $[t_1, t_2]$ such that $\alpha(t) < a < b < \beta(t)$ for all $t \in [t_1, t_2]$, the equality (355) holds in $\mathcal{D}'((a,b))$ for a.e. $t \in [t_1, t_2]$. Let $\delta > 0$ be such that $\alpha(t)+3\delta < a < b < \beta(t)-3\delta$ for all $t \in [t_1, t_2]$. By (290), $\alpha_k(t)+2\delta < a < b < \beta_k(t)-2\delta$ for all $t \in [t_1, t_2]$ and all $k \geq k^*$ for some k^*. Hence, by Proposition 41 there exist $\zeta > 0$ and $\hat{k} \geq k^*$ such that

$$h_k(t,x) \geq \zeta \quad \text{for all } t \in [t_1, t_2]\,,\ x \in [a-\delta, b+\delta]\,,\ \text{and } k \geq \hat{k}\,.$$

Letting $k \to \infty$, by (294) we obtain that

$$h(t,x) \geq \zeta \quad \text{for all } t \in [t_1, t_2]\,,\ x \in [a-\delta, b+\delta]\,.$$

In view of these inequalities and of (41) we can repeat the proof of [10, Theorem 3.8] to prove (355) in $\mathcal{D}'((a,b))$ for a.e. $t \in [t_1, t_2]$. We observe that since here we do not have periodic boundary conditions in x, the argument used in [10] needs to be modified accordingly. To be precise, the inequality (3.16) in [10] must be replaced by

$$\begin{aligned}\|h_k'(\tau_2,\cdot) - h_k'(\tau_1,\cdot)\|_{L^\infty((a,b))} &\leq C\|h_k''(\tau_2,\cdot) - h_k''(\tau_1,\cdot)\|^{3/4}_{L^2((a,b))}\|h_k(\tau_2,\cdot) - h_k(\tau_1,\cdot)\|^{1/4}_{L^2((a,b))} \\ &\quad + C\|h_k(\tau_2,\cdot) - h_k(\tau_1,\cdot)\|_{L^2((a,b))} \\ &\leq C|\tau_2 - \tau_1|^{1/32} + C|\tau_2 - \tau_1|^{1/8}\end{aligned}$$

for every $\tau_1, \tau_2 \in [t_1, t_2]$. Similarly, in the proof of [10, Corollary 3.7] the second displayed inequality must be replaced by

$$\begin{aligned}\int_{t_1}^{t_2} \|h_n'''(t,\cdot) - h_m'''(t,\cdot)\|^{12/5}_{L^\infty((a,b))} dt &\leq C \sup_{t\in[t_1,t_2]} \|h_n'(t,\cdot) - h_m'(t,\cdot)\|^{2/5}_{L^\infty((a,b))} \\ &\quad + \sup_{t\in[t_1,t_2]} \|h_n'(t,\cdot) - h_m'(t,\cdot)\|^{12/5}_{L^\infty((a,b))} \to 0\end{aligned}$$

as $n, m \to \infty$. □

The following theorem summarizes the main results obtained in this section.

Theorem 49. *Under the assumptions (285)–(288), there exists $T > 0$ such that the following items hold:*

(i) there exist two functions $\alpha, \beta \in H^1((0,T))$, with $\alpha(t) < \beta(t)$ for every $t \in [0,T]$, such that $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$, and

$$\beta(t) - \alpha(t) \geq \sqrt{2A_0/L_0} \quad \textit{for every } t \in [0,T]\,. \tag{356}$$

(ii) There exist $1 < p_1 < 6/5$ and a continuous function $h \geq 0$, defined for $t \in [0,T]$ and $x \in [\alpha(t), \beta(t)]$, such that $h(0,x) = h_0(x)$ for every $x \in [\alpha_0, \beta_0]$, $h(t,x) > 0$ for every $t \in [0,T]$ and every $x \in (\alpha(t), \beta(t))$, $h(t,\alpha(t)) = h(t,\beta(t)) = 0$, $h'(t,\alpha(t)) > 0$, and

$h'(t,\beta(t)) < 0$ for every $t \in [0,T]$, and

$$h(t,\cdot) \in H^2((\alpha(t),\beta(t))) \quad \textit{for every } t \in [0,T]\,, \tag{357}$$

$$h(t,\cdot) \in W^{4,p_1}((\alpha(t),\beta(t))) \ \textit{for a.e. } t \in [0,T]\,, \tag{358}$$

$$\int_{\alpha(t)}^{\beta(t)} h(t,x)\,dx = A_0 \quad \textit{for every } t \in [0,T]\,, \tag{359}$$

$$\operatorname{Lip} h(t,\cdot) < L_0 \quad \textit{for every } t \in [0,T]\,. \tag{360}$$

Moreover, if we denote by h_* the extension of h obtained by setting $h_*(t,x) := 0$ for $x \in \mathbb{R} \setminus (\alpha(t),\beta(t))$, then

$$h_* \in C^{0,3/10}([0,T];L^2(\mathbb{R})) \cap H^1((0,T);H^{-1}(\mathbb{R}))\,. \tag{361}$$

(iii) The function u introduced in (337) is such that

$$u(t,\cdot,\cdot) \in C^{3,1-1/p_1}(\overline{\Omega}_h^{a,b}) \quad \textit{for a.e. } t \in [0,T] \textit{ and } \alpha(t) < a < b < \beta(t)\,, \tag{362}$$

and $u(t,\cdot,\cdot)$ solves the boundary value problem

$$\begin{cases} -\operatorname{div} \mathbb{C}Eu(t,x,y) = 0 & \textit{in } \Omega_{h(t,\cdot)}\,, \\ \mathbb{C}Eu(t,x,h(t,x))\nu^h(t,x) = 0 & \textit{for } x \in (\alpha(t),\beta(t))\,, \\ u(t,x,0) = (e_0 x, 0) & \textit{for } x \in (\alpha(t),\beta(t))\,, \end{cases} \tag{363}$$

for a.e. $t \in (0,T)$.

(iv) Considering the functions s, κ, θ_α, θ_β, and ζ introduced in (345), (346), (347), and (356), respectively, we have

$$\sigma_0 \dot\alpha(t) = \gamma \cos\theta_\alpha(t) - \gamma_0 + \nu_0 \partial_s \kappa\big(t, s(t,\alpha(t))\big) \sin\theta_\alpha(t) \quad \textit{for a.e. } t \in [0,T]\,, \tag{364}$$

$$\sigma_0 \dot\beta(t) = -\gamma \cos\theta_\beta(t) + \gamma_0 - \nu_0 \partial_s \kappa\big(t, s(t,\beta(t))\big) \sin\theta_\beta(t) \quad \textit{for a.e. } t \in [0,T]\,, \tag{365}$$

$$\widetilde{V}(t,\cdot) = \partial_{ss}\zeta(t,\cdot) \quad \textit{in } \mathcal{D}\big((s(t,\alpha(t)), s(t,\beta(t))\big) \quad \textit{for a.e. } t \in [0,T]\,. \tag{366}$$

Proof. Item (i) follows from Proposition 30.

Let h and h_* be the functions introduced in Proposition 31 and let T be the constant T_1 introduced in Theorem 42. By Proposition 31 and Theorem 39, we have that $h(t,\cdot)$ is strictly positive in $(\alpha(t),\beta(t))$ and vanishes at the endpoints, with $h'(t,\alpha(t)) > 0$, and $h'(t,\beta(t)) < 0$, for every $t \in [0,T]$. Property (357) follows from (297); (358) from (308) and (341); (360) from (332); (359) from (296); and (361) from Proposition 31 and (353).

Let u be the function introduced in (337). Then (362) follows from elliptic regularity ([39, Theorem 9.3]), since $h(t,\cdot) \in C^{3,1-1/p_1}([\alpha(t),\beta(t)])$ for a.e. $t \in [0,T]$ by (358). In turn, by taking variations in (337), we obtain that (363) holds.

Finally, considering the functions s, κ, θ_α, θ_β, and ζ introduced in (345), (346), (347), and (356), respectively, we have that (364) and (365) follow from Theorem 46 and Remark 9 , while (366) follows from Theorem 48 and Remark 10. □

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Data availability

This research article does not involve empirical data. All results and analyses presented herein are derived from theoretical developments, mathematical proofs, and computational simulations. Therefore, there are no datasets to be made publicly available.

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