Homework 5 Solutions

3.6.7 Give a counterexample to show that the given transformation is not a linear transformation:

\[ T \left( \begin{array}{c} x \\ y \\ x^2 \end{array} \right) = \left( \begin{array}{c} y \\ x^2 \end{array} \right) \]

**Solution.** Note:

\[ T \left( \begin{array}{c} 0 \\ 1 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \\ \end{array} \right) \]

\[ T \left( \begin{array}{c} 0 \\ 2 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ 4 \\ \end{array} \right) \]

So:

\[ T \left( \begin{array}{c} 0 \\ 1 \\ \end{array} \right) + T \left( \begin{array}{c} 0 \\ 2 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ 5 \\ \end{array} \right) \]

But

\[ T \left( \begin{array}{c} 0 \\ 1 \\ \end{array} \right) + T \left( \begin{array}{c} 0 \\ 2 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ 9 \\ \end{array} \right) \]

3.6.44 Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a linear transformation. Show that \( T \) maps straight lines to a straight line or a point.

**Proof.** In \( \mathbb{R}^3 \) we can represent a line as:

\[ x = tm + b \]

where \( m \neq 0 \). So,

\[ T(tm + b) = t(Tm) + T(b) \]

If \( Tm = 0 \) (i.e. \( m \in \ker(T) \)) then \( T \) sends the line to a point, namely \( Tb \). Otherwise, \( Tm = k \neq 0 \) and \( Tb = c \) so we have the line gets sent to \( tk + c \), which is a line in \( \mathbb{R}^3 \).

3.6.53 Prove that \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation if and only if

\[ T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) \]

for all vectors \( v_1, v_2 \in \mathbb{R}^n \) and scalars \( c_1, c_2 \).

**Proof.** (\( \Leftarrow \)) We need to show that \( T \) respects scalar multiplication and scalar multiplication.

- First we show that for any \( x, y \) we have \( T(x + y) = Tx + Ty \). From the property (\( \ast \)) where \( c_1 = c_2 = 1 \) and \( v_1 = x \) and \( v_2 = y \) we have that

\[ T(1x + 1y) = 1Tx + 1Ty = Tx + Ty \]

- Need we show that for any \( x \) and scalar \( c \) we have \( T(cx) = cTx \). We use (\( \ast \)) for \( c_1 = c, c_2 = 0, v_1 = v_2 = x \) and we get:

\[ T(cx + 0x) = cTx + 0Tx = cTx \]

(\( \Rightarrow \)) Let \( v_1, v_2 \in \mathbb{R}^n \) and \( c_1, c_2 \) be scalars. Then we want to show

\[ T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) \]

Well, \( T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) \) by the sum property of linearity. Then \( T(c_1v_1) + T(c_2v_2) = c_1Tv_1 + c_2Tv_2 \) by the scalar property. This is what we wanted. \( \Box \)
4.1.12 Show that $\lambda = 3$ is an eigenvalue for the following matrix, and find one eigenvector.

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \\ 4 & 2 & 0 \end{pmatrix}$$

Solution. Consider $A - 3I$:

$$A - 3I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \\ 4 & 2 & -3 \end{pmatrix}$$

Doing the elementary row operation of adding $-2$ times row 1 to row 2 we get:

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 4 & 2 & -3 \end{pmatrix}$$

This show that the matrix does not have full rank, so therefore has a nontrivial null space. This is enough to know that $\lambda = 3$ is an eigenvalue. Continuing row reductions to rref:

$$\begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We now just need to find some vector in the null space. Viewing this has a homogenous equation, we see that the solutions look like:

$$t \begin{pmatrix} 1/4 \\ 1 \\ 1 \end{pmatrix}$$

These are all the eigenvectors. One particular eigenvector is $\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$

4.1.24 Use determinants of $2 \times 2$ matrices to find the spectrum of the given matrix. Find the eigenspaces and then give bases for each eigenspace.

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 6 \end{pmatrix}$$

Solution. Consider the following matrix:

$$A - \lambda I = \begin{pmatrix} -\lambda & 2 \\ 8 & 6 - \lambda \end{pmatrix}$$

Calculating the determinant:

$$\det(A - \lambda I) = -\lambda(6 - \lambda) - 16 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$$

The matrix is invertible if and only if the determinant is nonzero. Thus, this has a nontrivial null space only when the determinant is zero. Thus is has eigenvalues when the determinant is zero: $\lambda_1 = 8$, $\lambda_2 = -2$.

We can then find eigenvectors by calculating the actual nullspace of $A - 8I$ and $A + 2I$. The former is the matrix:

$$\begin{pmatrix} -8 & 2 \\ 8 & -2 \end{pmatrix}$$

The nullspace of this matrix is:

$$t \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$
This is just a one dimensional space, so a basis for the eigenspace corresponding to 8 is just $\left(\begin{array}{c} \frac{1}{4} \\ 1 \end{array}\right)$ (or any nonzero multiple of this vector will do) And for the other, we get the matrix:

$$\begin{pmatrix} 2 & 2 \\ 8 & 8 \end{pmatrix}$$

The nullspace of this matrix is:

$$t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This is just a one dimensional space, so a basis for the eigenspace corresponding to $-2$ is just $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any nonzero multiple of this vector will do)

4.1.37 Show that the eigenvalues of the upper triangular matrix:

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

are $\lambda = a$ and $\lambda = d$. Find the corresponding eigenspaces.

Solution. Consider $A - aI$.

$$A - aI = \begin{pmatrix} 0 & b \\ 0 & d - a \end{pmatrix}$$

This matrix has a nontrivial null space since it’s rank is less than the number of columns. Moreover, assuming that $b \neq 0$ or $d - a \neq 0$, then the rank of the matrix is 1, so it’s nullspace is one dimensional given by:

$$t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In the event both are zero, the rank is 0, and it’s nullspace is 2 dimensional and is given by:

$$t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly, $A - dI$ is:

$$A - dI = \begin{pmatrix} a - d & b \\ 0 & 0 \end{pmatrix}$$

if $a - d \neq 0$ or $b \neq 0$ we have that the rank of the matrix is 1, so it has a one dimensional nullspace. If $a - d \neq 0$ then it’s nullspace is given by:

$$t \begin{pmatrix} -b \\ a-d \end{pmatrix}$$

Otherwise, it’s simply

$$t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the even that both are 0 then the rank of the matrix is 0 in which case the null space is two dimensional, so it is:

$$t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$