We prove the following theorem:

**Theorem 1.** If a $K_n$ is drawn in the plane in such a way that it has a hamiltonian cycle that does not cross any edges (including its own), then it has at least $n^4 \left( \frac{1}{16} - \frac{1}{1024\pi} \right) + O(n^3)$ crossings.

The basic idea of the proof is as follows. Let the vertices of the graph be labeled 1, ..., $n$ in order along the non-crossing hamiltonian cycle. Each edge of the graph not on this cycle must be on either the inside or the outside. Consider a pair of such edges: $(a, c)$ and $(b, d)$. If $a$ and $c$ separate $b$ from $d$ along the circle, and if these two edges are drawn on the same side of the cycle, then they must cross. This reduces our problem to finding the solution to a certain MAX-CUT problem. We make the following definitions:

**Definition.** For $a, b, c, d \in \mathbb{Z}/n$, we say that $(a, c)$ crosses $(b, d)$ if, $a, b, c, d$ can be assigned representatives $a', b', c', d'$ so that either $a' < b' < c' < d' < a' + n$ or $a' > b' > c' > d' > a' - n$.

Note that $(a, c)$ crosses $(b, d)$ if and only if $(c, a)$ crosses $(b, d)$. This is because if (without loss of generality) $a' > b' > c' > d' > a' - n$, then $c' > d' > a' - n > b' - n > c' - n$. Similarly, $(a, c)$ crosses $(b, d)$ if and only if $(b, d)$ crosses $(a, c)$.

**Definition.** For positive integer $n$, let $G_n$ be the graph whose vertices are unordered pairs of distinct elements of $\mathbb{Z}/n$, and whose edges connect pairs $\{a, c\}$ and $\{b, d\}$ when $(a, c)$ crosses $(b, d)$.

**Lemma 2.** If a $K_n$ is drawn in the plane in such a way that it has a hamiltonian cycle that does not cross any edges (including its own), then it has at least $|E(G_n)| - \text{MAX-CUT}(G_n)$ crossings.

**Proof.** For every such drawing of a graph, label the vertices along the hamiltonian cycle by elements of $\mathbb{Z}/n$ in order. The edges of our $K_n$ now correspond to the vertices of $G_n$ in the obvious way. Let $S$ be the subset of the vertices of $G_n$ corresponding to edges of the $K_n$ that lie within the designated cycle. Note that any two vertices in $S$ or any two vertices not in $S$ connected by an edge, correspond to pairs of edges in the $K_n$ that must cross. Thus the number of crossings of our $K_n$ is at least

$$|E(S, S)| + |E(\bar{S}, \bar{S})| = |E(G_n)| - |E(S, \bar{S})| \geq |E(G_n)| - \text{MAX-CUT}(G_n).$$

We have thus reduced our problem to bounding the size of the solution of a certain family of MAX-CUT problems. We do this essentially by solving the Goemans-Williamson relaxation of a limiting version of this family of problems. To set things up, we need a few more definitions.

**Definition.** By $S^1$ here we will mean $\mathbb{R}/\mathbb{Z}$. Given $a, b, c, d \in S^1$ we say that $(a, c)$ crosses $(b, d)$ if $a, b, c$ and $d$ have representatives $a', b', c', d' \in \mathbb{R}$ respectively, so that either $a' > b' > c' > d' > a' - 1$ or $a' < b' < c' < d' < a' + 1$. 1
Define the indicator function

\[ C(a, b, c, d) := \begin{cases} 
1 & \text{if } (a, c) \text{ crosses } (c, d), \\
0 & \text{otherwise} 
\end{cases} \]

We now present the continuous version of our MAX-CUT problem:

**Proposition 3.** Let \( f : S^1 \times S^1 \to \{\pm 1\} \), then

\[
\int_{(S^1)^4} f(w, y)f(x, z)C(w, x, y, z)dwdxdydz \geq -\frac{1}{\pi^2}.
\]

We prove this by instead proving the following stronger result:

**Proposition 4.** Let \( f : S^1 \times S^1 \to \mathbb{C} \) satisfy \(|f(x, y)| \leq 1 \) for all \( x, y \), then

\[
\int_{(S^1)^4} f(w, y)f(x, z)C(w, x, y, z)dwdxdydz \geq -\frac{1}{\pi^2}.
\]

Furthermore, for any \( L^2 \) function \( f : S^1 \times S^1 \to \mathbb{C} \), we have that

\[
\int_{(S^1)^4} f(w, y)f(x, z)C(w, x, y, z)dwdxdydz \geq -\frac{2}{\pi^2} \int_{(S^1)^2} |f(x, y)|^2 \sin^2(\pi(x-y))dxdy.
\]

(1)

The proof of Proposition 4 will involve looking at the Fourier transforms of the functions involved. Before we can begin with this we need the following definition:

**Definition.** Define the function

\[ e(x) := e^{2\pi ix}. \]

We now express the Fourier transform of \( C \).

**Lemma 5.** We have that \( C(w, x, y, z) \) is equal to:

\[
\begin{align*}
&-\frac{1}{2\pi^2} \sum_{n, m \in \mathbb{Z}\setminus\{0\}} \frac{1}{nm} (e(nw - nx + my - mz) + e(nw - mx + my - nz)) \\
&+ \frac{1}{2\pi^2} \sum_{n, m \in \mathbb{Z}\setminus\{0\}} \frac{1}{nm} (e(-mx + (n + m)y - nz) + e(nw + my - (n + m)z)) \\
&+ \frac{1}{2\pi^2} \sum_{n, m \in \mathbb{Z}\setminus\{0\}} \frac{1}{nm} (e((n + m)w - nx - mz) - e(nw - (n + m)x + my)) \\
&+ \frac{1}{3}.
\end{align*}
\]
Proof. For \( x \in \mathbb{R}/\mathbb{Z} \) let \([x]\) be the representative of \( x \) lying in \([0, 1]\). For any \( w, x, y, z \in \mathbb{R}/\mathbb{Z} \), it is clear that \([x - w] + [y - x] + [z - y] + [w - z] \in \mathbb{Z} \). It is not hard to see that this number is odd if and only if \((w, y)\) crosses \((x, z)\). Thus,

\[
(-1)^C(w, x, y, z) = e\left(\frac{[x - w]}{2} + \frac{[y - x]}{2} + \frac{[z - y]}{2} + \frac{[w - z]}{2}\right)
\]

\[
= e\left(\frac{[x - w]}{2}\right) e\left(\frac{[y - x]}{2}\right) e\left(\frac{[z - y]}{2}\right) e\left(\frac{[w - z]}{2}\right).
\]

In order to compute the Fourier transform, we compute the Fourier transform of each individual term. Note that

\[
\int e\left(\frac{[x - w]}{2}\right) e(-n x - m w) dx dw = \int e\left(\frac{n}{2}\right) e(-m w - n(x + w)) dw d\alpha
\]

\[
= \int e(-m + n) w - (n - 1/2) \alpha) dw d\alpha
\]

\[
= \frac{\delta_{m, -n}}{\pi i (n - 1/2)}.
\]

Therefore, by standard Fourier analysis, we can say that

\[
e\left(\frac{[x - w]}{2}\right) = \frac{i}{\pi} \sum_{\alpha \in \mathbb{Z}} \frac{e(aw - ax)}{a + 1/2}.
\]

We have similar formulae for \(e\left(\frac{[w - x]}{2}\right), e\left(\frac{[z - y]}{2}\right), \) and \(e\left(\frac{[w - z]}{2}\right)\). Multiplying them together, we find that

\[
(-1)^C(w, x, y, z) = \frac{1}{\pi^4} \sum_{a, b, c, d \in \mathbb{Z}} \frac{e((a - d) w + (b - a) x + (c - b) y + (d - c) z)}{(a + 1/2)(b + 1/2)(c + 1/2)(d + 1/2)}.
\]

We now need to collect like terms. In particular, for every 4-tuple of integers \(\alpha, \beta, \gamma, \delta\), the coefficient of \(e(\alpha w + \beta x + \gamma y + \delta z)\) equals the sum over 4-tuples of integers \(a, b, c, d\) with \(\alpha = a - d, \beta = b - a, \gamma = c - b, \delta = d - c\) of

\[
\frac{1}{\pi^4 (a + 1/2)(b + 1/2)(c + 1/2)(d + 1/2)}.
\]

Clearly, there are no such \(a, b, c, d\) unless \(\alpha + \beta + \gamma + \delta = 0\). If this holds, then all such 4-tuples are of the form \(n, n + \beta, n + \beta + \gamma, n + \beta + \gamma + \delta\) for \(n\) an arbitrary integer. Thus, we need to evaluate

\[
\frac{1}{\pi^4} \sum_{n \in \mathbb{Z}} \frac{1}{(n + 1/2)(n + \beta + 1/2)(n + \beta + \gamma + 1/2)(n + \beta + \gamma + \delta + 1/2)}.
\]

Consider the complex analytic function

\[
g(z) = \frac{\pi \cot(\pi z)}{(z + 1/2)(z + \beta + 1/2)(z + \beta + \gamma + 1/2)(z + \beta + \gamma + \delta + 1/2)}.
\]
Note that along the contour \( \max(|\Re(z)|, |\Im(z)|) = m + 1/2 \) for \( m \) a large integer, \( |g(z)| = O(m^{-4}) \). Thus the limit over \( m \) of the integral of \( g \) over this contour is 0. This implies that the sum of all residues of \( g \) is 0. Note that \( g \) has poles only when either \( z \) is an integer or when \( (z + 1/2)(z + \beta + 1/2)(z + \beta + \gamma + 1/2)(z + \beta + \gamma + \delta + 1/2) = 0 \). At \( z = n \), \( g \) has residue

\[
\frac{1}{(n + 1/2)(n + \beta + 1/2)(n + \beta + \gamma + 1/2)(n + \beta + \gamma + \delta + 1/2)}
\]

Thus,

\[
\sum_{n \in \mathbb{Z}} \frac{1}{(n + 1/2)(n + \beta + 1/2)(n + \beta + \gamma + 1/2)(n + \beta + \gamma + \delta + 1/2)} = -\sum_{\rho \notin \mathbb{Z}} \text{Res}_\rho(g).
\]

Therefore, \((-1)^{C(w,x,y,z)}\) equals

\[
\frac{-1}{\pi} \sum_{\alpha+\beta+\gamma+\delta=0} e(\alpha w + \beta x + \gamma y + \delta z) \sum_{\rho \notin \mathbb{Z}} \text{Res}_\rho(f_{\alpha,\beta,\gamma,\delta}).
\]

Note that all other such residues are at half integers. Note furthermore that \( \cot(\pi z) \) is an odd function around half integers. Thus, \( g \) has a residue at \( z \notin \mathbb{Z} \) only if \( z \) is a root of \( (z + 1/2)(z + \beta + 1/2)(z + \beta + \gamma + 1/2)(z + \beta + \gamma + \delta + 1/2) \) of even order, and in particular order at least 2. In other words, we have residues only when some pair of elements of \((0, \beta, \beta + \gamma, \beta + \gamma + \delta)\) are the same, but no three of them are unless all four are 0. In particular, we get residues in the following cases:

- When \( \beta = 0 \), let \( \alpha = n, \gamma = m \). Then, for \( (\alpha, \beta, \gamma, \delta) = (n, 0, m, -(n+m)) \), we have a residue at \( \rho = -1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \gamma = 0 \), let \( \beta = -n, \delta = -m \). Then, for \( (\alpha, \beta, \gamma, \delta) = (n + m, -n, 0, -m) \), we have a residue at \( \rho = n-1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \delta = 0 \), let \( \alpha = n, \gamma = m \). Then, for \( (\alpha, \beta, \gamma, \delta) = (n, -(n+m), m, 0) \), we have a residue at \( \rho = n-1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \alpha = 0 \), let \( \beta = -n, \delta = -m \). Then, for \( (\alpha, \beta, \gamma, \delta) = (0, -n, n + m, -m) \), we have a residue at \( \rho = -1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \alpha + \beta = 0 \), let \( \alpha = n, \gamma = m \). Then for \( (\alpha, \beta, \gamma, \delta) = (n, -n, m, -m) \), we have a residue at \( \rho = n-1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \beta + \gamma = 0 \), let \( \alpha = n, \gamma = m \). Then for \( (\alpha, \beta, \gamma, \delta) = (n, -m, m, -n) \), we have a residue at \( \rho = -1/2 \) of \( \frac{\pi^2}{nm} \) so long as \( n, m \neq 0 \).
- When \( \alpha = \beta = \gamma = \delta \), we have a residue at \( \rho = -1/2 \).
Thus we have that \((-1)^{C(w,x,y,z)}\) equals
\[
\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e(nw - nx + my - mz) + e(nw - mx + my - nz) \right)
- \frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e(-mx + (n + m)y - nz) + e(nw + my - (n + m)z) \right)
- \frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e((n + m)w - nx - mz) - e(nw - (n + m)x + my) \right) + D.
\]

For some constant \(D\). Noting that \(C(w,x,y,z) = \frac{1-(-1)^{C(w,x,y,z)}}{2}\), we have that \(C(w,x,y,z)\) equals
\[
-\frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e(nw - nx + my - mz) + e(nw - mx + my - nz) \right)
+ \frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e(-mx + (n + m)y - nz) + e(nw + my - (n + m)z) \right)
+ \frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{1}{nm} \left( e((n + m)w - nx - mz) - e(nw - (n + m)x + my) \right) + D'.
\]

On the other hand, \(D' = \int_{S^1} C(w,x,y,z)\). Note that given any \(w,x,y,z\) distinct that of the three ways to partition \(\{w,x,y,z\}\) into two pairs, exactly one gives a set of crossing pairs. Thus \(C(w,x,y,z) + C(w,y,x,z) + C(w,x,z,y)\) equals 1 except on a set of measure 0. Thus, since the integral of each of these is \(D'\), we have that \(3D' = 1\), or that \(D' = 1/3\). This completes the proof. \(\square\)

**Proof of Proposition 4.** Since \(f\) is \(L^2\) we may write
\[
f(x,y) = \sum_{n,m \in \mathbb{Z}} a_{n,m} e(nx + my)
\]
for complex numbers \(a_{n,m}\) with \(\sum_{n,m} |a_{n,m}|^2 < \infty\). Notice that replacing \(f(x,y)\) by \(f(x,y) + f(y,x)\) does not effect the left hand side of Equation (1), and can only increase the right hand side. Thus we can assume that \(f(x,y) = f(y,x)\), and therefore that \(a_{n,m} = a_{m,n}\).

By Lemma 5, the left hand side of Equation (1) is
\[
-\frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{a_{n,m} a_{n,m} + a_{n,m} a_{0,n+m}}{nm} + \frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{a_{0,n+m} a_{0,n+m} + a_{n,m} a_{n,m}}{nm}
+ \frac{1}{2\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{a_{n,m} a_{n,m} + a_{n,m} a_{n+m,0}}{nm} + \frac{a_{0,0} a_{0,0}}{3}.
\]
Using $a_{n,m} = a_{m,n}$, this simplifies to

\[
-\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{a_{n,m}a_{n,m} - a_{n,m+1,m} - a_{n+1,m}a_{n,m}}{nm} + \frac{|a_{0,0}|^2}{3}
\]

\[
= -\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{|a_{n,m} - a_{n+1,m}|^2 - |a_{n,m+1}|^2}{nm} + \frac{|a_{0,0}|^2}{3}
\]

\[
= -\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{|a_{n,m} - a_{n+1,m}|^2}{nm} + \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}} |a_{k,0}|^2 \left( \sum_{n+m=k} \frac{1}{nm} \right) + \frac{|a_{0,0}|^2}{3}.
\]

We claim that for $k \neq 0$ that $\sum_{n+m=k} \frac{1}{nm} = 0$. This can be seen by considering the residues of the analytic function

\[
g(z) = \frac{\pi \cot(\pi z)}{z(k-z)}.
\]

Note that along the contour $\max(|\Re(z)|, |\Im(z)|) = m + 1/2$ for $m$ a large integer, $|g(z)| = O(m^{-2})$. Thus the limit over $m$ of the integral of $g$ over this contour is $0$. This implies that the sum of all residues of $g$ is $0$. It is clear that $g$ has residues only at integers. At $z = n$ for $n \neq 0, k$, it has residue $\frac{1}{nm}$. If $k = 0$, it has residue $0$ at $0$ and $k$. Thus, the sum of residues is exactly $\sum_{n+m=k} \frac{1}{nm}$.

Furthermore, if $k = 0$,

\[
\frac{1}{n^2} = -\frac{1}{n^2} = -2\zeta(2) = -\frac{\pi^2}{3}.
\]

Therefore, the left hand side of Equation (1) is

\[
-\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{|a_{n,m} - a_{n+1,m}|^2}{nm} + \frac{1}{\pi^2} |a_{0,0}|^2 \left( \frac{\pi^2}{3} \right) + \frac{|a_{0,0}|^2}{3}
\]

\[
= -\frac{1}{\pi^2} \sum_{n,m \in \mathbb{Z} \setminus \{0\}} \frac{|a_{n,m} - a_{n+1,m}|^2}{nm}.
\]

The right hand side of Equation (1) is

\[
-\frac{1}{\pi^2} \sum_{n,m} \frac{a_{n,m} (2a_{n,m} - a_{n+1,m} - a_{n-1,m+1})}{2}
\]

\[
= -\frac{1}{\pi^2} \sum_{n,m} a_{n,m} (a_{n,m} - a_{n+1,m-1})
\]

\[
= -\frac{1}{2\pi^2} \sum_{n,m} |a_{n,m} - a_{n+1,m-1}|^2
\]

\[
= -\frac{1}{2\pi^2} \sum_{n,m} \left( |a_{n,m} - a_{n+1,m}| - |a_{n+1,m-1} - a_{n+1,m}| \right)^2.
\]
We now let \( b_{n,m} = a_{n,m} - a_{n+m,0} \). Notice that \( b_{0,k} = b_{k,0} = 0 \). Equation (1) is now equivalent to
\[
\sum_{n,m \neq 0} \frac{-|b_{n,m}|^2}{nm} + \sum_{n,m} \frac{|b_{n,m} - b_{n+1,m-1}|^2}{2} \geq 0.
\]

We will in fact prove the stronger statement that
\[
\sum_{nm > 0} \frac{|b_{n,m}|^2}{nm} \leq \sum_{(n+1)m > 0} \frac{|b_{n,m} - b_{n+1,m-1}|^2}{2}.
\]

We note by symmetry that we can assume that \( n,m > 0 \). We also note that it suffices to prove for each \( k > 0 \) that
\[
\sum_{n,m > 0, n+m=k} |b_{n,m}|^2 \leq \sum_{n+1,m > 0, n+m=k} \frac{|b_{n,m} - b_{n+1,m-1}|^2}{2}.
\]

For fixed \( k \), let \( c_n = b_{n,k-n} - b_{n-1,k-n+1} \). By the symmetry exhibited by the \( b \)'s, the right hand side of Equation (2) is
\[
\sum_{n=1}^{\lfloor k/2 \rfloor} |c_n|^2.
\]

Meanwhile, the right hand side is
\[
\sum_{n=1}^{\lfloor k/2 \rfloor} \frac{|\sum_{i=1}^{n} c_i|^2}{n(k-n)} + \sum_{n=1}^{\lfloor (k-1)/2 \rfloor} \frac{|\sum_{i=1}^{n} c_i|^2}{n(k-n)}.
\]

Thus, the right hand side is given by a quadratic form in the \( c_1, \ldots, c_{\lfloor k/2 \rfloor} \) with positive coefficients. Therefore, the biggest ratio between the right and left sides is obtained by the unique eigenvector of this quadratic form for which all \( c_i \) are positive. We claim that this happens when \( c_n = k + 1 - 2n \). For these \( c \)'s, the derivative of the expression in Equation (3) with respect to \( c_m \) is
\[
2 \sum_{n=m}^{\lfloor k/2 \rfloor} \frac{(\sum_{i=1}^{n} c_i)^2}{n(k-n)} + 2 \sum_{n=m}^{\lfloor (k-1)/2 \rfloor} \frac{(\sum_{i=1}^{n} c_i)^2}{n(k-n)}.
\]

It is easy to verify that for this choice of \( c_i \) that
\[
\sum_{i=1}^{n} c_i = n(k-n).
\]

Thus, the above reduces to
\[
2 \sum_{n=m}^{\lfloor k/2 \rfloor} 1 + 2 \sum_{n=m}^{\lfloor (k-1)/2 \rfloor} 1 = 2([k/2] - m + 1) + 2([(k-1)/2] - m + 1)
\]
\[
= 2(k - 2m + 1) = 2c_m.
\]
Thus, these \(c_i\) give the unique positive eigenvector. Hence it suffices to check Equation (2) when \(c_m = k - 2n + 1\), or equivalently when \(b_{n,k} = n(k - n)\). In this case, the left hand side of Equation (2) is

\[
\sum_{n=1}^{k-1} n(k-n) = \sum_{n=1}^{k-1} (kn-n^2)
\]

\[
= \frac{k^2(k-1)}{2} - \frac{(k-1)(2k-1)}{6}
\]

\[
= \frac{k(k-1)(k+1)}{6}
\]

\[
= \frac{k^3 - k}{6}.
\]

For this choice, the right hand side is

\[
\sum_{n=1}^{k} \frac{(k+1-2n)^2}{2} = \sum_{n=1}^{k} \frac{k^2 + 2k - 4kn + 1 - 4n + 4n^2}{2}
\]

\[
= \frac{k^3}{2} + k^2 - k^2(k+1) + \frac{k}{2} - k(k+1) + \frac{k(k+1)(2k+1)}{3}
\]

\[
= \frac{3k^3 + 6k^2 - 6k^3 - 6k^2 + 3k - 6k^2 - 6k + 4k^3 + 6k^2 + 2k}{6}
\]

\[
= \frac{k^3 - k}{6}.
\]

Thus, the largest possible ratio between the left and right hand sides of Equation (2) is 1. This completes our proof.

We are now prepared to prove our main theorem.

**Proof of Theorem 1.** We will proceed by way of Lemma 2. We note that \(|E(G_n)| = n^4/24 + O(n^3)\). We have only to bound the size of the MAX-CUT of \(G_n\). Consider any subset \(S\) of the vertices of \(G_n\) defining a cut. We wish to bound the number of edges that cross this cut. Define the function \(f_S : S^1 \times S^1 \to \{\pm 1\}\) as follows:

\[
f_S(x,y) = \begin{cases} 1 & \text{if } ([nx], [ny]) \in S \\ -1 & \text{otherwise} \end{cases}
\]

Consider

\[
\int_{(S^1)^4} f_S(w,y)f_S(x,z)C(w,x,y,z)dwdx dydz. \tag{4}
\]

In order to evaluate this expression, we consider the integral over the region

\[
R_{a,b,c,d} = [a/n, (a+1)/n] \times [b/n, (b+1)/n] \times [c/n, (c+1)/n] \times [d/n, (d+1)/n]
\]

for some \(a, b, c, d \in \mathbb{Z}/n\). We note that over this region that \(f_S(w,y)f_S(x,z)\) is constant. In particular, it is 1 if \((a, c)\) and \((b, d)\) are either both in \(S\) or both
not in $S$, and $-1$ otherwise. It should also be noted that if $a, b, c, d$ are distinct then $C(w, x, y, z)$ is also constant on this region, and in particular is 1 if $G_n$ contains an edge between $(a, c)$ and $(b, d)$. Thus the expression in Equation (4) is

$$\sum_{a,b,c,d} \int_{R_{a,b,c,d}} f_S(w,y)f_S(x,z)C(w,x,y,z)dwdxdydz$$

$$= \sum_{a,b,c,d, \text{non-distinct}} \int_{R_{a,b,c,d}} O(1) + \sum_{\{a, c\}, \{b, d\} \in E(G_n)} \frac{f_S(a/n, c/n)f_S(b/n, d/n)}{n^4}$$

$$= 8n^{-4}(|\text{Edges not crossing the cut}| - |\text{Edges crossing the cut}|) + O(n^{-1}).$$

On the other hand, by Proposition 3, this is at least $-\frac{1}{\pi^2}$. Thus

$$|\text{Edges crossing the cut}| - |\text{Edges not crossing the cut}| \leq \frac{n^4}{8\pi^2} + O(n^3).$$

Adding the number of edges of $G_n$ and dividing by 2, we find that

$$|\text{Edges crossing the cut}| \leq n^4 \left( \frac{1}{16\pi^2} + \frac{1}{48} \right) + O(n^3).$$

This provides an upper bound on the size of MAX-CUT($G_n$). Thus by Lemma 2, the crossing number of $K_n$ is at least

$$n^4 \left( \frac{1}{48} - \frac{1}{16\pi^2} \right) + O(n^3).$$

This completes our proof. □