Problem 1 — 4.22 *

Verify that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{2x - 1}{2x(1-x)}$$

for all $x \in (0, 1)$ is a bijection.

Problem 2 — 4.29

Consider three functions $f, g, h : \mathbb{R} \to \mathbb{R}$, defined for all $x \in \mathbb{R}$ by

$$f(x) = \frac{x}{1 + x^2}, \quad g(x) = \frac{x^2}{1 + x^2}, \quad h(x) = \frac{x^3}{1 + x^2}$$

(a) Determine which of these functions are injective.

(b) Prove that $f$ and $g$ are not surjective.

(c) Graph all three functions.

Problem 3 — 4.30 *

Given real numbers $a, b, c, d$, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (ax + by, cx + dy)$ for all $(x, y) \in \mathbb{R}^2$. Prove that $f$ is injective if and only if $f$ is surjective.
Problem 4 — 4.34 *

Let \( f : A \to B \) and \( g : B \to C \) be functions, and define \( h = g \circ f \). Determine which of the following statements are true, giving proofs for the true statements and counterexamples for the false statements:

(a) If \( h \) is injective, then \( f \) is injective.
(b) If \( h \) is injective, then \( g \) is injective.
(c) If \( h \) is surjective, then \( f \) is surjective.
(d) If \( h \) is surjective, then \( g \) is surjective.

Problem 5 — 4.36

Consider functions \( f : A \to B \) and \( g : B \to A \). Prove that

(a) If \( f \circ g \) is the identity function on \( B \), then \( f \) is surjective.
(b) If \( g \circ f \) is the identity function on \( A \), then \( f \) is injective.

To remind you: given a set \( X \), the identity function on \( X \) is the function \( \text{id}_X : X \to X \) defined by \( \text{id}_X(x) = x \) for all \( x \in X \).

Problem 6 — 4.37

Consider a function \( f : A \to A \). Prove that if \( f \circ f \) is injective, then \( f \) is injective.

Problem 7 — 4.45 *

Let \( A \) be a set and let \( f : A \to A \) be a function. Prove that if \( A \) is finite, then \( f \) is injective if and only if \( f \) is surjective; and that if \( A \) is infinite, then this equivalence need not hold.

Problem 8 — 4.47

Prove that the set of all natural numbers, the set of all even natural numbers, and the set of all odd natural numbers all have the same cardinality.
Problem 9 — 4.51 *

Construct an explicit bijection from the open interval $(0, 1)$ to the closed interval $[0, 1]$.

Problem 10 *

Fix a prime number $p \in \mathbb{N}$, and define

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

Prove that the function $f : S \to S$ defined for all $(x, y, z) \in S$ by

$$f(x, y, z) = \begin{cases} 
(x + 2z, z, y - x - z) & \text{if } x < y - z \\
(2y - x, y, x - y + z) & \text{if } y - z < x < 2y \\
(x - 2y, x - y + z, y) & \text{if } x > 2y
\end{cases}$$

is a bijection and is its own inverse (i.e. $f^{-1} = f$).

[Optional: Prove that if $p = 4k + 1$ for some $k \in \mathbb{N}$, then $f(x, y, z) = (x, y, z)$ for exactly one triple $(x, y, z) \in S$.]

Problem 11 *

Let $f : A \to B$ be a function.

(a) Prove that there exists a set $X$ and functions $p : A \to X$ and $i : X \to B$, with $p$ surjective and $i$ injective, such that $f = i \circ p$.

(b) Prove that there exists a set $Y$ and functions $j : A \to Y$ and $q : Y \to B$, with $j$ injective and $q$ surjective, such that $f = q \circ j$. 