

AN AUTOMORPHISM GROUP OF AN ω -STABLE STRUCTURE THAT IS NOT LOCALLY (OB)

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ABSTRACT. We observe that there is an example of an automorphism group of a model of an ω -stable theory—in fact, the prime model of an uncountably categorical theory—that is not locally (OB), answering a question of C. Rosendal.

1. INTRODUCTION

Towards extending the techniques of geometric group theory to all topological groups, C. Rosendal, in [7], identifies, for a general topological group, the appropriate notion of “boundedness”. The sets with this property play the role of the compact subsets of a locally compact group and norm-bounded subsets of (the additive group of) a Banach space—and indeed, coincide with these examples for the above classes of groups. Here, the sets with the *relative property (OB)* are those that are inexorably bounded, in the sense that they take finite diameter with respect to *every* continuous, left-invariant pseudometric on the group.

Recall that a *coarse structure* on a set, X , is any family of subsets of X^2 extending the powerset of the diagonal and closed under subsets, unions, inverses, and compositions of relations. For example, a coarse structure naturally arising from a metric space, (X, d) , consists of those sets $E \subseteq X^2$ such that $\sup\{d(x, y) \mid (x, y) \in E\}$ is finite (see [5]). The family of sets with the relative property (OB) forms an ideal, stable under the group operations, and thereby induces a left-invariant coarse structure on the group generated by the entourages, $\{(x, y) \mid x^{-1}y \in A\}$, as A varies over relatively (OB) sets.

Associated to this concept are several attributes that a given topological group may possess. A group is *locally (OB)* if there is an open neighborhood of the identity element with the property (OB). For a broad class of topological groups, this completely coincides with the situation where the above coarse structure may be given by a metric. Additionally, the group simply *has the property (OB)* when every subset has the relative property (OB) as above, i.e., when the group has finite diameter with respect to every continuous, left-invariant pseudometric. These are the groups for which the above coarse structure is trivial.

Much of the motivation for better understanding “large” topological groups is that they often arise as transformations of important mathematical objects, e.g., homeomorphism groups of compact topological spaces, isometry groups of metric spaces, diffeomorphism groups of manifolds, and automorphism groups of

1991 *Mathematics Subject Classification*. Primary 03E15; Secondary 22A05, 22F50, 03C15.

Key words and phrases. coarse geometry of Polish groups, automorphism groups of countable structures, property (OB).

countable structures in model theory. The coarse geometry of the groups from this latter class received a more thorough treatment in [6]. One of the main results of that paper was,

Theorem 1 (Rosendal). *If \mathcal{M} is the countable, saturated model of an ω -stable theory, then $\text{Aut}(\mathcal{M})$ has the property (OB).*

Recall that a theory is ω -stable when there are only countably-many complete types over countable parameter sets, and that a countable model is *saturated* when it realizes all types over finite parameter sets. This theorem and its proof led to the conjecture,

Question (Rosendal). *If \mathcal{M} is any model of an ω -stable theory, must $\text{Aut}(\mathcal{M})$ be locally (OB)?*

Here we answer this question in the negative, namely,

Theorem 2. *There is a countable structure, \mathcal{M} , for which $\text{Th}(\mathcal{M})$ is uncountably categorical, \mathcal{M} is its prime model, and $\text{Aut}(\mathcal{M})$ is not locally (OB).*

As uncountably categorical theories are, in particular, ω -stable ([4] Theorem 3.8), such a group serves as a counterexample to the above question.

I am grateful to the anonymous referee for their helpful comments.

2. PRELIMINARIES

The important notions for the coarse geometry of automorphism groups were introduced above. Let us recall here, from [7], an important source of examples of groups that are not locally (OB).

Lemma 3. *A product of groups, $G = \prod_{i \in I} G_i$, is locally (OB) if and only if G_i has the (full) property (OB) for all-but-finitely-many $i \in I$.*

Proof. As it suffices for our needs here, we will show just (the contrapositive of) the “only if” direction, and for metrizable G . A full proof is found in Proposition 13 of [7]. Let $U \subseteq G$ be open. Then for some $j \in I$, the projection of U onto the j th coordinate is all of G_j , and G_j does not have the property (OB). Let d be a compatible, left-invariant metric on G_j of infinite diameter, and let ρ be any compatible, left-invariant metric on G . Then $\rho((g_i)_{i \in I}, (h_i)_{i \in I}) + d(g_j, h_j)$ is a compatible, left-invariant metric on G assigning infinite diameter to U . Therefore, U does not have the property (OB), and as it was an arbitrary open set, G is not locally (OB). \square

For the purposes here, “theory” will mean “full theory of an infinite structure in a countable language”. Recall that, for an infinite cardinal, κ , a theory is κ -categorical if it has exactly one model, up to isomorphism, of cardinality κ , and that by a foundational result of M. Morley, a theory is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinals. Such theories are then unambiguously termed *uncountably categorical*. If such a theory is also ω -categorical, then it is said to be *totally categorical*.

Given a structure, \mathcal{M} , and a tuple $\bar{a} \in M^n$, a formula $\varphi(x, \bar{a})$ is *strongly minimal* if it defines an infinite set, and in every elementary extension of \mathcal{M} , every definable subset of the set defined by φ is either finite or cofinite. Strongly minimal formulas

(and the *strongly minimal sets* they define) play a fundamental role in the structure theory of uncountably categorical theories.

Theorem 4 (Baldwin-Lachlan [1]). *If a theory, T , is uncountably categorical, then there is a strongly minimal $\varphi(x, \bar{a})$ (with parameters from the prime model), and models of T are determined, up to isomorphism, by the minimal cardinality of a set, $B \subseteq \varphi(\mathcal{M}, \bar{a})$ for which $\varphi(\mathcal{M}, \bar{a}) \subseteq \text{acl}(\bar{a} \cup B)$.*

Here, $\varphi(\mathcal{M}, \bar{a})$ is the set of points in M defined by $\varphi(x, \bar{a})$, and the *algebraic closure* of a set, $C \subset M$, is $\text{acl}(C) = \bigcup \{\varphi(\mathcal{M}, \bar{c}) \mid \bar{c} \subseteq C \text{ and } |\varphi(\mathcal{M}, \bar{c})| < \infty\}$, the union of all finite sets definable with parameters from C . A theory is *strongly minimal* if $x = x$ is strongly minimal (i.e., every definable subset of every model is finite or cofinite), and *almost strongly minimal* if every model is algebraic over a strongly minimal set. Such theories are uncountably categorical.

Let us remark that the example in the following section has a theory that is uncountably categorical, but not totally categorical, not almost strongly minimal, and for which \mathcal{M} is not the saturated model. In fact, for an uncountably categorical structure to be a counterexample, these additional properties are necessary.

Proposition 5. *Suppose \mathcal{M} is a countable structure and $T = \text{Th}(\mathcal{M})$ is uncountably categorical. Then if T is totally categorical, or more generally if \mathcal{M} is the countable, saturated model of T , or if T is almost strongly minimal, then $\text{Aut}(\mathcal{M})$ is locally (OB).*

Proof. The case where $\text{Th}(\mathcal{M})$ is ω -categorical is due to P. Cameron, and as mentioned above, was extended by Rosendal to saturated models of ω -stable theories. In both cases $\text{Aut}(\mathcal{M})$ has the property (OB). So suppose that \mathcal{M} is not saturated, but that $\text{Th}(\mathcal{M})$ is almost strongly minimal. Then there is a strongly minimal formula, $\varphi(x, \bar{a})$, and $M = \text{acl}(\varphi(\mathcal{M}, \bar{a}))$. As \mathcal{M} is not saturated, there is a finite $B \subseteq \varphi(\mathcal{M}, \bar{a})$ so that $\varphi(\mathcal{M}, \bar{a}) \subseteq \text{acl}(\bar{a} \cup B)$, for it follows from [1] that the model for which such a B must be infinite is saturated.

Therefore, $M = \text{acl}(\bar{a} \cup B)$. Let $V \leq \text{Aut}(\mathcal{M})$ be the stabilizer subgroup of $\bar{a} \cup B$. Then V is an open subgroup and as \mathcal{M} is algebraic over $\bar{a} \cup B$, every $c \in M$ has a finite V -orbit. As V naturally embeds as a closed subgroup of the compact group, $\prod_{c \in M} S_{V \cdot c}$, it too is compact (see also [3]), and so $\text{Aut}(\mathcal{M})$ is locally compact, and thus locally (OB). \square

We remark that all natural and frequently-cited examples of uncountably categorical structures possess at least one of the aforementioned properties, contributing to the plausibility of the conjecture refuted here.

3. THE EXAMPLE

The example here, of an uncountably categorical structure whose automorphism group is not locally (OB), is essentially that of section 4 of [1], with some modifications. Therefore, in the proof of categoricity we will provide only an outline, referring the reader to the above paper for more details on that aspect.

Let $\mathcal{L} = \{f, R, 0, 1\}$, a language with a ternary function symbol, a binary relation symbol, and two constants. Let $M = \mathbb{Q} \cup \mathbb{Q}^2$. Interpret the symbols 0 and 1 as the corresponding elements of \mathbb{Q} , and let $R^M = \{(p, (q, r)) \in M^2 \mid p = q\}$. We define f^M by cases:

- $f^M(p, q, r) = (q - p) + r$ for $p, q, r \in \mathbb{Q}$

- $f^{\mathcal{M}}((s, p), (s, q), (s, r)) = (s, (q - p) + r)$ for $p, q, r, s \in \mathbb{Q}$
- $f^{\mathcal{M}}(p, (p, q), r) = (p, (q - p) + r)$ for $p, q, r \in \mathbb{Q}$
- $f^{\mathcal{M}}(a, b, c) = c$ if (a, b, c) is not one of the above forms

One should keep in mind the following picture of \mathcal{M} : It consists of a “parent” copy of \mathbb{Q} , and corresponding to each of its elements, a “child” copy of \mathbb{Q} . The elements 0 and 1 of the parent copy are distinguished, and the relation $R^{\mathcal{M}}(a, b)$ holds precisely when a is a member of the parent copy and b is a member of the child copy associated to a .

The function $f^{\mathcal{M}}$ is best considered not as a three-variable function, but as a family of single-variable functions parameterized by pairs of elements of \mathcal{M} . That is, $f^{\mathcal{M}}(a, b, c)$ should be viewed as the value that c takes in the function determined by (a, b) . So the first condition says that if a and b are both in the parent copy of \mathbb{Q} , then $f^{\mathcal{M}}(a, b, \cdot)$ acts as a translation of the parent copy by $(b - a)$, and as the identity on the child copies. Similarly if a and b are in the same child copy, then $f^{\mathcal{M}}(a, b, \cdot)$ translates that child copy. The third case is probably the least intuitive, but if c is in the parent copy and $R^{\mathcal{M}}(a, b)$, then $f^{\mathcal{M}}(a, b, c)$ is best described as “where c would go if the parent copy was laid on top of the child copy corresponding to a , in such a way that a was made to line up with b ”.

Proposition 6 (Baldwin-Lachlan [1]). *$\text{Th}(\mathcal{M})$ is uncountably categorical.*

Proof. First, note that it suffices to show that \mathcal{M}' , the reduct of \mathcal{M} to the language $\mathcal{L}' = \{f, R\}$, is uncountably categorical. Next, we see that the structure (\mathbb{Q}, F) where $F(p, q, r) = (q - p) + r$ is strongly minimal. To see this, first verify, by induction, that for every term $t(x_1, \dots, x_n)$ there are $r_1, \dots, r_n \in \mathbb{Z}$, summing to 1, so that (\mathbb{Q}, F) interprets $t(a_1, \dots, a_n)$ as $r_1 a_1 + \dots + r_n a_n$ for every choice of a_1, \dots, a_n . Therefore, for every atomic formula $\varphi(x_1, \dots, x_n)$ there are $r_1, \dots, r_n \in \mathbb{Z}$, summing to 0, so that $(\mathbb{Q}, F) \models \varphi(a_1, \dots, a_n)$ if and only if $r_1 a_1 + \dots + r_n a_n = 0$. Then by induction on the construction of formulas, every \emptyset -definable relation in (\mathbb{Q}, F) is a Boolean combination of sets of the form

$$\{(a_1, \dots, a_n) \in \mathbb{Q}^n \mid r_1 a_1 + \dots + r_n a_n = 0\}$$

where the r_i 's sum to 0. So for every $\varphi(x_1, \dots, x_n)$, there is a $k_\varphi \in \mathbb{N}$ so that for any a_2, \dots, a_n , either $\varphi(x_1, a_2, \dots, a_n)$ or $\neg\varphi(x_1, a_2, \dots, a_n)$ has at most k_φ solutions. This fact is expressible in a first-order manner, and so in every model of $\text{Th}(\mathbb{Q}, F)$, every set defined by φ with parameters is either of size less than k_φ or has complement with this bound. Therefore, $\text{Th}(\mathbb{Q}, F)$ is strongly minimal.

Hence, $\psi(x) = \exists y R(x, y)$ is a strongly minimal formula in $\text{Th}(\mathcal{M}')$. Suppose \mathcal{N}_1 and \mathcal{N}_2 are \aleph_1 -models of $\text{Th}(\mathcal{M}')$. In \mathcal{M} , for every a and b with $R^{\mathcal{M}}(a, b)$, the restriction of $f^{\mathcal{M}}(a, b, \cdot)$ to $\psi(\mathcal{M})$ is one-to-one and onto $R(a, \mathcal{M})$ —in fact, it is an isomorphism of $\{f\}$ -structures. This is expressible in \mathcal{L}' , and is therefore known to $\text{Th}(\mathcal{M})$, and consequently for each $i = 1, 2$, $|\psi(\mathcal{N}_i)| = \aleph_1$ —otherwise, as each $R(a, \mathcal{N}_i)$ is in bijection with $\psi(\mathcal{N}_i)$, \mathcal{N}_i would be a countable union of countable sets. Therefore, as the $\psi(\mathcal{N}_i)$'s are models of the uncountably categorical theory, $\text{Th}(\mathbb{Q}, F)$, there is an $\{f\}$ -isomorphism $g : \psi(\mathcal{N}_1) \rightarrow \psi(\mathcal{N}_2)$. Extend g to all of \mathcal{N}_1 by choosing, for each $a \in \psi(\mathcal{N}_1)$, a point c_a with $R^{\mathcal{N}_1}(a, c_a)$, and likewise for each element of $\psi(\mathcal{N}_2)$. Then if $d \in \mathcal{N}_1 \setminus \psi(\mathcal{N}_1)$, say if $R^{\mathcal{N}_1}(a, d)$,

let $g(d) = f^{N_2}(g(a), c_{g(a)}, g((f^{N_1})^{-1}(a, c_a, d)))$, where $(f^{N_1})^{-1}(a, c_a, d)$ denotes the (unique) element, b , of $\psi(N_1)$, for which $f^{N_1}(a, c_a, b) = d$. One then verifies that this extension of g is an \mathcal{L}' -isomorphism.

For this, first note a few points that are clear from the construction. Namely, that it preserves R , that for $a, b, c \in \psi(N_1)$ it extends a map respecting f , and similarly for $b, c, d \in R(a, N_1)$, as it is a composition of the $\{f\}$ -isomorphisms $(f^{N_1})^{-1}(a, c_a, \cdot)$, $g \upharpoonright \psi(N_1)$, and $f^{N_2}(g(a), c_{g(a)}, \cdot)$. For the situation where $N_1 \models R(a, d) \wedge \psi(b)$, observe that in the special case of $d = c_a$,

$$g(f^{N_1}(a, c_a, b)) = f^{N_2}(g(a), c_{g(a)}, g((f^{N_1})^{-1}(a, c_a, f^{N_1}(a, c_a, b)))) = f^{N_2}(g(a), c_{g(a)}, g(b)),$$

and so for more general $d \in R(a, N_1)$,

$$\begin{aligned} f^{N_2}(g(a), g(d), g(b)) &= f^{N_2}(g(a), f^{N_2}(g(a), c_{g(a)}, g((f^{N_1})^{-1}(a, c_a, d))), g(b)) \\ &= f^{N_2}(g(a), c_{g(a)}, f^{N_2}(g(a), g((f^{N_1})^{-1}(a, c_a, d)), g(b))) \\ &= f^{N_2}(g(a), c_{g(a)}, g(f^{N_1}(a, (f^{N_1})^{-1}(a, c_a, d), b))) \\ &= g(f^{N_1}(a, c_a, f^{N_1}(a, (f^{N_1})^{-1}(a, c_a, d), b))) \\ &= g(f^{N_1}(a, f^{N_1}(a, c_a, (f^{N_1})^{-1}(a, c_a, d)), b)) \\ &= g(f^{N_1}(a, d, b)), \end{aligned}$$

where the second and second-to-last equalities follow from the fact that

$$\mathcal{M} \models \forall x \forall y \forall z \forall w (R(x, y) \wedge \psi(z) \wedge \psi(w)) \longrightarrow f(x, f(x, y, z), w) = f(x, y, f(x, z, w)).$$

Finally, the extension of g defined above preserves the cases used in the definition of f , so $g(f^{N_1}(a, b, c)) = g(c) = f^{N_2}(g(a), g(b), g(c))$ in the remaining one. Therefore, N_1 and N_2 are isomorphic. \square

So $\text{Th}(\mathcal{M})$ is uncountably categorical, and we will see below that every element of $\psi(\mathcal{M})$ is definable over \emptyset , so \mathcal{M} is its prime model.

Proposition 7. *Aut(\mathcal{M}) is isomorphic, as a topological group, to $\mathbb{Q}^{\mathbb{Q}}$, and therefore is not locally (OB). That is, the coarse structure associated to the relatively (OB) subsets of Aut(\mathcal{M}) cannot be given by a metric.*

Proof. Observe that every element of the structure (\mathbb{Q}, F) introduced in the proof of Proposition 6 is definable over $\{0, 1\}$. For this, let $F_{(a,b)}$ denote $F(a, b, \cdot)$, and observe that for $n \in \mathbb{N}$,

$$n = F_{(0,1)}^n(0) \text{ and } -n = F_{(1,0)}^n(0),$$

while $\frac{k}{n} \in \mathbb{Q}$ ($n \geq 1$) is the unique x for which,

$$k = F_{(0,x)}^n(0).$$

Therefore, every automorphism of (\mathbb{Q}, F) is determined by where it sends 0 and 1.

Now suppose $g \in \text{Aut}(\mathcal{M})$. Then as every point in the strongly minimal set, $\psi(\mathcal{M})$, is definable over \emptyset (recall \mathcal{L} contains symbols for 0 and 1), g must fix $\psi(\mathcal{M})$ pointwise, and so for every $a \in \psi(\mathcal{M})$, fixes $R(a, \mathcal{M})$ setwise. Let $g_a : \mathbb{Q} \rightarrow \mathbb{Q}$ be the automorphism of (\mathbb{Q}, F) induced by the action of g on $R(a, \mathcal{M})$, satisfying $g(a, b) = (a, g_a(b))$.

Recall that $f^{\mathcal{M}}(p, (p, q), r) = (p, (q - p) + r)$. So for any $x \in \mathbb{Q}$,

$$\begin{aligned}
(a, g_a(x)) &= g(a, x) \\
&= g(a, (0 - a) + (x + a)) \\
&= g(f^{\mathcal{M}}(a, (a, 0), x + a)) \\
&= f^{\mathcal{M}}(g(a), g(a, 0), g(x + a)) \\
&= f^{\mathcal{M}}(a, (a, g_a(0)), x + a) \text{ (since } g \upharpoonright \psi(\mathcal{M}) = \text{id)} \\
&= (a, (g_a(0) - a) + (x + a)) \\
&= (a, x + g_a(0))
\end{aligned}$$

and g_a is a translation by $g_a(0)$.

So every $g \in \text{Aut}(\mathcal{M})$ fixes $\psi(\mathcal{M})$ and restricts to a translation on each $R(a, \mathcal{M})$. In this way, it can be naturally identified with a point in $\mathbb{Q}^{\mathbb{Q}}$ given by $(g_a(0))_{a \in \mathbb{Q}}$. Conversely, suppose $h \in \mathbb{Q}^{\mathbb{Q}}$. Let $\widehat{h} : M \rightarrow M$ fix $\psi(\mathcal{M})$ and send $(p, q) \mapsto (p, q + h(p))$. Then \widehat{h} is easily seen to respect $R, 0, 1$, and the first, second, and fourth parts of the definition of f , while for the third,

$$\begin{aligned}
\widehat{h}(f^{\mathcal{M}}(p, (p, q), r)) &= \widehat{h}(p, (q - p) + r) \\
&= (p, (q - p) + r + h(p)) \\
&= f^{\mathcal{M}}(p, (p, q + h(p)), r) \\
&= f^{\mathcal{M}}(\widehat{h}(p), \widehat{h}(p, q), \widehat{h}(r)),
\end{aligned}$$

and $\widehat{h} \in \text{Aut}(\mathcal{M})$. So $\text{Aut}(\mathcal{M})$ can be identified with $\mathbb{Q}^{\mathbb{Q}}$, and as a basic open set in $\text{Aut}(\mathcal{M})$ is determined by its action on finitely-many points (i.e., fixes the values of $h(a)$ for finitely-many a), they are isomorphic as topological groups when the base \mathbb{Q} is given the discrete group topology.

Therefore, $\text{Aut}(\mathcal{M})$ is an infinite product of groups that are not (OB), and so is not locally (OB), by Lemma 3. \square

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