Solutions for Problem Set Three

Problem 1. (1) The supermartingale property with respect to the filtration \( \{ F_{\tau \wedge t} \} \) is a direct consequence of the optional sampling theorem for bounded stopping times. As for the original filtration, first observe that

\[
X_{\tau \wedge s} \geq \mathbb{E}[X_{\tau \wedge t}|F_{\tau \wedge s}] = \mathbb{E}[X_{\tau \wedge t}1_{\{\tau \leq s\}}|F_{\tau \wedge s}] + \mathbb{E}[X_{\tau \wedge t}1_{\{\tau > s\}}|F_{\tau \wedge s}]
\]

for \( s \leq t \). The first term equals \( \mathbb{E}[X_{\tau \wedge t}1_{\{\tau \leq s\}}|F_s] \) since \( X_{\tau \wedge t}1_{\{\tau \leq s\}} = X_{\tau \wedge s}1_{\{\tau \leq s\}} \) is \( F_{\tau \wedge s} \)-measurable. The second term equals \( \mathbb{E} [1_{\{\tau > s\}} \mathbb{E}[X_{\tau \wedge t}|F_s]|F_{\tau \wedge s}] \), where the integrand

\[
1_{\{\tau > s\}} \mathbb{E}[X_{\tau \wedge t}|F_s] \in F_{\tau \wedge s}.
\]

Therefore,

\[
X_{\tau \wedge s} \geq \mathbb{E}[X_{\tau \wedge t}1_{\{\tau \leq s\}}|F_s] + 1_{\{\tau > s\}} \mathbb{E}[X_{\tau \wedge t}|F_s] = \mathbb{E}[X_{\tau \wedge t}|F_s].
\]

(2) Let \( s < t \) and \( A \in F_s \). Define \( \sigma = s1_A + t1_{A^c} \) and \( \tau = t \). It is obvious that \( \sigma, \tau \) are bounded \( \{F_t\} \)-stopping times. Therefore,

\[
\mathbb{E}[X_{\sigma}] = \mathbb{E}[X_s1_A] + \mathbb{E}[X_t1_{A^c}] \leq \mathbb{E}[X_t] = \mathbb{E}[X_t],
\]

which implies the desired submartingale property.

Problem 2. (1) Let \( s < t \) and \( A \in F_s \). Since \( F_s \subseteq F_t \), we have

\[
\int_A M_t d\mathbb{P} = \mathbb{Q}(A) = \int_A M_s d\mathbb{P}.
\]

Therefore, \( \{M_t, F_t\} \) is a martingale.

(2) Necessity. Suppose that \( \{M_t\} \) is uniformly integrable. Then \( M_t \to M_\infty \) almost surely and in \( L^1 \) for some \( M_\infty \in F_\infty \). Let \( A \in F_t \) for some \( t > 0 \). Then for any \( u > t \), we have \( A \in F_u \) and thus

\[
\mathbb{Q}(A) = \int_A M_u d\mathbb{P}.
\]
By letting \( u \to \infty \), we obtain that
\[
Q(A) = \int_A M_\infty d\mathbb{P}.
\]
This is indeed true for all \( A \in \mathcal{F}_\infty \) by the monotone class theorem, since \( \mathcal{F}_\infty \) is generated by the \( \pi \)-system \( \cup_{t \geq 0} \mathcal{F}_t \). Therefore, \( Q \ll \mathbb{P} \) when restricted on \( \mathcal{F}_\infty \) with the Radon-Nikodym derivative given by \( M_\infty \).

Sufficiency. Suppose that \( Q \ll \mathbb{P} \) when restricted on \( \mathcal{F}_\infty \) with \( dQ/d\mathbb{P} = Z \) for some \( Z \in \mathcal{F}_\infty \). Then for each \( t \geq 0 \) and \( A \in \mathcal{F}_t \), we have
\[
Q(A) = \int_A M_t d\mathbb{P} = \int_A Z d\mathbb{P}.
\]
Therefore, \( M_t = \mathbb{E}[Z|\mathcal{F}_t] \) which implies that \( \{M_t\} \) is uniformly integrable.

Apparently, from the above argument we have already proved that \( M_\infty \triangleq \lim_{t \to \infty} M_t \) is the Radon-Nikodym derivative of \( Q \) against \( \mathbb{P} \) on \( \mathcal{F}_\infty \). To see the final part, since in this case \( M_t \) is an \( \{\mathcal{F}_t\} \)-martingale with a last element \( M_\infty \), from the optional sampling theorem, we know that
\[
Q(A) = \int_A M_\infty d\mathbb{P} = \int_A M_\tau d\mathbb{P}, \quad \forall A \in \mathcal{F}_\tau.
\]
Therefore, \( Q \ll \mathbb{P} \) when restricted on \( \mathcal{F}_\tau \) and \( M_\tau = dQ/d\mathbb{P} \) on \( \mathcal{F}_\tau \).

**Problem 3.** Since \( |X_t| \) is a right continuous submartingale, Doob’s \( L^p \)-inequality implies that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s|^p \right] \leq q^p \mathbb{E}[|X_t|^p] \leq q^p M,
\]
where \( M \triangleq \sup_{t \geq 0} \mathbb{E}[|X_t|^p] \) and \( q = p/(p-1) \). In particular, Fatou’s lemma implies that \( \sup_{t \geq 0} |X_t|^p \in L^p \). On the other hand, since \( \{X_t\} \) is uniformly integrable (because it is bounded in \( L^p \)), \( X_t \) converges to some \( X_\infty \) almost surely and in \( L^1 \).

Now
\[
|X_t - X_\infty|^p \leq 2^p (|X_t|^p + |X_\infty|^p) \leq 2^{p+1} \sup_{t \geq 0} |X_t|^p \in L^1.
\]
The dominated convergence theorem then implies that
\[
\lim_{t \to \infty} \mathbb{E}[|X_t - X_\infty|^p] = 0.
\]
Problem 4. (1) Let \( f(t) = \log t - t/e \) \((t > 0)\), then \( f'(t) = 1/t - 1/e \). Therefore, \( f(t) \leq f(e) = 0 \). Now we prove that \( a \log^+ b \leq a \log^+ a + b/e \) for \( a, b > 0 \). If \( b \leq 1 \), this is trivial. If \( b > 1, a \leq 1 \), then

\[
a \log^+ b = a \log b \leq \frac{b}{e} = a \log^+ a + \frac{b}{e}.
\]

If \( a, b > 1 \), then the desired inequality follows from the fact that \( \log(b/a) \leq (b/a)/e \).

(2) Similar to the proof of Doob's \( L^p \)-inequality, we have

\[
\mathbb{E}[\rho(X_T^*)] \leq \mathbb{E} \left[ \int_0^{X_T^*} \rho(d\lambda) \right] \\
= \mathbb{E} \left[ \int_0^{X_T^*} \mathbb{1}_{\{X_T^* \geq \lambda\}} \rho(d\lambda) \right] \\
= \int_0^{X_T^*} \mathbb{P}(X_T^* \geq \lambda) \rho(d\lambda) \\
\overset{\text{Doob}}{\leq} \int_0^{\infty} \frac{1}{\lambda} \mathbb{E}[X_T^* \mathbb{1}_{\{X_T^* \geq \lambda\}}] \rho(d\lambda) \\
= \mathbb{E} \left[ X_T \int_0^{X_T^*} \lambda^{-1} \rho(d\lambda) \right].
\]

(3) Let \( \rho(t) = (t - 1)^+ \) \((t \geq 0)\). Then from the second part, we have

\[
\mathbb{E}[X_T^*] - 1 \leq \mathbb{E}[(X_T^* - 1) ; X_T^* \geq 1] \\
\leq \mathbb{E} \left[ X_T \int_0^{X_T^*} \lambda^{-1} \rho(d\lambda) \right] \\
= \mathbb{E} \left[ X_T \int_1^{X_T^*} \lambda^{-1} \rho(d\lambda) ; X_T^* \geq 1 \right] \\
= \mathbb{E}[X_T \log X_T^* ; X_T^* \geq 1] \\
= \mathbb{E}[X_T \log^+ X_T^*] \\
\leq \mathbb{E}[X_T \log^+ X_T^*] + \frac{1}{e} \mathbb{E}[X_T^*].
\]

Rearranging the terms yields the desired inequality.

Problem 5. From the assumption that

\[
\sup_{0 \leq t < \infty} X_t(\omega) = \infty, \quad \inf_{0 \leq t < \infty} X_t(\omega) = -\infty,
\]

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it is apparent that every $\tau_n$ is well-defined finitely. Since $X_t$ is $\{F_t\}$-adapted and has continuous sample paths, according to Proposition 2.7 in the lecture notes, we know that $\tau_1$ is an $\{F_t\}$-stopping time. To see why $\tau_2$ is also an $\{F_t\}$-stopping time, define $\tilde{X}_t \triangleq X_{\tau_1 + t} - X_{\tau_1}$ and $G_t \triangleq F_{\tau_1 + t}$. It follows that $\tilde{X}_t$ is $\{G_t\}$-adapted and has continuous sample paths. Therefore, the same reason implies that $\tau_2 - \tau_1$ is a $\{G_t\}$-stopping time. According to Problem Sheet 2, Problem 4, (2), (ii), we conclude that $\tau_2$ is an $\{F_t\}$-stopping time. Inductively, we know that every $\tau_n$ is an $\{F_t\}$-stopping time.

Now we study the distribution of the random sequence $\{X_{\tau_n} : n \geq 1\}$. Define $\sigma_n \triangleq \inf\{t \geq 0 : |X_t| > 2n\}$. Then $\tau_n < \sigma_n$ (in fact, $|X_t| \leq n$ for all $t \in [0, \tau_n]$) and $X_{t \wedge \sigma_n}$ is a bounded $\{F_t\}$-martingale. In particular, $X_t$ has a last element $X_\infty = \lim_{t \to \infty} X_t$. By the optional sampling theorem, we conclude that

$$E[X_{\tau_n} - X_{\tau_{n-1}} | F_{\tau_{n-1}}] = E[X_{\tau_n} - X_{\tau_{n-1}}^\sigma_n | F_{\tau_{n-1}}] = 0.$$ 

Now let $A^+_n \triangleq \{X_{\tau_n} - X_{\tau_{n-1}} = 1\}$ and $A^-_n \triangleq \{X_{\tau_n} - X_{\tau_{n-1}} = -1\}$ respectively. It follows that

$$P(A^+_n | F_{\tau_{n-1}}) = P(A^-_n | F_{\tau_{n-1}}) = \frac{1}{2} \text{ a.s.}$$

Therefore, for any $i_1, \ldots, i_n = \pm 1$, we have

$$P(X_{\tau_1} = i_1, X_{\tau_2} - X_{\tau_1} = i_2, \ldots, X_{\tau_n} - X_{\tau_{n-1}} = i_n) = \int_{\{X_{\tau_1} = i_1, \ldots, X_{\tau_{n-1}} - X_{\tau_{n-2}} = i_{n-1}\}} P(\{X_{\tau_n} - X_{\tau_{n-1}} = i_n\} | F_{\tau_{n-1}}) dP$$

$$= \frac{1}{2} P(X_{\tau_1} = i_1, \ldots, X_{\tau_{n-1}} - X_{\tau_{n-2}} = i_{n-1}).$$

Recursively, in the end this will imply that $X_{\tau_1}, X_{\tau_2} - X_{\tau_1}, \ldots, X_{\tau_n} - X_{\tau_{n-1}}$ are independent and identically distributed with distribution $P(X_{\tau_1} = \pm 1) = 1/2$. Therefore, $\{X_{\tau_n} : n \geq 1\}$ is distributed as the standard simple random walk.

**Problem 6.** (1) We first prove a claim:

$$E[|X_\tau - X_\sigma| | F_\sigma] \leq M_X,$$

for any $\{F_t\}$-stopping times $\sigma \leq \tau$. Indeed, since $\{X_t : 0 \leq t \leq \infty\}$ is a continuous martingale with a last element, the optional sampling theorem and the assumption imply that

$$E[|X_\tau - X_\sigma| | F_\sigma] = E[|E[X_\infty | F_\tau] - X_\sigma| | F_\sigma] \leq E[|E[X_\infty - X_\sigma | F_\tau]| F_\sigma] = E[|X_\infty - X_\sigma| | F_\sigma] \leq M_X.$$
Now for $\lambda, \mu > 0$, let
\[
\sigma \triangleq \inf\{t \geq 0 : |X_t| \geq \lambda\},
\tau \triangleq \inf\{t \geq 0 : |X_t| \geq \lambda + \mu\}.
\]
According to (1), we have
\[
\int_{\{\sigma < \infty\}} |X_\tau - X_\sigma| d\mathbb{P} \leq M_X \mathbb{P}(\sigma < \infty) \leq M_X \mathbb{P}(X^* \geq \lambda).
\]
But since $\{X^* \geq \lambda + \mu\} \subseteq \{\sigma < \infty\}$ and $|X_\tau - X_\sigma| = \mu$ on $\{X^* \geq \lambda + \mu\}$, it follows that
\[
\int_{\{\sigma < \infty\}} |X_\tau - X_\sigma| d\mathbb{P} \geq \mu \mathbb{P}(X^* \geq \lambda + \mu).
\]
Therefore,
\[
\mathbb{P}(X^* \geq \lambda + \mu) \leq \frac{M_X}{\mu} \mathbb{P}(X^* \geq \lambda).
\]

(2) Let $\lambda > 0$. Note that for any $k \geq 1$, from (1) we have
\[
\mathbb{P}(X^* \geq keM_X) \leq e^{2 - \lambda \frac{\lambda}{eM_X}} \leq e^{-k}.
\]
Now if $\lambda \geq eM_X$, let $k$ be the unique positive integer such that $keM_X \leq \lambda < (k + 1)eM_X$. Then
\[
\mathbb{P}(X^* \geq \lambda) \leq \mathbb{P}(X^* \geq keM_X) \leq e^{-k} \leq e^{1 - \lambda \frac{\lambda}{eM_X}} \leq e^{2 - \lambda \frac{\lambda}{eM_X}}.
\]
The inequality is trivial for $0 < \lambda < eM_X$ since in this case $e^{2 - \lambda \frac{\lambda}{eM_X}} > 1$.

To see the exponential integrability, first note that the first part implies that $X^* < \infty$ almost surely, and
\[
\mathbb{P}(e^{aX^*} \geq e^{a\lambda}) \leq e^{2 - \lambda \frac{\lambda}{eM_X}}, \quad \forall \lambda > 0.
\]
Therefore,
\[
\mathbb{E}[e^{aX^*}] = \int_0^\infty \mathbb{P}(e^{aX^*} \geq \mu) d\mu
\]
\[
\leq 1 + \alpha \int_0^\infty \mathbb{P}(e^{aX^*} \geq e^{a\lambda}) e^{a\lambda} d\lambda
\]
\[
\leq 1 + \alpha \int_0^\infty e^{2 - \lambda \frac{\lambda}{eM_X}} d\lambda,
\]

which is finite if $0 < \alpha < (eM_X)^{-1}$. The $L^p$-integrability follows from then the exponential integrability.

**Problem 7.** (1) Let $\tau \in S_T$. By the optional sampling theorem,

$$E[X_\tau 1_{\{X_\tau > \lambda\}}] \leq E[X_T 1_{\{X_\tau > \lambda\}}].$$

But

$$\mathbb{P}(X_\tau > \lambda) \leq \frac{E[X_\tau]}{\lambda} \leq \frac{E[X_T]}{\lambda} \to 0$$

uniformly in $\tau \in S_T$ as $\lambda \to \infty$. Therefore, $E[X_\tau 1_{\{X_\tau > \lambda\}}] \to 0$ uniformly in $\tau \in S_T$ as $\lambda \to \infty$, which proves the claim that $X_t$ is of class (DL). Suppose further that $X_t$ is continuous. Let $\tau_n \uparrow \tau \in S_T$. Then $X_{\tau_n} \to X_\tau$ almost surely as $n \to \infty$. But $X_t$ is of class (DL), so $\{X_{\tau_n}\}$ is uniformly integrable. Therefore, $X_{\tau_n} \to X_\tau$ in $L^1$, which implies that $X_t$ is regular.

(2) If $X_t$ is non-negative and uniformly integrable, then $X_t$ converges to some $X_\infty$ almost surely and in $L^1$. Moreover, we have

$$X_t \leq E[X_\infty 1_{\mathcal{F}_t}]$$

for every $t \geq 0$. The optional sampling theorem then implies that

$$X_\tau \leq E[X_\infty 1_{\mathcal{F}_\tau}]$$

for every finite $\{\mathcal{F}_t\}$-stopping time $\tau$. The uniform integrability of $\{X_\tau\}$ follows from the same argument as in the first part of the problem.

Since $X_t = M_t + A_t$ by the Doob-Meyer decomposition, we know that $E[X_t] = E[M_0] + E[A_t]$. By letting $t \to \infty$, we conclude that $E[X_\infty] = E[M_0] + E[A_\infty]$. In particular, $E[A_\infty] < \infty$. 

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