Problem Sheet 6

Due for Submission: 12/07 Wednesday

Problem 1. Show that the following SDEs are all exact. Solve them explicitly with the given initial data. Here $B_t$ is a one dimensional Brownian motion.

1. The stochastic harmonic oscillator model:

$$\begin{cases} 
    dX_t = Y_t dt, \\
    m dY_t = -kX_t dt - cY_t dt + \sigma dB_t, 
\end{cases}$$

where $m, k, c, \sigma$ are positive constants. Initial data is arbitrary.

2. The stochastic RLC circuit model:

$$\begin{cases} 
    dX_t = Y_t dt, \\
    L dY_t = -R Y_t - \frac{1}{C} X_t + G(t) + \alpha dB_t, 
\end{cases}$$

where $R, C, L, \alpha$ are positive constants and $G(t)$ is a given deterministic function. Initial data is arbitrary.

3. The stochastic population growth model:

$$dX_t = r X_t (K - X_t) dt + \beta X_t dB_t,$$

where $r, K, \beta$ are positive constants. Initial data is $X_0 = x > 0$.

Problem 2. Let $B_t$ be a one dimensional Brownian motion on a filtered Probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\})$ which satisfies the usual conditions.

1. Define $X_t \triangleq B_t - t B_1$ ($0 \leq t \leq 1$). Show that $X_t$ is a Gaussian process. Compute its mean and covariance function $\rho(s, t) \triangleq \mathbb{E}[X_s X_t]$ ($0 \leq s, t \leq 1$).

2. Find the solution $Y_t$ ($0 \leq t < 1$) to the SDE

$$\begin{cases} 
    dY_t = dB_t - \frac{Y_t}{1-t} dt, \\
    Y_0 = 0. 
\end{cases}$$

Show that $Y_t$ has the same law as $X_t$ ($0 \leq t < 1$). In particular, $\lim_{t \uparrow 1} Y_t = 0$ almost surely and we can define $Y_1 \triangleq 0$. This defines a process $Y_t$ ($0 \leq t \leq 1$) which has the same law as $X_t$ ($0 \leq t \leq 1$).

3. Show that

$$\mathbb{P} \left( \sup_{0 \leq t \leq 1} Y_t \geq x \right) = e^{-2x^2}, \quad x \geq 0.$$

Problem 3. Consider the one dimensional SDE

$$dY_t = 3Y_t^2 dt - 2|Y_t|^2 dB_t.$$
(1) Show that this SDE is exact (in the context with possible explosion).
(2) Show that if $Y_0 \geq 0$, then $Y_t \geq 0$ for all $t$ up to its explosion time $e$.
(3) Suppose that $Y_0 = 1$. Compute $\mathbb{P}(e > t)$ for $t \geq 0$. Conclude that $\mathbb{P}(e < \infty) = 1$ but $\mathbb{E}[e] = \infty$.

**Problem 4.** (1) Let $H, G$ be continuous semimartingales with $\langle H, G \rangle = 0$ and $H_0 = 0$. Show that

$$Z_t \triangleq \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} dH_s,$$

where $\mathcal{E}_t^G$ is the stochastic exponential of $G$ defined by

$$\mathcal{E}_t^G \triangleq e^{X_t - \frac{1}{2} \langle X \rangle_t},$$

then $Z_t$ satisfies

$$Z_t = H_t + \int_0^t Z_s dG_s.$$

(2) Consider the following two SDEs on $\mathbb{R}^1$:

$$dX_t^i = \sigma(t, X_t)dB_t + b^i(X_t)dt, \quad i = 1, 2,$$

where $\sigma: [0, \infty) \times \mathbb{R}^1 \to \mathbb{R}^1$, $b^i : \mathbb{R}^1 \to \mathbb{R}^1$ are bounded continuous, $\sigma$ is Lipschitz continuous and one of $b^1$, $b^2$ is Lipschitz continuous. Suppose further that $b^1 < b^2$ everywhere. Let $X_t^i$ be a solution to the above SDE with $i = 1, 2$ respectively, defined on the same filtered probability space with the same Brownian motion, such that $X_0^1 \leq X_0^2$ almost surely. By putting $Z_t = X_t^2 - X_t^1$, and choosing a suitable positive bounded variation process $H_t$ and a continuous semimartingale $G_t$ in the first part of the question, show that

$$\mathbb{P}(X_t^1 < X_t^2 \quad \forall t \geq 0) = 1.$$ 

Give an example to show that if $\sigma$ is not Lipschitz continuous, then the conclusion can false even $b^1, b^2$ are Lipschitz continuous.