Problem Sheet 5

Due for Submission: 11/11 Friday

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1. Let \( M \in \mathcal{M}_{loc}^{0} \) be a continuous local martingale vanishing at \( t = 0 \).

1. Recall that \( H_0^2 \) is the space of \( L^2 \)-bounded continuous martingales vanishing at \( t = 0 \). Show that \( M \in H_0^2 \) if and only if \( \mathbb{E}[\langle M \rangle_{\infty}] < \infty \), where \( \langle M \rangle_{\infty} \triangleq \lim_{t \to \infty} \langle M \rangle_t \).

2. Show that \( \langle M \rangle_t \) is deterministic (i.e. there exists a function \( f : [0, \infty) \to \mathbb{R} \) such that with probability one, \( \langle M \rangle_t(\omega) = f(t) \) for all \( t \geq 0 \)) if and only if \( M_t \) is a Gaussian martingale, in the sense that it is a martingale and \( (M_{t_1}, \cdots, M_{t_n}) \) is Gaussian distributed in \( \mathbb{R}^n \) for every \( 0 \leq t_1 < \cdots < t_n \). In this case, \( M_t \) has independent increments.

3. Show that there exists a measurable set \( \tilde{\Omega} \in \mathcal{F} \), such that \( \mathbb{P}(\tilde{\Omega}) = 1 \) and

\[
\tilde{\Omega} \cap \{ \langle M \rangle_{\infty} < \infty \} = \tilde{\Omega} \cap \left\{ \lim_{t \to \infty} M_t \text{ exists finitely} \right\},
\]

\[
\tilde{\Omega} \cap \{ \langle M \rangle_{\infty} = \infty \} = \tilde{\Omega} \cap \left\{ \limsup_{t \to \infty} M_t = \infty, \liminf_{t \to \infty} M_t = -\infty \right\}.
\]

Problem 2. Let \( B_t \) be the three dimensional Brownian motion with \( \{ \mathcal{F}_B t \} \) being its augmented natural filtration. Define \( X_t \triangleq 1/|B_1 + t| \).

1. Show that \( X_t \) is a continuous \( \{ \mathcal{F}_B t \} \)-local martingale which is uniformly bounded in \( L^2 \) (and hence uniformly integrable) but it is not an \( \{ \mathcal{F}_B t \} \)-martingale.

2. Show that if a uniformly integrable continuous submartingale \( Y_t \) has a Doob-Meyer decomposition, it has to be of class \( (D) \) in the sense that \( \{ Y_{\tau} : \tau \text{ is a finite stopping time} \} \) is uniformly integrable. By showing that \( X_t \) is not of class \( (D) \), conclude that \( X_t \) does not have a Doob-Meyer decomposition.

Problem 3. This problem is the stochastic counterpart of Fubini’s theorem. Give up if you don’t like this question—it is hard and boring. I have to include it because we need to use it in the lecture notes when we study local times and I don’t want to waste time proving it in class.

1. A set \( \Gamma \subseteq [0, \infty) \times \Omega \) is called \( \textit{progressive} \) if the stochastic process \( 1_\Gamma(t, \omega) \) is progressively measurable. Show that the family \( \mathcal{P} \) of progressive sets forms a sub-\( \sigma \)-algebra of \( \mathcal{B}([0, \infty)) \otimes \mathcal{F} \), and a stochastic process \( X \) is progressively measurable if and only if it is measurable with respect to \( \mathcal{P} \).

2. Let \( \Phi = \{ \Phi^a : a \in \mathbb{R}^1 \} \) be a family of real valued stochastic processes parametrized by \( a \in \mathbb{R}^1 \). Viewed as a random variable on \( \mathbb{R}^1 \times [0, \infty) \times \Omega \), suppose that \( \Phi \) is uniformly bounded and \( \mathcal{B}(\mathbb{R}^1) \otimes \mathcal{P} \)-measurable. Let \( X_t \) be a continuous semimartingale. Show that there exists a \( \mathcal{B}(\mathbb{R}^1) \otimes \mathcal{P} \)-measurable

\[
Y : \mathbb{R}^1 \times [0, \infty) \times \Omega \to \mathbb{R}^1,
\]

\[
(a, t, \omega) \mapsto Y^a_t(\omega),
\]
such that for every $a \in \mathbb{R}^1$, $Y^a$ and $I^X(\Phi^a)$ are indistinguishable as stochastic processes in $t$, and for every finite measure $\mu$ on $(\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$, with probability one, we have

$$\int_{\mathbb{R}^1} Y^a_t \mu(da) = \int_0^t \left( \int_{\mathbb{R}^1} \Phi^a_s \mu(da) \right) dX_s, \quad \forall t \geq 0.$$ 

**Problem 4.** Let $B_t$ be an $\{F_t\}$-Brownian motion defined on a filtered probability space which satisfies the usual conditions. Let $\mu_t$ and $\sigma_t$ be two uniformly bounded, $\{F_t\}$-progressively measurable processes.

1. By using Itô’s formula, find a continuous semimartingale $X_t$ explicitly, such that

$\lim_{t \to 0} \frac{X_t}{\mu_t} = 1 + \int_0^t X_s \mu_s ds + \int_0^t X_s \sigma_s dB_s, \quad t \geq 0.$

By using Itô’s formula again, show that such $X_t$ is unique.

2. Assume further that $\sigma \geq C$ for some constant $C > 0$. Given $T > 0$, construct a probability measure $\mathbb{P}_T$, equivalent to $\mathbb{P}$, under which $\{X_t, F_t: 0 \leq t \leq T\}$ is a continuous local martingale.

**Problem 5.** Let $B_t$ be a one dimensional Brownian motion and let $\{\mathcal{F}_t^B\}$ be the augmented natural filtration.

1. Fix $T > 0$. For $\xi = B^2_T$ and $B_1^3$, find the unique progressively measurable process $\Phi$ on $[0, T]$ with $\mathbb{E} \left[ \int_0^T \Phi^2_t dt \right] < \infty$, such that $\xi = \mathbb{E}[\xi] + \int_0^T \Phi_t dB_t$.

2. Construct a process $\Phi \in L^2_{\mathcal{F}}(B)$ with $\int_0^\infty \Phi^2_t dt < \infty$ almost surely (so $\int_0^\infty \Phi_t dB_t$ is well defined), such that $\int_0^\infty \Phi_t dB_t = 0$ but with probability one, $0 < \int_0^\infty \Phi^2_t dt < \infty$.

3. Consider $S_t \triangleq \max_{0 \leq t \leq 1} B_t$. By writing $\mathbb{E}[S_t | \mathcal{F}_t^B]$ as a function of $(t, S_t, B_t)$, find the unique progressively measurable process $\Phi$ on $[0, 1]$ with $\mathbb{E} \left[ \int_0^1 \Phi^2_t dt \right] < \infty$, such that $S_t = \mathbb{E}[S_1] + \int_0^1 \Phi_t dB_t$.

**Problem 6.** (1) Let $B_t$ be the $d$-dimensional Brownian motion. Define $\tau \triangleq \inf\{t \geq 0: |B_t| = 1\}$. What is the distribution of $B_\tau$? Show that $B_\tau$ and $\tau$ are independent.

(2) Let $c \in \mathbb{R}^d$ and define $X_t \triangleq B_t + ct$ to be the $d$-dimensional Brownian motion with drift vector $c$. Define $\tau$ in the same way as before but for the process $X_t$. By using Girsanov’s theorem under a suitable framework, show that $X_\tau$ and $\tau$ are independent.

**Problem 7.** Let $B$ be a one dimensional Brownian motion and let $l$ be its local time at 0.

1. Let $X_t = B_t + cl_t$ where $c \in \mathbb{R}^1$. Define $L^a$ to be the local time at $a$ of $X$. Show that for every $T > 0$ and $k \geq 1$, there exists some constant $C_{T,k}$ such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |L^a_t - L^b_t|^{2k} \right] \leq C_{T,k}|a - b|^k.$$ 

Conclude that $\{L^a_t: a \in \mathbb{R}^1, t > 0\}$ has a modification which is locally $\gamma$-Hölder continuous in $a$ uniformly on every finite $t$-interval for every $\gamma \in (0, 1/2)$.

2. Let $\lambda, \mu > 0$ with $\lambda \neq \mu$. After taking the modification given by Theorem 5.18 in the lecture notes, show that the local time $L^a_t$ of the continuous semimartingale $X_t \triangleq \lambda B^a_t - \mu B^a_t$ is discontinuous at $a = 0$. Compute this jump (at any given $t > 0$).