Carnegie Mellon University
Department of Mathematical Sciences
21-122 Review Final Exam, Spring, 2006

Part 1

1. Integrate and simplify your result.

$$\int_0^{\pi/3} \cos^4 t \sin^2 t \; dt$$

Solution: $\pi/48 + \sqrt{3}/64$.

2. Integrate

$$\int \frac{dx}{x^2\sqrt{9-x^2}}$$

Solution: $-\frac{\sqrt{9-x^2}}{9x} + C$

3. Integrate

$$\int \frac{1}{x^2(x^2+4)} \; dx$$

Solution: $-\frac{1}{4x} - \frac{1}{8}\arctan\frac{x}{2} + C$

4. Determine if the following integral converges or diverges, and evaluate the integral if it converges.

$$\int_0^{\infty} x^2 e^{-x^3} \; dx$$

Solution: Converges, to $\frac{1}{3}$

Part 2

1. Find the gerneral solution of $\frac{dy}{dx} + \sin x + y\cos x = \cos x + y \sin x$.

   Solution: $y = 1 + Ce^{-\sin x - \cos x}$

2. Find the general solution of $\frac{dy}{dx} + \frac{1}{x}y = 1$

   Solution: $\frac{1}{2}x + \frac{C}{x}$

3. Find the solution to $y'' - 8y' + 17y = 0$ that satisfy the conditions $y(0) = 1$, $y'(0) = -1$.

   Solution: $y = e^{4x}(\cos x - 5\sin x)$

4. Suppose it takes 6 days for 25% of a radioactive substance to decay.

   (a) What is the half-life of the substance? **Solution:** $\frac{6\ln 2}{2\ln 2 - \ln 3}$

   (b) How long does it take for 75% of the substance to decay? **Solution:** $\frac{12\ln 2}{2\ln 2 - \ln 3}$

Part 3

1. Determine if the following series converges conditionally, converges absolutely, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-\pi)^n}{n+n!}$$

Solution: Converges absolutely. Take absolute value. Compare to $b_n = \frac{\pi^n}{n!}$, series which converges by the ratio test.

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{1 \cdot 4 \cdot 7 \cdots (3n+1)}$$

Solution: Converges by the ratio test.

3. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$$

Solution: Diverges by L.C.T, with $b_n = \frac{1}{n}$.

4. Approximate the sum of the following series to within 0.1

$$\sum_{n=1}^{\infty} \frac{1}{n+4^n}$$

Solution: Use comparison test, $b_n = \frac{1}{4^n}$. Need $n = 1$. $s \approx 0.2$.

Part 4

1. Find the interval of convergence (*you must test endpoints, if applicable*) of the following series.

$$\sum_{n=0}^{\infty} 2^{-n}(2x-1)^n$$

Solution: $-1/2 < x < 3/2$

2. Find a power series representation, and the interval on which it represents the function, for

$$f(x) = \frac{x^2}{4-x^2}$$

Solution: $\sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}}$, $-2 < x < 2$

3. Find the Taylor series centered at $a = 8$ for

$$f(x) = \sqrt[3]{x}$$

Solution:

$$2 + \frac{1}{12}(x-8) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 2 \cdot 5 \cdots (3n-4)}{3^n 2^{3n-1} n!}(x-8)^n$$

4. Using Taylor's Inequality, determine an upper bound for the error in approximating $f(x) = e^{-2x}$ by its Taylor polynomial $T_2(x)$, centered at $a = 0$, on the interval $[-1/4, 1/4]$.

Solution: $\frac{\sqrt{e}}{48}$