21-484 Graph Theory
Assignment # 5
Due: Friday, March 14

1. Prove the following version of the max-flow/min-cut theorem.

   Let $D = (V, A)$ be a digraph and let $S, T \subset V$ be disjoint sets of sources and sinks. Let $c : A \to \mathbb{N}$ give capacities on the arcs in D and consider flows $f : A \to \mathbb{R}^+$ with the property that $f^+(v) = f^-(v)$ for all $v \in V \setminus (S \cup T)$. The value $v(f)$ of such a flow is the net flow out of the source vertices, which equals the net flow into the sink vertices. We have

   $$\max_{\text{flows } f} v(f) = \min_{S \subseteq X \subseteq V \setminus T} c(X, \overline{X}).$$

2. Prove the following version of Menger's Theorem using the max-flow min-cut theorem we proved in class.

   Menger's Theorem, local vertex version. Let $G = (V, E)$ be a graph. If u and v are distinct nonadjacent vertices of G, then the minimum number of vertices is a set X that does not contain u, v and separates u from v is equal to the maximum number of independent paths from u to v.

3. Suppose that a graph G is a union of k trees. Prove that $\chi(G) \leq 2k$.

4. Supppose that a graph G has the property that every pair of odd cycles has a vertex in common. Prove that $\chi(G) \leq 5$.

5. Recall that the clique number $\omega(G)$ is the maximum number of vertices in a complete subgraph of G while the independence number $\alpha(G)$ is the maximum number of vertices in an independent set in G. Use these parameters to prove

   $$\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$$

   for any graph G on n vertices.

6. Prove that for any graph G on n vertices we have

   $$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

   Hint. Consider an ordering $x_1, x_2, \ldots, x_n$ of the vertices such that $d(x_i) \geq d(x_{i+1})$ for $i = 1, \ldots, n-1$ and color in this order.