21-484 Graph Theory
Assignment # 4
Due: Friday, February 21

1. Let $G = (V, E)$ be a 3-regular graph with the property that all bridges in G are traversed by a single path. Prove that G has a perfect matching.

2. Let $G = (V, E)$ be a connected graph with an even number of edges. Use Tutte's 1-factor Theorem to prove that the edge set of G can be partitioned into 2-edge paths.

   Hint: Consider the line graph G. This is the graph $\mathcal{L}(G)$ that has vertex set E and edge set $E(\mathcal{L}(G)) = \{\{e, f\} \in \binom{E}{2} : e \cap f \neq \emptyset\}$.

3. Let $n \geq 4$. Determine the minimum number of edges in a 3-connected graph on n vertices.

4. Without using Menger's theorem, prove that if G is k-connected ($k \geq 2$) and has at least $2k$ vertices, then G contains a cycle of length at least $2k$.

5. Derive the following version of Menger's Theorem from König's Theorem.

   > Let $D = (V, A)$ be a digraph. If $X, Y \subseteq V$ such that $X \cap Y = \emptyset$ then the minimum number of vertices in a set T with the property that every path from X to Y intersects T is equal to the maximum size of a collection of pairwise disjoint directed paths from X to Y.

6. Let $G = (V, E)$ be a graph. Given $U \subset V$ and a vertex $x \in V \setminus U$, an x-U fan is a set $|U|$ paths from x to U any two of which have only the vertex x in common. Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any set $U \in \binom{V}{k}$ and any vertex $x \in V \setminus U$ there is an x-U fan in G.

   Hint. Apply Menger's Theorem.