21-484 Graph Theory
Assignment # 3
Due: Friday, February 7

1. The complete bipartite graph $K_{n,m}$ has vertex set $A\dot{\cup}B$ where $|A| = n$ and $|B| = m$ and edge set consisting of all pairs with one vertex in A and one vertex in B. Use the Matrix Tree Theorem to determine the number of spanning trees of $K_{n,m}$.

2. Let $G = (V, E)$ be a graph and $M \subseteq E$ be a matching. We begin with two definitions.
   - A path $x_0, x_1, \ldots, x_k$ is an **augmenting path** if
     - $x_{i-1}x_i \notin M$ for all odd $i \in \{1, \ldots, k\}$,
     - $x_{i-1}x_i \in M$ for all even $i \in \{1, \ldots, k\}$, and
     - Neither x_0 nor x_k is incident with any edge in M.
   - The matching M is a **maximum matching** if there is no matching M' such that $|M'| > |M|$.

   Prove that M is a maximum matching if and only if there no augmenting path.

3. Show that a graph $G = (V, E)$ has a matching of size k if and only if

$$q(G - S) \leq |S| + |V| - 2k \quad \text{for all } S \subseteq V.$$

4. Deduce the Marriage Theorem from Tutte's 1-factor Theorem.

For the remaining questions make use of the following definitions. If G is a graph on more than one vertex and $G - F$ is connected for every set F of fewer than ℓ edges then we say that G is **ℓ-edge-conected**. The greatest integer ℓ such that G is ℓ-edge-connected is the **edge connectivity** of G, which is denoted $\lambda(G)$.

5. Let G be an n vertex graph such that

$$d(x) + d(y) \geq n - 1 \quad \text{for all non-adjacent } x, y \in V(G).$$

   Prove that $\lambda(G) = \delta(G)$.

6. Suppose $G = (V, E)$ is k-edge-connected and the deletion of any edge of G gives a graph that is not k-edge-connected. Show that G has minimum degree k.

   Hint: Consider $X \subseteq V$ with $|E(X, V \setminus X)| = k$ and $|X|$ minimum.