21-484 Graph Theory
Assignment # 2
Due: Friday, January 31

1. Show that every automorphism of a tree fixes a vertex or fixes an edge.
2. Use the Matrix Tree Theorem to show that the number of spanning trees on a set of n labelled vertices is n^{n-2}. *Hint. Start by considering the eigenvalues of* A_{K_n}.
3. The complete bipartite graph $K_{n,m}$ has vertex set $A\dot{\cup}B$ where $|A| = n$ and $|B| = m$ and edge set consisting of all pairs with one vertex in A and one vertex in B. Use the Matrix Tree Theorem to determine the number of spanning trees of $K_{n,m}$.
4. Give an example of 3-regular graph that does not have a 1-factor. (A 3-regular graph is called a cubic graph. A 1-factor is a collection of edges with the property that each vetex is in exactly one of the edges in the collection.)
5. Let $G = (V, E)$ be a bipartite graph with biparitition $V = A\dot{\cup}B$. Show that there is a matching M in G that saturates every vertex $v \in A$ such that $d(v) = \Delta(G)$. (We say that matching M saturates a vertex v if v is contained in the union of the edges in M.)
6. Let X be a set with $|X| = n$. Suppose $\mathcal{F} \subset 2^X$ is a collection of sets with the property

   $$A, B \in \mathcal{F} \quad \text{implies} \quad A \not\subset B.$$

   Prove

   $$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

   Hint: Define a chain to be a collection of sets $A_1, \ldots, A_k$ *such that* $A_1 \subset A_2 \subset \cdots \subset A_k$. *Prove that there is decomposition of the collection of all subsets of* X *into* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ *chains.*