21-484 Graph Theory
Assignment # 1
Due: Friday, January 24

1. Let $n \geq 1$. Prove that the number of graphs on vertex set $\{1, 2, \ldots, n\}$ with all even degrees is $2^{\binom{n-1}{2}}$.

2. Let $G = (V, E)$ be a connected graph such that

$$V = \bigcup_{i=1}^{k} V_i$$

is a partition of V into non-empty parts such that $G[V_i]$ is connected for $i = 1, 2, \ldots, k$. Prove that that are indices $i \neq j$ such that $G - V_i = G[V \setminus V_i]$ and $G - V_j = G[V \setminus V_j]$ are connected.

3. Let $G = (V, E)$ be a graph. Recall that $\alpha(G)$ denotes the cardinality of the largest independent set in G and that $\Delta(G)$ is the maximum degree in G. Prove that if G does not contain a copy of K_3 then $\Delta(G) \leq \alpha(G)$. Conclude that we have

$$|E| \leq \frac{|V|\alpha(G)}{2}.$$

4. Is the following statement True or False? Explain your answer.

   If $n \geq 4$ then the complete graph K_n is the union of an edge, a path of length 2, and the cycles $C_3, C_4, \ldots, C_{n-1}$.

5. Let X be a set such that $|X| = n \geq 6$. A Steiner triple system is a collection of sets

$$\mathcal{F} \subset \binom{X}{3}$$

with the property that every set $A \in \binom{X}{2}$ is a subset of exactly one of the sets in $\mathcal{F}$. Show that if a Steiner triple system on the set X exists then $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.

6. Let G be a graph of average degree $d > 0$. Show that there is a vertex x of G with the property that the average of the degrees of the neighbors of x is at least d.