21-484 Graph Theory
Review sheet for test 2

Definitions. The test will assume that you know the following definitions.

- Independent collection of paths
- Vertex set that seperates $A, B \subset V(G)$
- If $D = (V, A)$ is a directed graph then the **out-neighborhood** and **in-neighborhood**, respectively, are

$$N^+(v) = \{y \in V : (v, y) \in A\} \quad \text{and} \quad N^-(v) = \{y \in V : (y, v) \in A\}.$$

- If $D = (V, A)$ is a directed graph and $f : V \to \mathbb{R}$ is a function then we define

$$f^+(v) = \sum_{y \in N^+(v)} f(v, y) \quad \text{and} \quad f^-(v) = \sum_{y \in N^-(v)} f(y, v).$$

- If $D = (V, A)$ is a directed graph and $X, Y \subseteq V$ then

$$A(X, Y) = \{(x, y) \in A : x \in X \text{ and } y \in Y\}$$

  Furthemore, if $g : A \to \mathbb{R}$ then

$$g(X, Y) = \sum_{(x,y) \in A(X,Y)} g(x, y).$$

- Integral flow.
- Line graph.
- Proper vertex coloring, k-coloring, $\chi(G)$.
- Idependence number, clique number.
- Minimum degree, maximum degree, $\delta(G), \Delta(G)$.
- Proper edge coloring, k-edge-coloring, $\chi'(G)$.

Theorems. The test assumes knowledge of the following theorems

- **Menger's Theorem.** Let $G = (V, E)$ be a graph and let $A, B \subset V$. A set S of vertices **seperates** A and B if every path from A to B traverses a vertex in S. (N.b. If $A \cap B \neq \emptyset$ then $A \cap B$ is a subset of any seperating set.)

  > **Theorem.** qThe minimum number of vertices in a set seperating A and B is the maximum number of disjoint path from A to B.

- **Menger's Theorem, global version.** Let $G = (V, E)$ be a graph.

  (i) G is k-connected if and only for every pair of distinct vertices in G is joined by a collection of k independent paths.

  (ii) G is k-edge-connected if and only if every pair of distinct vertices in G is joined by a collection of k edge-disjoint paths.

- **Max-flow/Min-cut,** Directed, arc-weight version. Let $D = (V, A)$ be a directed graph and let $c : A \to \mathbb{N}$ specify capacities for each arc. Furthermore, let a source $s \in V$ and a sink $t \in V$ be specified. A function $f : A \to \mathbb{R}^+$ is a flow if

  - $f^+(v) = f^-(v)$ for all $v \in V \setminus \{s, t\}$.
  - $f(x, y) \leq c(x, y)$ for all $(x, y) \in A$.

  The value of a flow is $v(f) = f^+(s) - f^-(s) = f^-(t) - f^+(t)$. A set $S \subset V$ is a cut if $s \in S$ and $t \notin S$. The capacity of the cut S is $c(S, V \setminus S)$.

  > **Theorem.** The maximum value of a flow from s to t equals the minimum capacity of a cut that seperates s from t.

- **Max-flow/Min-cut,** Directed, vertex weight version. Let $D = (V, A)$ be a digraph with a specified source $s \in V$ and sink $t \in V$ and a function $c : V \setminus \{s, t\} \to \mathbb{N}$ that specifies vertex capacities. A flow is now a map $f : A \to \mathbb{R}^+$ such that

  $$f^+(v) = f^-(v) \leq c(v) \quad \text{for all } \ v \in V \setminus \{s, t\}.$$

  The value of a flow is again $v(f) = f^+(s) - f^-(s) = f^-(t) - f^+(t)$. A cut is now a set $X \subseteq V \setminus \{s, t\}$ such that there is no flow of positive value from s to t in $G - X$. The capacity of a cut X is

  $$c(X) = \sum_{x \in X} c(x).$$

  > **Theorem.** The maximum value of a flow from s to t is equal to the maximum capacity of a cut X.

- **Fact.** In the context of Max-flow/Min-cut, there is a maximum value flow that is an integral flow.

- **Claim.** If G and H are graphs and $G \cap H$ is complete then $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$.

- **Claim.** Every graph G has an induced subgraph H such that

$$\delta(H) \geq \chi(G) - 1.$$

- **Brook's Theorem.** Let $G = (V, E)$ be a connected graph. If G is neither an odd cycle nor complete then

$$\chi(G) \leq \Delta(G).$$

- **Theorem.** If $G = (V, E)$ is a bipartite graph then $\chi'(G) = \Delta(G)$.
- **Vizing's Theorem.** If $G = (V, E)$ is a graph then

$$\chi'(G) \leq \Delta(G) + 1.$$

Review Problems. Doing these problems should help in preparation for the third test.

1. A circulation in a directed graph $D = (V, A)$ is a flow without a source and sink. Suppose that we specify a lower capacity $\ell(x, y)$ and an upper capacity $u(x, y)$ for each arc $(x, y) \in A$ such that $0 \leq \ell(x, y) \leq u(x, y)$. We say that a circulation f is feasible if
$$\ell(x, y) \leq f(x, y) \leq u(x, y) \quad \forall (x, y) \in A.$$
Prove that there is feasible circulation if and only if
$$\ell(S, V \setminus S) \leq u(V \setminus S, S) \quad \forall A \subset V.$$

2. Let $D = (V, A)$ be a digraph with source $s \in V$ and sink $t \in V$ and arc capacities $C : A \to \mathbb{N}$. If f is a flow on D then we say that a directed cycle $v_1, v_2, \ldots, v_k$ carries positive flow if
$$f(v_1, v_2), f(v_2, v_3), \ldots, f(v_{k-1}, v_k), f(v_k, v_1) > 0.$$
Prove that there is a flow of maximum value such that no directed cycle in D carries positive flow and $f^-(s) = f^+(t) = 0$.

3. Let G be a k-connected graph. Prove that any k vertices in G are contained in a common cycle.

4. Show that if $G = (V, E)$ is a 5-connected graph and u, v, w are distinct vertices in G then there is a collection of six cycles in G such that all cycles in the collection contain u and v but no cycle in the collection contains w.

5. Let $[n] = \{1, 2, \ldots, n\}$. For $2k \leq n$ define the graph $K = K(n, k)$ to be the graph whose vertex set $\binom{[n]}{k}$ with an edge between vertices A and B if A and B are disjoint.
   (a) Given an explicit vertex coloring of $K(n, k)$ that uses at most n colors.
   (b) Let $X = \{n - 2k + 1, n - 2k + 2, \ldots, n\}$ and consider the induced graph $H = K[\binom{X}{k}]$. Determine the chromatic number of this graph.
   (c) Using the previous part show that the chromatic number of $K(n, k)$ is at most $n - 2k + 2$.

6. Let $G = (V, E)$ be a connected graph. We define the distance between two vertices $x, y \in V$ to be the number of edges in the shortest path from x to y.

   Let v be a vertex and for $r = 0, 1, 2, \ldots$ let V_r be the set of vertices of distance exactly r from v. Show that
$$\chi(G) \leq \max\{\chi(G_r) + \chi(G_{r+1}) : r \geq 0\}.$$

7. A graph G with $\chi(G) = k$ is *critically k-chromatic* if $\chi(G - v) < k$ for all $v \in V(G)$. Note that every k-chromatic graph G has an induced graph H that is critically k-chromatic and shiw that the minimum degree of such a subgraph is $k - 1$.

(a) Determine the critically 3-chromatic graphs.

(b) Show that every critically k-chromatic graph is $(k-1)$-edge-connected.

8. Prove that every graph G with $\chi(G) \geq 3$ contains a cycle of length at least $\chi(G)$.

9. Use Hall's Theorem to prove that if G is bipartite then $\chi'(G) = \Delta(G)$.

10. Let H be a $2k$-regular graph such that $|V(H)| = 4k+1$. Let M be a matching in H such that $|M| = k-1$ and set

$$G = H - M.$$

Prove $\chi'(G) = 2k+1$.