21-484 Graph Theory
Review sheet test 1

Definitions. The test will assume that you know the following definitions.

- graph, vertex set, and edge set
- adjacent vertices
- path, cycle, and complete graph
- complement of a graph
- graph isomorphism
- subgraph, spanning subgraph, and induced subgraph
- degree of a vertex, minimum degree, maximum degree and average degree
- r-regular graph
- connected graph and connected component of a graph
- bipartite graph
- tree, forest, and leaf
- adjacency matrix and Laplacian
- multigraph and edge contraction for multigraphs
- matching, perfect matching, and A-perfect matching
- vertex cover and edge cover
- matching number and vertex cover number
- neighborhood of a vertex
- k-factor
- cut vertex and bridge
- k-connected and the connectivity of a graph
- ℓ-edge-connnected and the edge connectivity of a graph
- vertex cut and edge cut
- edge contraction for graphs

Theorems. The test assumes knowledge of the following theorems

- **Handshaking Lemma.** If $G = (V, E)$ is graph then
$$\sum_{v \in V} d(v) = 2|E|.$$
- If $G = (V, E)$ is a graph with at least one edge then G has a subgraph H such that
$$\delta(H) \geq \frac{\bar{d}(G)}{2},$$
where $\bar{d}(G)$ is the average degree of G.
- G is bipartite if and only if G contains no odd cycle.
- **characterizations of trees.** Let $G = (V, E)$ be a graph. The following are equivalent
  - G is a tree.
  - For every pair of vertices $u, v \in V$ there is a unique path from u to v.
  - G is an edge-maximal acyclic graph.
  - G is an edge-minimal connected graph.
  - G is connected and $|E| = |V| - 1$.
  - G is acyclic and $|E| = |V| - 1$.
- **Matrix Tree Theorem.** If $G = (V, E)$ is a graph then the number of spanning trees is equal to $det(L_G[v])$ for all $v \in V$. (Where L_G is the Laplacian of G and $L_G[v]$ is the matrix we get by deleting the row and column on L_G corresponding to vertex v.)
- **Hall's Theorem.** Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \dot{\cup} B$. G contains an A-perfect matching if and only
$$|N(X)| \geq |X| \quad \text{for all } X \subseteq A.$$
- **König's Theorem.** If $G = (V, E)$ is a bipartite graph then $\nu(G) = \tau(G)$, where $\nu(G)$ is matching number of G and $\tau(G)$ is the vertex cover number of G.
- **Tutte's 1-factor Theorem.** Let $G = (V, E)$ be a graph. G contains a 1-factor if and only if
$$q(S) \leq |S| \quad \text{for all } S \subseteq V,$$
where $q(S)$ is the number of odd components of $G - S$.
- **Handle decomposition of 2-connected graphs.** If $G = (V, E)$ is a 2-connected graph then there exists a sequence of graphs
$$G_1 \subset G_2 \subset \ldots G_k = G$$
such that G_1 is a cycle and we optain G_{i+1} from G_i by adding a path that joins two vertices of G_i

- **Tutte's characterization of 3-connected graphs.** A graph $G = (V, E)$ is 3-connected if and only if there is a sequence of graphs

$$K_4 = G_0, G_1, \ldots, G_k = G$$

such that for $i = 1, \ldots, k$ there is an $xy \in E(G_i)$ such that

  - $d(x), d(y) \geq 3$, and
  - $G_{i-1} = G_i/xy$.

Review Problems. Doing these problems should help in preparation for the third test.

1. Let $n \geq 8$. Prove that every n-vertex graph iwth at least $6n - 20$ edges contains a subgraph with minimum degree 7.

2. Prove that any two paths of maximum length in a connected graph have a vertex in common.

3. Prove that a regular bipartite graph with degree at least two does not have a bridge.

4. Let $G = (V.E)$ be a graph. Prove that there is a paritition $V = A \dot{\cup} B$ such that $|E(A, B)| \geq |E|/2$.

5. A **tournament** is a complete oriented graph; that is, a directed graph in which for any two distinct vertics u, v either the arc (u, v) is in the digraph or the arv (v, u) is in the digraph. Shoe that every tournament has a directed path that includes all of the vertices in the tournament. (Such a path is called a Hamiltonian path.)

6. Let G be a bipartite graph with bipartition $V = A \dot{\cup} B$ such that $|A| = |B| = n$. Show that if the minimum degree of G is at $n/2$ then G has a perfect matching.

7. A Latin square is an $n \times n$ matrix A such that all entries in the matrix are in the set $[n] := \{1, 2, \ldots, n\}$ and every number in $[n]$ appears exactly once in each row and column of A. An $r \times a$ matrix B is a Latin rectangle on $[n]$ if all the entries in B are from the set $[n]$ and no integer appears more than once in any row or column of B.

   Show that an $r \times n$ Latin rectangle on $[n]$ can be extended to give a full $n \times n$ Latin square.

8. Let $G = (V, E)$ be graph such that $\kappa(G) = k \geq 1$. Suppose that the set X is a minimum vertex-cut. So X is a set of k vertices and there is a partition $V = A \dot{\cup} X \dot{\cup} B$ such that $A, B \neq \emptyset$ ans there are no edges joining A and B.

   Consider the bipartite graph on vertex set $A \cup X$ that consists of all edges in G that have one edge in A and one edge in X. Prove that H has an A-perfect matching or an X-perfect matching.

9. Let G be an r-regular bipartite graph and let F be a collection of $r - 1$ edges in G. Show that $G - F$ has a perfect matching.

10. Show that König's Theorem implies Hall's Theorem.

11. Let $G = (V, E)$ be a 2-connected graph and let X, Y be disjoint sets of vertices such that $|X| = |Y| = 2$. Use the handle theorem to show that there are two disjoint paths joining X and Y.

12. Prove that following strengthening of Tutte's Theorem for Trees:

    A tree T has a perfact matching if and only if $q(T - v) = 1$ for all $v \in T$.

Prove this directly (rather than by applying Tutte or the proof of Tutte we gave in class).

13. Let $k \leq \ell$ be positive integers. Show that there is a graph G such that $\kappa(G) = k$ and $\lambda(G) = \ell$. (Recall that κ is the (vertex-)connectivity of G and λ is the edge-connectivity of G.

14. Describe all maximal graphs on $n = 2k$ vertices that do not contain a 1-factor. In other words, describe the set of all graphs $G = (V, E)$ with the property that G does not have a 1-factor but $G + \{x, y\}$ does have a 1-factor for all $\{x, y\} \in \binom{V}{2} \setminus E$.

15. Let $n \geq 4$. What is the minimum number of edges in 3-connected graph on n vertices?

True/False questions. The test will include some questions in the following format.

Are the following statements True or False? For each statement give a brief justification for your answer.

i. A 3-regular graph that contains a cut-vertex has a bridge.

ii. If G is a graph such that $\kappa(G) = k$ and X is a vertex cut such that $|X| = k$ then $G - X$ has exactly two connected components.

iii. If $G = (V, E)$ is a graph with three cycles then $|E| \geq |V| + 2$.

iv. If G and H are graphs and there is a map $\varphi : V(G) \to V(H)$ such that $d(x) = d(\varphi(x))$ for all $x \in V(G)$ then $G \cong H$.

v. Every vertex cover has a subset that is a minimum vertex cover.