1 Modica–Mortola Functional

2 Γ-Convergence

Let (X, d) be a metric space and consider a sequence $\{F_n\}$ of functionals $F_n : X \to [-\infty, \infty]$. We say that $\{F_n\}$ Γ-converges to a functional $F : X \to [-\infty, \infty]$ if the following properties hold:

(i) **(Liminf Inequality)** For every $x \in X$ and every sequence $\{x_n\} \subset X$ such that $x_n \to x$,

$$F(x) \le \liminf_{n\to+\infty} F_n(x_n). \tag{1}$$

(ii) **(Limsup Inequality)** For every $x \in X$ there exists $\{x_n\} \subset X$ such that $x_n \to x$ and

$$\limsup_{n\to\infty} F_n(x_n) \le F(x). \tag{2}$$

Remark 1 *An important property of Γ-convergence is that if each F_n is bounded from below and admits a minimizer x_n, that is,*

$$F_n(x_n) = \min_{y\in X} F_n(y),$$

if $x_n \to x$ for some $x \in X$, and if $\{F_n\}$ Γ-converges to F, then x is a minimizer of F and

$$\min_{y\in X} F(y) = F(x) = \lim_{n\to+\infty} F_n(x_n) = \lim_{n\to+\infty} \min_{y\in X} F_n(y).$$

To see this, let $y \in Y$ and apply property (ii) to find a sequence $\{y_n\} \subset X$ such that $y_n \to y$ and

$$\limsup_{n\to\infty} F_n(y_n) \le F(y).$$

Using the fact that $F_n(x_n) = \min_{y\in X} F_n(y)$ and property (i) for the sequence $\{x_n\}$, we have that

$$F(x) \le \liminf_{n\to+\infty} F_n(x_n) \le \limsup_{n\to\infty} F_n(x_n) \le \limsup_{n\to\infty} F_n(y_n) \le F(y),$$

which shows that x is a minimizer of F. Taking $y = x$, gives that there exists $\lim_{n\to+\infty} F_n(x_n) = F(x)$.

3 Compactness

For $\varepsilon > 0$ consider the functional

$$F_\varepsilon : W^{1,2}\left(\Omega; \mathbb{R}^d\right) \to [0, \infty]$$

defined by

$$F_\varepsilon(u) := \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \right) dx,$$

where the double well potential $W : \mathbb{R}^d \to [0, \infty)$ satisfies the following hypotheses:

(H_1) W is continuous, $W(z) = 0$ if and only if $z \in \{\alpha, \beta\}$ for some $\alpha, \beta \in \mathbb{R}^d$ with $\alpha \neq \beta$.

(H_2) There exist $L > 0$ and $R > 0$ such that

$$W(z) \geq L|z|.$$

for all $z \in \mathbb{R}^d$ with $|z| \geq R$.

Theorem 2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that the double-well potential W satisfies conditions (H_1) and (H_2). Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}(\Omega; \mathbb{R}^d)$ be such that*

$$M := \sup_n F_{\varepsilon_n}(u_n) < \infty. \tag{3}$$

Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in BV(\Omega; \{\alpha, \beta\})$ such that

$$u_{n_k} \to u \text{ for in } L^1(\Omega; \mathbb{R}^d).$$

Proof. We begin by showing that $\{u_n\}$ is bounded in $L^1(\Omega; \mathbb{R}^d)$ and equi-integrable. By (3) and (H_2), for every $t \geq R$,

$$\int_{\{|u_n| \geq t\}} |u_n| \, dx \leq \frac{1}{L} \int_{\{|u_n| \geq t\}} W(u_n(x)) \, dx \leq M \varepsilon_n,$$

which implies that $\{u_n\}$ is equi-integrable. Moreover, since Ω has finite measure,

$$\begin{aligned} L \int_\Omega |u_n| \, dx &= L \int_{\{|u_n| \geq R\}} |u_n| \, dx + L \int_{\{|u_n| < R\}} |u_n| \, dx \\ &\leq L \int_{\{|u_n| \geq R\}} W(u_n(x)) \, dx + LR|\Omega| \leq M\varepsilon_n + LR|\Omega|. \end{aligned}$$

In view of the Vitali's convergence theorem, to obtain strong convergence of a subsequence, it suffices to prove convergence in measure or pointwise $\mathcal{L}^N$ a.e. in Ω. We divide the proof in two steps.

Step 1: Assume first that $d = 1$. Define

$$W_1(z) := \min\{W(z), 1\}, \quad z \in \mathbb{R}.$$

Since $0 \leq W_1 \leq W$, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} F_{\varepsilon_n}(u_n) &\geq \frac{1}{2} \int_\Omega \left(\sqrt{W_1(u_n(x))} |\nabla u_n(x)| \right) dx \\ &= \int_\Omega \left(|\nabla (\Phi_1 \circ u_n)(x)| \right) dx, \end{aligned}$$

where

$$\Phi_1(t) := \frac{1}{2}\int_0^t \sqrt{W_1(s)}\,ds, \quad t \in \mathbb{R}. \tag{4}$$

Then by (3),

$$\sup_n \int_\Omega \left(|\nabla(\Phi_1 \circ u_n)|\right) dx \leq M. \tag{5}$$

Moreover, since $\sqrt{W_1} \leq 1$,

$$|\Phi_1(u_n(x))| \leq |u_n(x)|$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and for all $n \in \mathbb{N}$. Since $\{u_n\}$ is bounded in $L^1(\Omega)$, it follows that the sequence $\{\Phi_1 \circ u_n\}$ is bounded in $L^1(\Omega)$. By the Rellich–Kondrachov theorem, there exist a subsequence $\{u_{n_k}\}$ and a function $w \in BV(\Omega)$ such that

$$w_k := \Phi_1 \circ u_{n_k} \to w \text{ in } L^1_{\text{loc}}(\Omega).$$

By taking a further subsequence, if necessary, without loss of generality, we may assume that $w_k(x) \to w(x)$ and that $W(u_{n_k}(x)) \to 0$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Since the function $W_1(t) > 0$ for all $t \neq \alpha, \beta$, it follows from (4) that the function Φ_1 is strictly increasing and continuous. Thus, its inverse Φ_1^{-1} is continuous and

$$u_{n_k}(x) = \Phi_1^{-1}(w_k(x)) \to \Phi_1^{-1}(w(x)) := u(x)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$. It follows by (H_1) and the fact that $W(u_{n_k}(x)) \to 0$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, that $u(x) \in \{\alpha, \beta\}$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. In turn, $w(x) \in \{\Phi_1(\alpha), \Phi_1(\beta)\}$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, and since $w \in BV(\Omega)$, we may write

$$w = \Phi_1(\alpha)\chi_E + \Phi_1(\beta)(1 - \chi_E) \tag{6}$$

for a set $E \subset \Omega$ of finite perimeter. Hence,

$$u = \alpha\chi_E + \beta(1 - \chi_E) \tag{7}$$

belongs to $BV(\Omega)$.

Step 2: Assume that $d \geq 2$ and that $|\alpha| \neq |\beta|$. For every $t \geq 0$ define

$$V(t) := \min_{|z|=t} W(z).$$

Then V is upper semicontinuous, $V(t) > 0$ for $t \neq |\alpha|, |\beta|$, $V(|\alpha|) = V(|\beta|) = 0$, and $V(t) \geq Lt$ for $t \geq R$. For every $u \in W^{1,2}(\Omega; \mathbb{R}^d)$ define

$$G_\varepsilon(u) := \int_\Omega \left(\frac{1}{\varepsilon}V(|u|) + \varepsilon|\nabla|u||^2\right) dx \leq F_\varepsilon(u).$$

Then by (3),

$$\sup_n G_{\varepsilon_n}(u_n) \leq \sup_n F_{\varepsilon_n}(u_n; \Omega) < \infty,$$

and so by the compactness in the scalar case $d = 1$, there exist a subsequence $\{u_{n_k}\}$ and $w \in BV(\Omega)$ such that

$$w_k := \Phi_2 \circ |u_{n_k}| \to w \text{ in } L^1_{\text{loc}}(\Omega),$$

where

$$\Phi_2(t) := \frac{1}{2}\int_0^t \sqrt{V_1(s)}\, ds, \quad t \in \mathbb{R}$$

and

$$V_1(z) := \min\{V(z), 1\}, \quad z \in \mathbb{R}.$$

Hence,

$$|u_{n_k}| \to v := \Phi_2^{-1} \circ w \text{ in } L^1(\Omega).$$

By taking a further subsequence, if necessary, without loss of generality, we may assume that $|u_{n_k}(x)| \to v(x)$ and that $W(u_{n_k}(x)) \to 0$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. This implies that $v \in BV(\Omega; \{\alpha, \beta\})$. Define

$$u(x) := \begin{cases} \alpha & \text{if } v(x) = |\alpha|, \\ \beta & \text{if } v(x) = |\beta|. \end{cases}$$

We claim that

$$u_{n_k} \to u \text{ in } L^1\left(\Omega; \mathbb{R}^d\right).$$

To see this, fix $x \in \Omega$ such that $|u_{n_k}(x)| \to v(x)$ and that $W(u_{n_k}(x)) \to 0$. Then, by (H_1), necessarily, $u_{n_k}(x) \to u(x)$.

Step 3: If $d \geq 2$ and $|\alpha| = |\beta|$, let e_i be a vector of the canonical basis of $\mathbb{R}^d$ such that $\alpha \cdot e_i \neq \beta \cdot e_i$. Then $|\alpha + e_i| \neq |\beta + e_i|$. It suffices to apply the previous step with W replaced by

$$\hat{W}(z) := W(z - e_i), \quad z \in \mathbb{R}^d,$$

and u_n by $u_n + e_i$. ■

4 Gamma Convergence of the Modica–Mortola Functional

In view of the previous theorem, the metric convergence in the definition of Γ-convergence should be $L^1\left(\Omega; \mathbb{R}^d\right)$. Thus, we extend F_ε to $L^1\left(\Omega; \mathbb{R}^d\right)$ by setting

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right) dx & \text{if } u \in W^{1,2}\left(\Omega; \mathbb{R}^d\right), \\ \infty & \text{if } u \in L^1\left(\Omega; \mathbb{R}^d\right) \setminus W^{1,2}\left(\Omega; \mathbb{R}^d\right). \end{cases}$$

Let $\varepsilon_n \to 0^+$. Under appropriate hypotheses on W and Ω, we will show that the sequence of functionals $\{F_{\varepsilon_n}\}$ Γ-converges to the functional

$$F(u) := \begin{cases} c_W |Du|(\Omega) & \text{if } u \in BV(\Omega; \{\alpha, \beta\}), \\ \infty & \text{if } u \in L^1\left(\Omega; \mathbb{R}^d\right) \setminus BV(\Omega; \{\alpha, \beta\}). \end{cases}$$

We begin with the liminf inequality. Consider a sequence $\{u_n\} \subset L^1\left(\Omega;\mathbb{R}^d\right)$ such that $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^d\right)$ for some $u \in L^1\left(\Omega;\mathbb{R}^d\right)$. If

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) = \infty,$$

then there is nothing to prove, thus we assume that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) < \infty. \tag{8}$$

Let $\{\varepsilon_{n_k}\}$ be a subsequence of $\{\varepsilon_n\}$ such that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) = \lim_{k\to+\infty} F_{\varepsilon_{n_k}}(u_{n_k}) < \infty.$$

Then $F_{\varepsilon_{n_k}}(u_{n_k}) < \infty$ for all k sufficiently large. Hence, $u_{n_k} \in W^{1,2}\left(\Omega;\mathbb{R}^d\right)$ for all k sufficiently large. Moreover, if hypotheses (H_1) and (H_2), then by Theorem 2, $u \in BV(\Omega;\{\alpha,\beta\})$. Finally, by extracting a further subsequence, not relabelled, we can assume that $u_n(x) \to u(x)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Hence, in what follow, without loss of generality, we will assume that (8) holds, that $\{u_n\} \subset W^{1,2}\left(\Omega;\mathbb{R}^d\right)$, that $u \in BV(\Omega;\{\alpha,\beta\})$, that $\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n)$ is actually a limit, and that $\{u_n\}$ converges to u in $L^1\left(\Omega;\mathbb{R}^d\right)$ and pointwise $\mathcal{L}^N$ a.e. in Ω.

5 Liminf Inequality, $N = 1$, $d = 1$

We begin with the case $N = d = 1$ and assume that (H_1) and (H_2) hold with $\alpha < \beta$ and that $\Omega = (a,b)$. Consider $u \in BV(\Omega;\{\alpha,\beta\})$. Without loss of generality we may assume that there exists a partition

$$a = t_0 < t_1 < \cdots < t_m = b$$

such that $u(x) = \alpha$ in (t_{2i-2}, t_{2i-1}) and $u(x) = \beta$ in (t_{2i-1}, t_{2i}). Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}(\Omega)$ be such that $u_n \to u$ in $L^1(\Omega)$ and pointwise $\mathcal{L}^1$ a.e. in Ω. Fix $\delta > 0$ small. Then

$$\int_\Omega \left(\frac{1}{\varepsilon_n}W(u_n) + \varepsilon_n|u_n'|^2\right)\,dx \geq \sum_{i=1}^m \int_{t_i-\delta}^{t_i+\delta} \left(\frac{1}{\varepsilon_n}W(u_n) + \varepsilon_n|u_n'|^2\right)\,dx.$$

Consider one term

$$\int_{t_i-\delta}^{t_i+\delta} \left(\frac{1}{\varepsilon}W(u_n) + \varepsilon_n|u_n'|^2\right)\,dx.$$

For simplicity we can assume that $t_i = 0$ and, by taking δ smaller, that $u_n(-\delta) \to \alpha$ and $u_n(\delta) \to \beta$. Then

$$\begin{aligned}\int_{-\delta}^{\delta} \left(\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n|u_n'(x)|^2\right)\,dx &= \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} \left(W(u_n(\varepsilon_n y)) + \varepsilon_n^2|u_n'(\varepsilon_n y)|^2\right)\,dy \qquad (9)\\ &= \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} \left(W(v_n(y)) + |v_n'(y)|^2\right)\,dy,\end{aligned}$$

where we have made the change of variables $x = \varepsilon_n y$ and $v_n(y) := u_n(\varepsilon_n y)$. It follows that

$$\begin{aligned}\int_{-\delta}^{\delta}\left(\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n|u_n'(x)|^2\right)dx &\geq \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n}\left(W(v_n(y)) + (v_n'(y))^2\right)dy \\ &\geq \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} 2\sqrt{W(v_n)}v_n'(y)\,dy \\ &= \int_{u_n(-\delta)}^{u_n(\delta)} 2\sqrt{W(s)}\,ds \to \int_{\alpha}^{\beta} 2\sqrt{W(s)}\,ds\end{aligned}$$

as $n \to \infty$, where we have made the change of variables $s = w_n(y)$. In turn,

$$\liminf_{n\to\infty}\int_{\Omega}\left(\frac{1}{\varepsilon_n}W(u_n) + \varepsilon_n|u_n'|^2\right)dx \geq \int_{\alpha}^{\beta} 2\sqrt{W(s)}\,ds \text{ (number of jumps of } u).$$

6 Limsup Inequality, $N = 1$, $d = 1$

Assume in addition to (H_1) and (H_2) that W is of class C^1. Let g be the solution of the Cauchy problem

$$g' = \sqrt{W(g)}, \quad g(0) = \frac{\alpha+\beta}{2}. \tag{10}$$

Then g is globally defined, strictly increasing, $\alpha < g(t) < \beta$ for all t, and

$$\lim_{t\to-\infty} g(t) = \alpha \quad \lim_{t\to\infty} g(t) = \beta.$$

Define

$$c_W := \int_{\mathbb{R}}\left(W(g(t)) + |g'(t)|^2\right)dt. \tag{11}$$

Note that by (10) we have

$$\begin{aligned}c_W &= \lim_{n\to\infty}\int_{-n}^{n}\left(W(g(t)) + |g'(t)|^2\right)dt = 2\lim_{n\to\infty}\int_{-n}^{n}\sqrt{W(g(t))}g'(t)\,dt \\ &= 2\lim_{n\to\infty}\int_{g(-n)}^{g(n)}\sqrt{W(s)}\,ds = 2\int_{\alpha}^{\beta}\sqrt{W(s)}\,ds.\end{aligned}$$

Hence, if $\Omega = \mathbb{R}$ and

$$u(y) = \begin{cases} \alpha & \text{if } y < 0, \\ \beta & \text{if } y \geq 0, \end{cases}$$

then the right sequence would be

$$u_n(x) := g\left(\frac{x}{\varepsilon_n}\right),$$

since

$$\int_{\mathbb{R}} \left(\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n |u_n'(x)|^2 \right) dx = \int_{\mathbb{R}} \left(\frac{1}{\varepsilon_n} W(g\left(\frac{x}{\varepsilon_n}\right)) + \varepsilon_n \left| \frac{1}{\varepsilon_n^2} g'\left(\frac{x}{\varepsilon_n}\right) u_n'(x) \right|^2 \right) dx$$
$$= \int_{\mathbb{R}} \left(W(g(y)) + |g'(y)|^2 \right) dy = c_W.$$

In the case of a general $u \in BV(\Omega; \{\alpha, \beta\})$, we need to glue g to α and β. Let u be as in the previous section. Fix $\rho > 0$ and let b_ρ be such that $g(b_\rho) = \beta - \rho$ and let a_ρ be such that $g(a_\rho) = \alpha + \rho$. Define

$$g_\rho(y) := \begin{cases} \beta & \text{if } y \geq b_\rho + 1, \\ \rho(y - b_\rho) + \beta - \rho & \text{if } b_\rho < y < b_\rho + 1, \\ g(y) & \text{if } a_\rho \leq y \leq b_\rho, \\ \rho(y - a_\rho) + \alpha + \rho & \text{if } a_\rho - 1 < y < a_\rho, \\ \alpha & \text{if } y \leq a_\rho - 1, \end{cases}$$

and

$$u_{n,\rho}(x) := \begin{cases} g_\rho\left(\frac{x - t_{2i}}{\varepsilon_n}\right) & \text{if } t_{2i} - r < x < t_{2i} + r, \\ g_\rho\left(\frac{t_{2i+1} - x}{\varepsilon_n}\right) & \text{if } t_{2i+1} - r < x < t_{2i+1} + r, \\ u(x) & \text{otherwise,} \end{cases}$$

where $2r < \min\{t_{i+1} - t_i : i = 0, \ldots, m-1\}$. Then using the change of variables $\frac{x - t_{2i}}{\varepsilon_n} = y$ or $\frac{t_{2i+1} - x}{\varepsilon_n} = y$ we get

$$\int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u_{n,\rho}'|^2 \right) dx = \sum_{i=1}^m \int_{t_i - r}^{t_i + r} \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u_{n,\rho}'|^2 \right) dx$$
$$= \sum_{i=1}^m \int_{-r/a_n}^{r/\varepsilon_n} \left(W(g_\rho(y)) + |g_\rho'(y)|^2 \right) dx$$
$$= \sum_{i=1}^m \int_{a_\rho - 1}^{b_\rho + 1} \left(W(g_\rho(y)) + |g_\rho'(y)|^2 \right) dx$$

for all n sufficiently large.

Since $W(\beta) = 0$, by the mean value theorem

$$W(t) = W(\beta) + W'(\theta)(t - \beta) = 0 + W'(\theta)(t - \beta)$$

for some θ between t and β. Hence, in $[b_\rho, b_\rho + 1]$,

$$W(g_\rho(y)) \leq (\beta - g_\rho(y)) \max_{[\alpha,\beta]} |W'| = (-(\rho(y - b_\rho) - \rho)) \max_{[\alpha,\beta]} |W'| \leq 2\rho \max_{[\alpha,\beta]} |W'|,$$
$$g_\rho'(y) = \rho.$$

It follows that

$$\int_{a_\rho - 1}^{b_\rho + 1} \left(W(g_\rho(y)) + |g_\rho'(y)|^2 \right) dx \leq C\rho,$$

where we have made the change of variables $s = w_n(y)$. On the other hand,

$$\int_{a_\rho}^{b_\rho} \left(W(g_\rho(y)) + |g_\rho'(y)|^2\right) dx = \int_{a_\rho}^{b_\rho} \left(W(g(y)) + |g'(y)|^2\right) dx$$
$$\leq \int_{\mathbb{R}} \left(W(g(y)) + |g'(y)|^2\right) dy = c_W.$$

Hence, we get

$$\int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u_{n,\rho}'|^2\right) dx \leq c_W m + C\rho m.$$

In turn,

$$\limsup_{\rho\to 0^+} \limsup_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u_{n,\rho}'|^2\right) dx \leq c_W \,(\text{number of jumps of } u).$$

Moreover, using the change of variables $\frac{x-t_{2i}}{\varepsilon_n} = y$ or $\frac{t_{2i+1}-x}{\varepsilon_n} = y$,

$$\int_\Omega |u_{n,\rho} - u|\, dx = \sum_{i=1}^m \int_{t_i-r}^{t_i+r} |u_{n,\rho}(x) - u(x)|\, dx = \sum_{i=1}^m \varepsilon_n \int_{-r/a_n}^{r/\varepsilon_n} |g_\rho(y) - v(y)|\, dy \tag{12}$$
$$= \sum_{i=1}^m \varepsilon_n \int_{a_\rho-1}^{b_\rho+1} |g_\rho(y) - v(y)|\, dy \leq C_\rho \varepsilon_n \to 0$$

as $n \to \infty$, where

$$v(y) := \begin{cases} \beta & \text{if } y > 0, \\ \alpha & \text{if } y \leq 0. \end{cases}$$

We now diagonalize the sequence $\{u_{n,\rho}\}$ to obtaining a sequence $\{u_{n,\rho_n}\}$ converging to u in $L^1(\Omega)$ and such that

$$\lim_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho_n}) + \varepsilon_n |u_{n,\rho_n}'|^2\right) dx \leq c_W \,(\text{number of jumps of } u).$$

7 Liminf Inequality, $N = 1$, $d \geq 1$

In this case the constant c_W should be replaced by

$$c_W := \inf\left\{ \int_{\mathbb{R}} \left(W(g(t)) + |g'(t)|^2\right) dt : \right. \tag{13}$$
$$\left. g \in H^1_{\text{loc}}\left(\mathbb{R}; \mathbb{R}^d\right) \text{ such that } \lim_{t\to-\infty} g(t) = \alpha \quad \lim_{t\to\infty} g(t) = \beta \right\}.$$

We proceed as in the case $d = 1$ up to (9). Fix $\rho > 0$ small. Since $u_n(-\delta) \to \alpha$ and $u_n(\delta) \to \beta$, we have that

$$|v_n(-\delta/\varepsilon_n) - \alpha| = |u_n(-\delta) - \alpha| < \rho, \quad |v_n(\delta/\varepsilon_n) - \beta| = |u_n(\delta) - \beta| < \rho$$

for all n large.

We now extend v_n to $\mathbb{R}$ by setting

$$w_n(y) := \begin{cases} \beta & \text{if } y \geq \delta/\varepsilon_n + 1, \\ (\beta - v_n(\delta/\varepsilon_n))(y - \delta/\varepsilon_n) + v_n(\delta/\varepsilon_n) & \text{if } \delta/\varepsilon_n < y < \delta/\varepsilon_n + 1, \\ v_n(y) & \text{if } -\delta/\varepsilon_n \leq y \leq \delta/\varepsilon_n, \\ (v_n(-\delta/\varepsilon_n) - \alpha)(y + \delta/\varepsilon_n) + v_n(-\delta/\varepsilon_n) & \text{if } -\delta/\varepsilon_n - 1 < y < -\delta/\varepsilon_n, \\ \alpha & \text{if } y \leq -\delta/\varepsilon_n - 1, \end{cases}$$

Since $W(\beta) = 0$ by the mean value theorem

$$W(z) = W(\beta) + \nabla W(\theta) \cdot (z - \beta) = 0 + \nabla W(\theta) \cdot (z - \beta)$$

for some θ in the segment between z and β. Hence, in $[\delta/\varepsilon_n, \delta/\varepsilon_n + 1]$,

$$\begin{aligned} W(w_n(y)) &\leq |v_n(y) - \beta| \max_{B(\beta,2)} |\nabla W| \leq C\rho, \\ w_n'(y) &= \rho. \end{aligned}$$

A similar estimate can be made in the interval $[-\delta/\varepsilon_n - 1, -\delta/\varepsilon_n]$. It follows that

$$\begin{aligned} \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} \left(W(v_n(y)) + |v_n'(y)|^2\right) dy &\geq \int_{\mathbb{R}} \left(W(w_n) + |w_n'|^2\right) dy - C\rho, \\ &\geq c_W - C\rho. \end{aligned}$$

In turn,

$$\liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u_n'|^2\right) dx \geq c_W \text{ (number of jumps of } u) - C\rho.$$

Letting $\rho \to 0$ gives

$$\liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u_n'|^2\right) dx \geq c_W \text{ (number of jumps of } u).$$

8 Limsup Inequality, $N = 1$, $d \geq 1$

Fix $\eta > 0$ and by (13) find $g \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^d)$ such that $\lim_{t\to-\infty} g(t) = \alpha$, $\lim_{t\to\infty} g(t) = \beta$ and

$$\int_{\mathbb{R}} \left(W(g(t)) + |g'(t)|^2\right) dt \leq c_W + \eta. \tag{14}$$

Let u be as in the previous section. Fix $\rho > 0$ and let $b_\rho >> 1$ be such that $|g(b_\rho) - \beta| < \rho$ and let $a_\rho << -1$ be such that $|g(a_\rho) - \alpha| < \rho$. Define

$$g_\rho(y) := \begin{cases} \beta & \text{if } y \geq b_\rho + 1, \\ (\beta - g(b_\rho))(y - b_\rho) + g(b_\rho) & \text{if } b_\rho < y < b_\rho + 1, \\ g(y) & \text{if } a_\rho \leq y \leq b_\rho, \\ (g(a_\rho) - \alpha)(y - a_\rho) + g(a_\rho) & \text{if } a_\rho - 1 < y < a_\rho, \\ \alpha & \text{if } y \leq a_\rho - 1, \end{cases} \tag{15}$$

and

$$u_{n,\rho}(x) := \begin{cases} g_\rho\left(\frac{x - t_{2i}}{\varepsilon_n}\right) & \text{if } t_{2i} - r < x < t_{2i} + r, \\ g_\rho\left(\frac{t_{2i+1} - x}{\varepsilon_n}\right) & \text{if } t_{2i+1} - r < x < t_{2i+1} + r, \\ u(x) & \text{otherwise,} \end{cases}$$

where $2r < \min\{t_{i+1} - t_i : i = 0, \ldots, m-1\}$. Reasoning as in the previous sections we get

$$\begin{aligned} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u'_{n,\rho}|^2\right) dx &= \sum_{i=1}^m \int_{t_i - r}^{t_i + r} \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u'_{n,\rho}|^2\right) dx \\ &= \sum_{i=1}^m \int_{-r/a_n}^{r/\varepsilon_n} \left(W(g_\rho(y)) + |g'_\rho(y)|^2\right) dx \\ &= \sum_{i=1}^m \int_{a_\rho - 1}^{b_\rho + 1} \left(W(g_\rho(y)) + |g'_\rho(y)|^2\right) dx \\ &\leq m \int_{\mathbb{R}} \left(W(g(t)) + |g'(t)|^2\right) dt + C\rho \\ &\leq (c_W + \eta) m + C\rho \end{aligned}$$

for all n sufficiently large. Note that we can take $\eta = \rho$. It follows that

$$\limsup_{\rho \to 0^+} \limsup_{n \to \infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u'_{n,\rho}|^2\right) dx \leq c_W m$$

and as in (12), $\int_\Omega |u_{n,\rho} - u| \, dx \to 0$ as $n \to \infty$. Again we can diagonalize to get a sequence $\{u_{n,\rho_n}\}$.

9 Liminf Inequality, $N \geq 1$, $d = 1$

10 Liminf Inequality, $N \geq 1$, $d \geq 1$

11 Limsup Inequality, $N \geq 1$, $d \geq 1$

For $N \geq 1$, given $u \in BV(\Omega; \{\alpha, \beta\})$, we write

$$u = \alpha \chi_E + \beta \chi_{\Omega \setminus E}.$$

Assume first that E is a regular set, that is, that E is an open set with $\Omega \cap \partial E$ of class C^2, and that E meets the boundary of Ω transversally, that is, $\mathcal{H}^{N-1}(\partial E \cap \partial\Omega) = 0$. Let g and g_ρ be as in (14) and (15). In this case we take

$$u_{n,\rho}(x) := \begin{cases} \alpha & \text{if } \operatorname{dist}(x, \partial E) < -\varepsilon_n L_\rho \\ g_\rho(\operatorname{dist}(x, \partial E)/\varepsilon_n) & \text{if } |\operatorname{dist}(x, \partial E)| \le \varepsilon_n L_\rho, \\ \beta & \text{if } \operatorname{dist}(x, \partial E) > \varepsilon_n L_\rho, \end{cases}$$

where dist is the signed distance, and $L_\rho > 0$ is such that $g_\rho(L_\rho) = \beta$ and $g_\rho(-L_\rho) = \alpha$. Note that the function $\operatorname{dist}(x, \partial E)$ is of class C^2 in the set

$$\left\{x \in \mathbb{R}^N : |\operatorname{dist}(x, \partial E)| < r\right\}$$

for r small, with $|\nabla \operatorname{dist}(x, \partial E)| = 1$. Moreover,

$$\lim_{r \to 0^+} \mathcal{H}^{N-1}\left(\left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial E) = r\right\}\right) = \mathcal{H}^{N-1}(\partial E).$$

Hence, by the coarea formula

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |\nabla u_{n,\rho}|^2\right) dx \\
&\quad = \frac{1}{\varepsilon_n} \int_{|\operatorname{dist}(x,\partial E)| \le \varepsilon_n L_\rho} \left(W(g_\rho(\operatorname{dist}(x, \partial E)/\varepsilon_n)) + |g'_\rho(\operatorname{dist}(x, \partial E)/\varepsilon_n)|^2\right) dx \\
&\quad = \frac{1}{\varepsilon_n} \int_{-\varepsilon_n L_\rho}^{\varepsilon_n L_\rho} \left(W(g_\rho(r/\varepsilon_n)) + |g'_\rho(r/\varepsilon_n)|^2\right) \mathcal{H}^{N-1}\left(\left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial E) = r\right\}\right) dr \\
&\quad = \int_{-L_\rho}^{L_\rho} \left(W(g_\rho(r)) + |g'_\rho(r)|^2\right) \mathcal{H}^{N-1}\left(\left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial E) = \varepsilon_n r\right\}\right) dr \\
&\quad \to \int_{-L_\rho}^{L_\rho} \left(W(g_\rho(r)) + |g'_\rho(r)|^2\right) dr \mathcal{H}^{N-1}(\partial E).
\end{aligned}$$

In turn,

$$\begin{aligned}
\limsup_{n \to \infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |\nabla u_{n,\rho}|^2\right) dx \le \int_{\mathbb{R}} \left(W(g(t)) + |g'(t)|^2\right) dt\, \mathcal{H}^{N-1}(\partial E) \\
+ C\rho \mathcal{H}^{N-1}(\partial E),
\end{aligned}$$

and so taking $\eta = \rho$ and using (14),

$$\limsup_{\rho \to 0^+} \limsup_{n \to \infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |\nabla u_{n,\rho}|^2\right) dx \le c_W \mathcal{H}^{N-1}(\partial E).$$