Problem 1.1. Let $(\mathbb{P}, \Omega, \mathcal{F})$ be a probability space. Prove that

1. If $C, D \in \mathcal{F}$ then $C \cap D, C \setminus D, C \Delta D, \in \mathcal{F}$

2. If $X_1, X_2, \cdots \in \mathcal{F}$ are disjoint, then

$$\mathbb{P}(\bigcup_{i=1}^{n} X_i) = \Sigma_{i=1}^{n} \mathbb{P}(X_i)$$

3. $\mathbb{P}(\bigcup_{i=1}^{\infty} Z_i) \leq \Sigma_{i=1}^{\infty} \mathbb{P}(Z_i)$

4. If $Z_1 \subset Z_2 \subset \cdots$, then $\lim_{n \to \infty} \mathbb{P}(Z_n) = \mathbb{P}(\bigcup_{i=1}^{\infty} Z_i)$

Problem 1.2. A certain planet has $n$ days in one year. What is the probability that among $k$ people on that planet there are (at least) two who have the same birthday?

(no points) What is this probability for $n = 365, k = 30$?

Problem 1.3. Two identical decks of cards, each containing $N$ cards, are shuffled randomly. We say that a $k$-matching occurs if the two decks agree in exactly $k$ places. Show that the probability that there is a $k$-matching is

$$\pi_k = \frac{1}{k!} \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!} \right)$$

for $k = 0, 1, 2, \cdots, N$. We note that $\pi_k \approx 1/(k!e)$ for large $N$ and fixed $k$. Such matching probabilities are used in testing departures from randomness in circumstances such as psychological tests and wine-tasting competitions. (The convention is that $0! = 1$.)

Problem 1.4. The buses which stop at the end of my road do not keep to the timetable. They should run every quarter hour, at 08.30, 08.45, 09.00, \cdots, but in fact each bus is either five minutes early or five minutes late, the two possibilities being equally probable and different buses being independent. Other people arrive at the stop in such a way that, $t$ minutes after the departure of one bus, the probability that no one is waiting for the next
one is $e^{-t/5}$. What is the probability that no one is waiting at 9.00? One day, I come to the bus stop at 9.00 and find no one there; show that the chances are more than four to one that I have missed the nine o’clock bus.

You may use an approximation $e^3 \approx 20$.

**Problem 1.5.** There are $n$ socks in a drawer, three of which are red and the rest black. John chooses his socks by selecting two at random from the drawer, and puts them on. He is three times more likely to wear socks of different colours than to wear matching red socks. Find $n$.

For this value of $n$, what is the probability that John wears matching black socks?

**Problem 1.6.** Show that the axiom that $\mathbb{P}$ is countably additive is equivalent to the axiom that $\mathbb{P}$ is finitely additive and continuous. That is to say, let $\Omega$ be a set and $\mathcal{F}$ an event space of subsets of $\omega$. If $\mathbb{P}$ is a mapping from $\mathcal{F}$ into $[0, 1]$ satisfying

1. $\mathbb{P}(\omega) = 1, \mathbb{P}(\emptyset) = 0$,

2. if $A, B \in \mathcal{F}$, and $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$,

3. if $A_1, A_2, \cdots \in \mathcal{F}$ and $A_i \subseteq A_{i+1}$ for $i = 1, 2, \cdots$, then

$$\mathbb{P}(A) = \lim_{i \to \infty} \mathbb{P}(A_i),$$

where $A = \bigcup_{i=1}^{\infty} A_i$,

then $\mathbb{P}$ satisfies $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$ for all sequences $A_1, A_2, \cdots$ of disjoint events.