# Independent transversals in locally sparse graphs

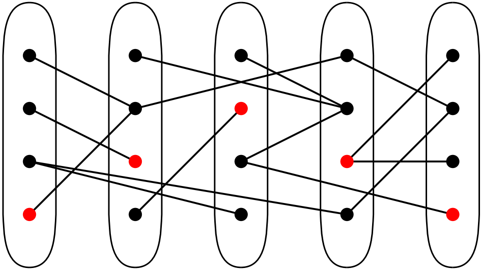
Po-Shen Loh Benny Sudakov

Mathematics Department
Princeton University

# Independent Transversals

Let $G$ be a multipartite graph with maximum degree $\Delta$.

Independent transversal: *One vertex from each part, with no adjacencies between the vertices.*



Bollobás, Erdős, Szemerédi (1975) *What ratios between the part sizes and $\Delta$ will guarantee an independent transversal?*

# Past Results

Alon (1988) *Sufficient: part sizes* $\geq 2e\Delta$.

# Past Results

Alon (1988) *Sufficient: part sizes* $\geq 2e\Delta$.

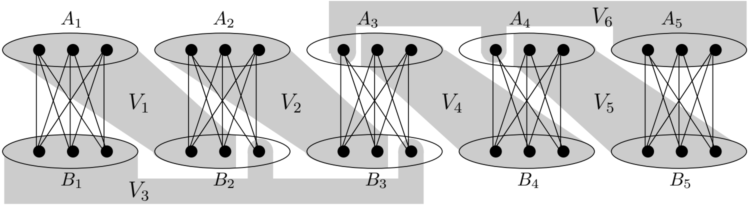
Haxell (2001) *Sufficient: part sizes* $\geq 2\Delta$.

# Past Results

Alon (1988) *Sufficient: part sizes* $\geq 2e\Delta$.

Haxell (2001) *Sufficient: part sizes* $\geq 2\Delta$.

Jin (1992); Yuster (1997); Szabó, Tardos (2005) $2\Delta$ *is tight.*



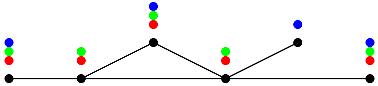

$\Delta = 3$, part sizes $2\Delta - 1 = 5$, and no indep. trans.

# Applications

- Linear arboricity
- Strong chromatic number
- Partitioning into graphs with bounded components
- List coloring
- Cooperative colorings

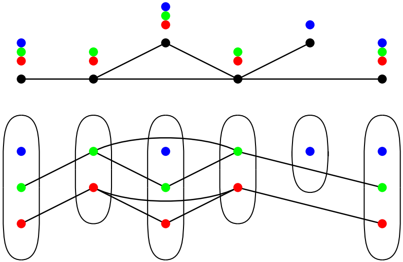
# Application: list coloring

Each vertex of graph $G$ has a list of permitted colors.
Does $G$ have a proper coloring with respect to the lists?



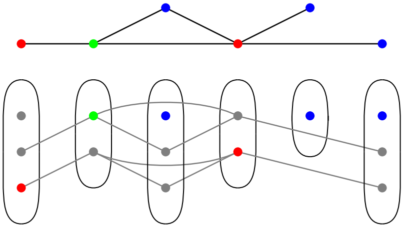
# Application: list coloring

Each vertex of graph $G$ has a list of permitted colors.
Does $G$ have a proper coloring with respect to the lists?



# Application: list coloring

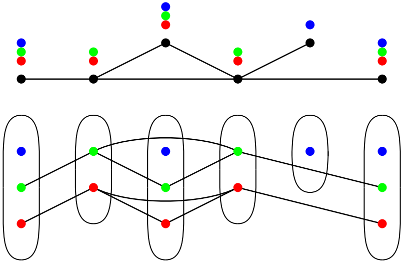
Each vertex of graph $G$ has a list of permitted colors.
Does $G$ have a proper coloring with respect to the lists?



independent transversal $\longleftrightarrow$ proper coloring

# Application: list coloring

Each vertex of graph $G$ has a list of permitted colors.
Does $G$ have a proper coloring with respect to the lists?

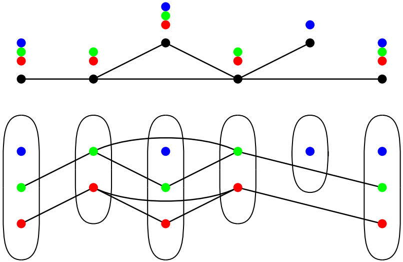


$$\Delta = \max_{v,c} \left\{ \begin{array}{l} c \text{ is in } v\text{'s color list} \;: \\ \#\text{ of neighbors of } v \text{ with color } c \text{ in their list} \end{array} \right\}$$

Reed (1999) *Conj: lists of size* $\Delta + 1 \Rightarrow \exists$ *proper coloring.*

# Application: list coloring

Each vertex of graph $G$ has a list of permitted colors.
Does $G$ have a proper coloring with respect to the lists?



$$\Delta = \max_{v,c} \left\{ \begin{array}{l} c \text{ is in } v\text{'s color list} \ : \\ \#\text{ of neighbors of } v \text{ with color } c \text{ in their list} \end{array} \right\}$$

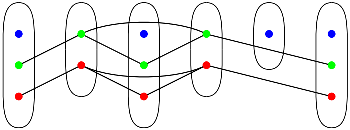
Reed (1999) *Conj: lists of size* $\Delta + 1 \Rightarrow \exists$ *proper coloring.*

Reed, Sudakov (2002) *Lists of size* $(1 + o(1))\Delta$ *are sufficient.*

# Local degree

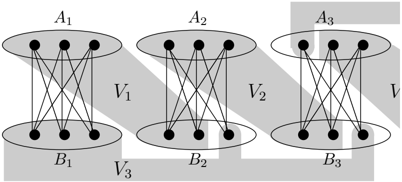
Let $G$ be a multipartite graph with maximum degree $\Delta$.

$$\text{local degree} = \max_{v,i} \left\{ \text{\# of nbrs of } v \text{ in } i^{\text{th}} \text{ part} \right\}$$



local degree $= 1$

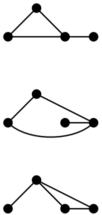
local degree $= \Delta$



# Cooperative colorings

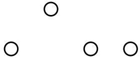
Family of graphs with max. degree $\Delta$, sharing same vertex set.
Color the vertex set with one independent set from each graph.

Q: *What number of graphs will guarantee this is possible?*



$\Longrightarrow$

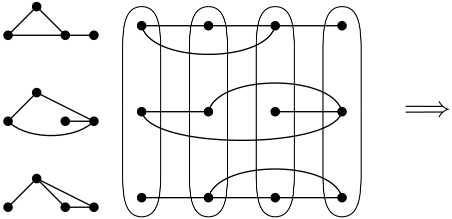
Vertex set:



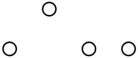
# Cooperative colorings

Family of graphs with max. degree $\Delta$, sharing same vertex set.
Color the vertex set with one independent set from each graph.

Q: *What number of graphs will guarantee this is possible?*



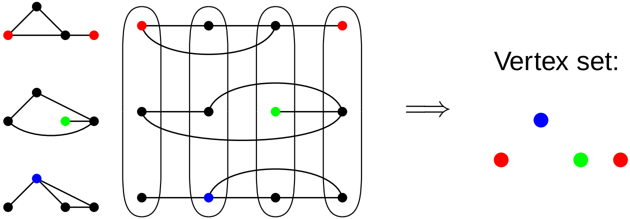
Vertex set:



# Cooperative colorings

Family of graphs with max. degree $\Delta$, sharing same vertex set.
Color the vertex set with one independent set from each graph.

Q: *What number of graphs will guarantee this is possible?*

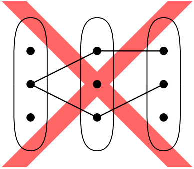



independent transversal $\longleftrightarrow$ cooperative coloring

# Cooperative colorings

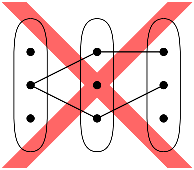
Aharoni, Berger, Holzman, Kfir (2005)

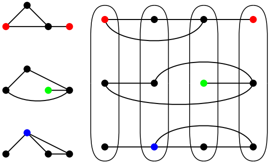
*If there is no path of length* $\leq 4$ *between a pair of vertices in the same part, then parts of size* $(1 + o(1))\Delta$ *are sufficient.*



# Cooperative colorings

Aharoni, Berger, Holzman, Kfir (2005)
*If there is no path of length* $\leq 4$ *between a pair of vertices in the same part, then parts of size* $(1 + o(1))\Delta$ *are sufficient.*

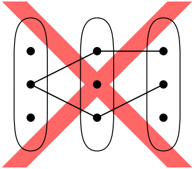


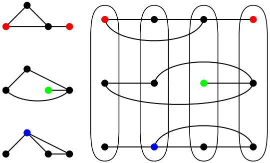


Corollary: $(1 + o(1))\Delta$ graphs are enough to ensure that a cooperative coloring exists.

# Cooperative colorings

Aharoni, Berger, Holzman, Kfir (2005)
*If there is no path of length* $\leq 4$ *between a pair of vertices in the same part, then parts of size* $(1 + o(1))\Delta$ *are sufficient.*





Corollary: $(1 + o(1))\Delta$ graphs are enough to ensure that a cooperative coloring exists.

ABHK Conjecture: *If the local degree is 1, then parts of size* $(1 + o(1))\Delta$ *guarantee an independent transversal.*

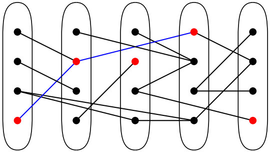
# New results

L., Sudakov (2005) *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\Delta$. Then $G$ has an independent transversal.*

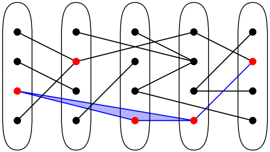
# New results

L., Sudakov (2005) *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1+o(1))\Delta$. Then $G$ has an independent transversal.*

$K_s$-free transversal: *One vertex from each part, with no subset inducing a $K_s$.*



$K_3$-free transversal



Transversal containing $K_3$.

Our main result is for $K_s$-free transversals with $s = 2$.

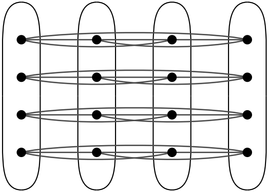
# New results

$K_s$-free transversals: *Let G be a multipartite graph with maximum degree* $\Delta$, *local degree* $o(\Delta)$, *and all parts of size* $(1+o(1))\frac{\Delta}{s-1}$. *Then G has a* $K_s$*-free transversal.*

# New results

$K_s$-free transversals: *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1+o(1))\frac{\Delta}{s-1}$. Then $G$ has a $K_s$-free transversal.*

Construction to establish asymptotic tightness:



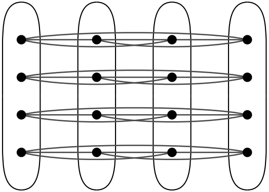
No $K_s$-free transversal

- $\Delta + 1$ parts, each of size $\left\lfloor \frac{\Delta}{s-1} \right\rfloor$.
- Each level is a clique.

# New results

$K_s$-free transversals: *Let G be a multipartite graph with maximum degree* $\Delta$, *local degree* $o(\Delta)$, *and all parts of size* $(1+o(1))\frac{\Delta}{s-1}$. *Then G has a* $K_s$*-free transversal.*

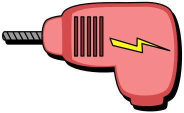
Construction to establish asymptotic tightness:

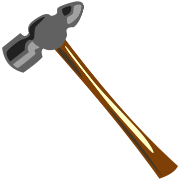


No $K_s$-free transversal

- $\Delta + 1$ parts, each of size $\left\lfloor \frac{\Delta}{s-1} \right\rfloor$.
- Each level is a clique.
- To be $K_s$-free, a transversal can only take $\leq s-1$ vertices from each level.
- But it needs 1 from each part.

# Probabilistic tools





# Talagrand's Inequality

- Probability space $\Omega = \prod_{i=1}^{n} \Omega_i$.
- Random variable $X : \Omega \to \mathbb{R}$.

# Talagrand's Inequality

- Probability space $\Omega = \prod_{i=1}^{n} \Omega_i$.
- Random variable $X : \Omega \to \mathbb{R}$.
- c-Lipschitz: *If* $\omega, \omega' \in \Omega$ *differ only in 1 coordinate, then* $|X(\omega) - X(\omega')| \le c$.

# Talagrand's Inequality

- Probability space $\Omega = \prod_{i=1}^{n} \Omega_i$.
- Random variable $X : \Omega \to \mathbb{R}$.
- c-Lipschitz: *If $\omega, \omega' \in \Omega$ differ only in 1 coordinate, then $|X(\omega) - X(\omega')| \leq c$.*
- Certifiable: *For any $s$ and $\omega$ with $X(\omega) \geq s$, $\exists I \subset \{1, \ldots, n\}$, $|I| \leq s$, such that all $\omega'$ that agree with $\omega$ on coordinates in $I$ have $X(\omega') \geq s$ as well.*

# Talagrand's Inequality

- Probability space $\Omega = \prod_{i=1}^{n} \Omega_i$.
- Random variable $X : \Omega \to \mathbb{R}$.
- c-Lipschitz: *If $\omega, \omega' \in \Omega$ differ only in 1 coordinate, then $|X(\omega) - X(\omega')| \leq c$.*
- Certifiable: *For any $s$ and $\omega$ with $X(\omega) \geq s$, $\exists I \subset \{1, \ldots, n\}$, $|I| \leq s$, such that all $\omega'$ that agree with $\omega$ on coordinates in $I$ have $X(\omega') \geq s$ as well.*

Then for $t \gg \sqrt{\mathbb{E}[X]}$,

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] < 4e^{-\frac{t^2}{9c^2\mathbb{E}[X]}}.$$

# Lovász Local Lemma

- Family of events $\{B_i\}_1^N$.
- Unlikely: all $\mathbb{P}[B_i] \leq p$.
- Mostly independent: each $B_i$ is mutually independent of all but $\leq d$ other events.

Then:

$$ep(d+1) \leq 1 \implies \mathbb{P}[\text{none of } B_i \text{ occur}] > 0.$$

# Parts of size $2e\Delta$ are enough

Alon (1988) *Let $G$ be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then $G$ has an independent transversal.*

# Parts of size $2e\Delta$ are enough

Alon (1988) *Let $G$ be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then $G$ has an independent transversal.*

Proof via Local Lemma:

- Randomly select 1 vertex from each part.
- For each edge $w$, let $B_w$ be when both endpoints are selected.

# Parts of size $2e\Delta$ are enough

Alon (1988) *Let G be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then G has an independent transversal.*

Proof via Local Lemma:

- Randomly select 1 vertex from each part.
- For each edge $w$, let $B_w$ be when both endpoints are selected.
- $p = \mathbb{P}[B_w] = \left(\frac{1}{2e\Delta}\right)^2$.

# Parts of size $2e\Delta$ are enough

Alon (1988) *Let G be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then G has an independent transversal.*

Proof via Local Lemma:

- Randomly select 1 vertex from each part.
- For each edge $w$, let $B_w$ be when both endpoints are selected.
- $p = \mathbb{P}[B_w] = \left(\frac{1}{2e\Delta}\right)^2$.
- Dependency $d \leq 2(2e\Delta)\Delta - 2$.

# Parts of size $2e\Delta$ are enough

Alon (1988) *Let $G$ be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then $G$ has an independent transversal.*

Proof via Local Lemma:

- Randomly select 1 vertex from each part.
- For each edge $w$, let $B_w$ be when both endpoints are selected.
- $p = \mathbb{P}[B_w] = \left(\frac{1}{2e\Delta}\right)^2$.
- Dependency $d \leq 2(2e\Delta)\Delta - 2$.
- $ep(d+1) < 1$, so there is a case when none of $B_w$ occur.

$\square$

# Proof of second result

$K_s$-free transversals: *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\frac{\Delta}{s-1}$. Then $G$ has a $K_s$-free transversal.*

# Proof of second result

$K_s$-free transversals: *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\frac{\Delta}{s-1}$. Then $G$ has a $K_s$-free transversal.*

Proof using our main theorem:

- Color the vertices of $G$ with $s - 1$ colors, minimizing monochromatic edges.
- Delete all edges whose endpoints are different colors.
- The new maximum degree is $\left\lfloor \frac{\Delta}{s-1} \right\rfloor$, so our main theorem applies.

# Proof of second result

$K_s$-free transversals: *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\frac{\Delta}{s-1}$. Then $G$ has a $K_s$-free transversal.*

Proof using our main theorem:

- Color the vertices of $G$ with $s - 1$ colors, minimizing monochromatic edges.
- Delete all edges whose endpoints are different colors.
- The new maximum degree is $\left\lfloor \frac{\Delta}{s-1} \right\rfloor$, so our main theorem applies.
- An indep. trans. here is an $(s - 1)$-colorable subgraph of the original graph, so it is a $K_s$-free transversal there.

$\square$

# Proof of main theorem

L., Sudakov (2005) *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\Delta$. Then $G$ has an independent transversal.*

# Proof of main theorem

L., Sudakov (2005) *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\Delta$. Then $G$ has an independent transversal.*

Steps:

- Reduce to the case when local degree is less than 10.

- Use the "Rödl Nibble" to solve that final case.

# Proof of main theorem

L., Sudakov (2005) *Let $G$ be a multipartite graph with maximum degree $\Delta$, local degree $o(\Delta)$, and all parts of size $(1 + o(1))\Delta$. Then $G$ has an independent transversal.*

Steps:

- Reduce to the case when local degree is less than 10.

  Lemma: *Given the above setup, there exists an induced subgraph $G'$ of $G$ with maximum degree $\Delta'$, local degree $\leq 10$, and parts of size $(1 + o(1))\Delta'$, with respect to the same partition.*

- Use the "Rödl Nibble" to solve that final case.

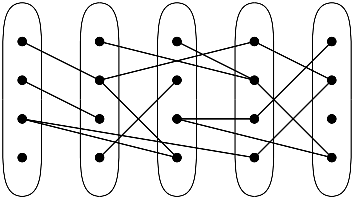
# Proof of main theorem

Remains to show: *Fix* $\epsilon > 0$*. For sufficiently large* $\Delta$*, every multipartite graph G with maximum degree* $\Delta$*, local degree 1, and parts of size* $(1+\epsilon)\Delta$ *has an independent transversal.*

# Proof of main theorem

Remains to show: *Fix $\epsilon > 0$. For sufficiently large $\Delta$, every multipartite graph G with maximum degree $\Delta$, local degree 1, and parts of size $(1+\epsilon)\Delta$ has an independent transversal.*

Algorithm (one iteration):

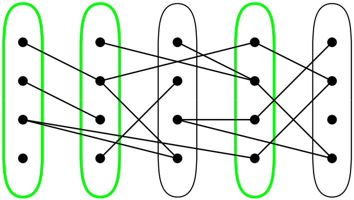


1. Activate each part w.p. $\frac{1}{\log \Delta}$.
2. Independently select a random vertex in each activated part.
3. Delete all neighbors of selected vertices.
4. Add the remaining selected vertices to the indep. trans., and delete their entire parts.

# Proof of main theorem

Remains to show: *Fix $\epsilon > 0$. For sufficiently large $\Delta$, every multipartite graph G with maximum degree $\Delta$, local degree 1, and parts of size $(1+\epsilon)\Delta$ has an independent transversal.*

Algorithm (one iteration):

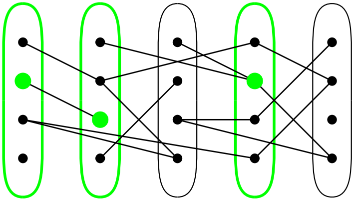


1. Activate each part w.p. $\frac{1}{\log \Delta}$.
2. Independently select a random vertex in each activated part.
3. Delete all neighbors of selected vertices.
4. Add the remaining selected vertices to the indep. trans., and delete their entire parts.

# Proof of main theorem

Remains to show: *Fix $\epsilon > 0$. For sufficiently large $\Delta$, every multipartite graph G with maximum degree $\Delta$, local degree 1, and parts of size $(1+\epsilon)\Delta$ has an independent transversal.*

Algorithm (one iteration):

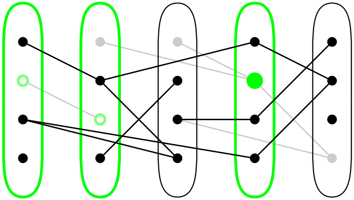


1. Activate each part w.p. $\frac{1}{\log \Delta}$.
2. Independently select a random vertex in each activated part.
3. Delete all neighbors of selected vertices.
4. Add the remaining selected vertices to the indep. trans., and delete their entire parts.

# Proof of main theorem

Remains to show: *Fix $\epsilon > 0$. For sufficiently large $\Delta$, every multipartite graph G with maximum degree $\Delta$, local degree 1, and parts of size $(1+\epsilon)\Delta$ has an independent transversal.*

Algorithm (one iteration):

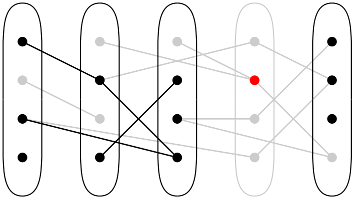


1. Activate each part w.p. $\frac{1}{\log \Delta}$.
2. Independently select a random vertex in each activated part.
3. Delete all neighbors of selected vertices.
4. Add the remaining selected vertices to the indep. trans., and delete their entire parts.

# Proof of main theorem

Remains to show: *Fix $\epsilon > 0$. For sufficiently large $\Delta$, every multipartite graph G with maximum degree $\Delta$, local degree 1, and parts of size $(1+\epsilon)\Delta$ has an independent transversal.*

Algorithm (one iteration):



1. Activate each part w.p. $\frac{1}{\log \Delta}$.
2. Independently select a random vertex in each activated part.
3. Delete all neighbors of selected vertices.
4. Add the remaining selected vertices to the indep. trans., and delete their entire parts.

# Main idea

Let $s_t(i)$ = size of part $i$ at start of iteration $t$.
Let $d_t(v)$ = degree of $v$ at start of iteration $t$.

Claim: *Let $T = \frac{10}{\epsilon} \log \Delta$. Then for all $t \leq T$ and all $i$ and $v$, we can arrange to have:*

$$s_t(i) \geq S_t, \qquad d_t(v) \leq D_t, \qquad \frac{S_T}{D_T} \geq 2e.$$

# Main idea

Let $s_t(i)$ = size of part $i$ at start of iteration $t$.
Let $d_t(v)$ = degree of $v$ at start of iteration $t$.

Claim: *Let $T = \frac{10}{\epsilon} \log \Delta$. Then for all $t \le T$ and all $i$ and $v$, we can arrange to have:*

$$s_t(i) \ge S_t, \qquad d_t(v) \le D_t, \qquad \frac{S_T}{D_T} \ge 2e.$$

Alon (1988) *Let $G$ be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then $G$ has an independent transversal.*

# Main idea

Let $s_t(i)$ = size of part $i$ at start of iteration $t$.
Let $d_t(v)$ = degree of $v$ at start of iteration $t$.

Claim: *Let $T = \frac{10}{\epsilon} \log \Delta$. Then for all $t \leq T$ and all $i$ and $v$, we can arrange to have:*

$$s_t(i) \geq S_t, \qquad d_t(v) \leq D_t, \qquad \frac{S_T}{D_T} \geq 2e.$$

$$S_1 = (1+\epsilon)\Delta, \qquad S_{t+1} \approx S_t \left(1 - \frac{1}{1+\epsilon} \frac{1}{\log \Delta}\right)$$
$$D_1 = \Delta, \qquad D_{t+1} \approx D_t \left(1 - \frac{1}{\log \Delta}\right)$$

Alon (1988) *Let $G$ be a multipartite graph with maximum degree $\Delta$ and all parts of size $2e\Delta$. Then $G$ has an independent transversal.*

# Induction strategy

Property $P(t)$: At the start of iteration $t$, all part sizes are $\geq S_t$ and all degrees are $\leq D_t$.

# Induction strategy

Property $P(t)$: At the start of iteration $t$, all part sizes are $\geq S_t$ and all degrees are $\leq D_t$.

To show that $P(t) \Rightarrow P(t+1)$ for $t < T$:

- Let $A_i$ be the event that $s_{t+1}(i) < S_{t+1}$.
- Let $B_v$ be the event that $d_{t+1}(v) > D_{t+1}$.
- Vertices that are far apart give independent events, so the dependency $d \leq O(\Delta^{100})$.
- Local Lemma:
  $ep(d+1) \leq 1 \implies \mathbb{P}[\text{none of } A_i, B_v \text{ occur}] > 0$.

*Suffices to show that $\mathbb{P}[A_i]$, $\mathbb{P}[B_v]$ are exponentially small.*

# On average, parts remain large enough . . .

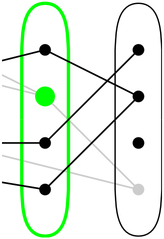
Goal: *With very high probability, the size of the $i^{th}$ part is $\geq S_{t+1}$ at the end of iteration $t$, given $P(t)$.*

$$S_{t+1} \approx S_t \left(1 - \frac{1}{1+\epsilon} \frac{1}{\log \Delta}\right)$$

# On average, parts remain large enough . . .

Goal: *With very high probability, the size of the $i^{th}$ part is $\geq S_{t+1}$ at the end of iteration $t$, given $P(t)$.*

$$S_{t+1} \approx S_t \left(1 - \frac{1}{1+\epsilon}\frac{1}{\log \Delta}\right)$$

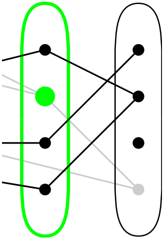


$$\mathbb{E}\left[s_{t+1}(i)\right] \quad \geq \quad s_t(i)\left(1 - \frac{1}{\log \Delta}\frac{1}{S_t}\right)^{D_t}$$

# On average, parts remain large enough ...

Goal: *With very high probability, the size of the $i^{th}$ part is $\geq S_{t+1}$ at the end of iteration $t$, given $P(t)$.*

$$S_{t+1} \approx S_t \left(1 - \frac{1}{1+\epsilon}\frac{1}{\log \Delta}\right)$$



$$\begin{aligned}\mathbb{E}\left[s_{t+1}(i)\right] &\geq s_t(i)\left(1 - \frac{1}{\log \Delta}\frac{1}{S_t}\right)^{D_t} \\ &\geq s_t(i)\left(1 - \frac{1}{\log \Delta}\frac{D_t}{S_t}\right) \\ &\geq s_t(i)\left(1 - \frac{1}{\log \Delta}\frac{1}{1+\epsilon}\right)\end{aligned}$$

# . . . and part sizes are tightly concentrated

Define the random variable $X = s_t(i) - s_{t+1}(i)$.

This is the number of vertices lost in part $i$ due to the selection of a neighbor.

- 1-Lipschitz: *Every choice we make affects $X$ by $\leq 1$, because the local degree is 1.*
- Certifiable: *For every vertex that the $i^{th}$ part loses, there is one culprit.*

Talagrand's Inequality:

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \frac{s_t(i)}{\log^2 \Delta}\right] < 4e^{-\frac{s_t(i)^2}{(\log^4 \Delta) 9\mathbb{E}[X]}}$$

# Degrees shrink quickly enough

Recall $D_{t+1} \approx D_t \left(1 - \frac{1}{\log \Delta}\right)$.

Goal: Almost definitely, degree of $v$ shrinks by a factor $\approx \frac{1}{\log \Delta}$.

# Degrees shrink quickly enough

Recall $D_{t+1} \approx D_t\left(1 - \frac{1}{\log \Delta}\right)$.

Goal: Almost definitely, degree of $v$ shrinks by a factor $\approx \frac{1}{\log \Delta}$.

Let $Z = \#$ of parts *adjacent* to $v$ that are deleted in Step 4.
(A part is *adjacent* to $v$ if it contains a neighbor of $v$.)

Suffices to show: Almost definitely, $Z \approx \frac{d_t(v)}{\log \Delta}$.

# Degrees shrink quickly enough

Recall $D_{t+1} \approx D_t \left(1 - \frac{1}{\log \Delta}\right)$.

Goal: Almost definitely, degree of $v$ shrinks by a factor $\approx \frac{1}{\log \Delta}$.

Let $Z = \#$ of parts *adjacent* to $v$ that are deleted in Step 4.
(A part is *adjacent* to $v$ if it contains a neighbor of $v$.)

Suffices to show: Almost definitely, $Z \approx \frac{d_t(v)}{\log \Delta}$.

First activate parts adjacent to $v$.
Let $X = \#$ of activated parts adjacent to $v$.

- $\mathbb{E}[X] = \frac{d_t(v)}{\log \Delta}$.
- $X \approx \mathbb{E}[X]$ because it is Binomial.

# Degrees shrink quickly enough

$Z$ = # of parts adjacent to $v$ deleted in Step 4.
$X$ = # of activated parts adjacent to $v$.

Randomly select a vertex from each of the $X$ parts above.

- Since local degree is bounded, $\exists$ independent set of selected vertices of size $Y \approx X$ almost definitely.
- Almost definitely, every vertex in the graph is adjacent to $\leq \log \Delta$ vertices in $Y$, since local degree is bounded.

# Degrees shrink quickly enough

$Z$ = # of parts adjacent to $v$ deleted in Step 4.
$X$ = # of activated parts adjacent to $v$.

Randomly select a vertex from each of the $X$ parts above.

- Since local degree is bounded, $\exists$ independent set of selected vertices of size $Y \approx X$ almost definitely.
- Almost definitely, every vertex in the graph is adjacent to $\leq \log \Delta$ vertices in $Y$, since local degree is bounded.

Activate the remaining parts and choose a random vertex in each. $Y - Z$ is the number that are lost due to the selection of a neighbor.

- This difference is $(\log \Delta)$-Lipschitz and Certifiable, so Talagrand gives $Z \approx Y$.

$\square$

# Conclusion

Question: What conditions ensure that $G$ can be partitioned into a disjoint union of independent transversals?

# Conclusion

Question: What conditions ensure that $G$ can be partitioned into a disjoint union of independent transversals?

Alon (1992) *Parts of size $C\Delta$ are sufficient ($C$ very large).*

Haxell (2004) *$C = 3$ is sufficient.*

Szabó, Tardos (2005) *$C$ cannot be reduced below 2.*

Folklore: *$C$ should be 2.*

# Conclusion

Question: What conditions ensure that $G$ can be partitioned into a disjoint union of independent transversals?

Alon (1992) *Parts of size $C\Delta$ are sufficient ($C$ very large).*

Haxell (2004) *$C = 3$ is sufficient.*

Szabó, Tardos (2005) *$C$ cannot be reduced below 2.*

Folklore: *$C$ should be 2.*

Corollary of our result $+$ trick of Aharoni, Berger, and Ziv:
*If the local degree is $o(\Delta)$, then parts of size $(2 + o(1))\Delta$ are sufficient.*

L., Sudakov Conjecture: *If the local degree is $o(\Delta)$, then parts of size $(1 + o(1))\Delta$ are sufficient.*