Extremal Combinatorics

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1 Problems and famous results

1. Let G be a graph with all degrees at least 20. Prove that it is possible to partition the vertices into two groups such that for each vertex, at least 10 of its neighbors ended up in the other group.

   Solution: Take a max-cut: the bipartition which maximizes the number of crossing edges.

2. (Mantel.) Prove that every graph with strictly more than $\frac{n^2}{4}$ edges contains a triangle.

   Solution: Assume no triangles. Then for every edge uv, must have $d(u) + d(v) \leq n$. Sum over all edges. Then

   $$nE \geq \sum d(v)^2 \geq n \cdot \left(\frac{2E}{n}\right)^2 = \frac{4E^2}{n},$$

   i.e., $E \leq n^2/4$, contradiction.

3. (Turán.) The n-vertex K_t-free graphs with the most edges are the complete $(t-1)$-partite graphs with all parts of size as equal as possible.

   Solution: Zykov symmetrization. First show any non-adjacent vertices have same degree. Indeed, if one has more degree, then delete the smaller and clone the larger, and get strictly more edges without making any bigger cliques.

   Next show non-adjacency is an equivalence relation, to get a complete multipartite graph (and then use convexity). So suppose we had x, y_1, y_2 such that x is not adjacent to either y_i, but the y_i are adjacent to each other. By previous, $d(y_1) = d(x) = d(y_2)$. But note that the degree of y_i includes $+1$ for their internal edge (and this is double-counted when adding $d(y_1) + d(y_2)$, so if we delete both y_i and clone x twice, then we get at least $+1$ in total edges.

4. (Moscow, 1964, from GDC 2008.) King Arthur summoned $2n$ knights to his court. Each knight has at most $n-1$ enemies among the other knights present. Prove that the knights can sit at the Round Table so that no two enemies sit next to each other. (The "enemy" relation is symmetric.)

   Solution: This is Dirac's theorem in disguise. Suppose the longest path has t vertices $x_1, \ldots, x_t$. We will show there is a cycle of t vertices as well. Suppose not. All neighbors of x_1 and x_t must lie on the path or else it is not longest. Minimum degree condition implies that both have degree $\geq t/2$. But if $x_1 \sim x_k$, then $x_t \not\sim x_{k-1}$ or else we can re-route to get a cycle. So, each of x_1's $t/2$ neighbors on the path prohibit a potential neighbor of x_t. Yet x_t's neighbors come from indices $1 \ldots t-1$, so there is not enough space for x_t to have $t/2$ neighbors there, avoiding the prohibited ones.

   Now if this longest path is not the full n vertices, then we get a cycle C missing some vertex x. But min-degree $n/2$ implies that the graph is connected (smallest connected component is $n/2+1$), so there is a shortest path from x to C, and adding this to the cycle gives a longer path than t, contradiction.

5. Let G be a connected graph with 100 vertices, in which all vertices have degree at least 10. Prove that G contains a path with at least 21 vertices.

6. (China, 1986, from GDC 2008.) In a chess tournament with n players, every pair of players plays once and there are no draws. Show that there must be some player A such that for every other player B, either A beat B or A beat some other player C who in turn beat B.

   Solution: (From Havet and Thomassé, 2000.) A vertex v in a directed graph is a *king* if every other vertex can be reached from it via directed paths of length at most 2. That is, $V = \{v\} \cup N^+(v) \cup N^{++}(v)$. Show that every tournament has a king.

   Take the first vertex v_1 in a median order, and consider v_k. If already $\overrightarrow{v_1 v_k}$, then done. Otherwise, by the feedback property, at least half of the edges from v_1 to $v_2 \dots v_k$ point forward. These give at least $\frac{k-1}{2}$ landing points in $v_2 \dots v_{k-1}$, because we assumed that v_1 does not go to v_k. Similarly, there are at least $\frac{k-1}{2}$ vertices in $v_2 \dots v_{k-1}$ which are origination points for edges directed to v_k. Yet there are only $k-2$ vertices in this window, so by pigeonhole some vertex is both an origination point and a landing point, giving a directed path of length 2.

7. (Romania.) Given n points in the plane, prove that there exists a set of $\sqrt{n}$ points such that no 3 points in the set form an equilateral triangle.

   Solution: First we prove the Erdős-Szekeres theorem, that every sequence of n^2 distinct numbers contains a subsequence of length n which is monotone (i.e. either always increasing or always decreasing). Indeed, for each of the n^2 indices in the sequence, associate the ordered pair (x, y) where x is the length of the longest increasing subsequence ending at x, and y is the length of the longest decreasing one. All ordered pairs must obviously be distinct. But if they only take values with $x, y \in \{1, \dots, n-1\}$, then there are not enough for the total n^2 ordered pairs. Thus n appears somewhere, and we are done.

   Next, we find an x-axis direction such that all points have distinct projection on both x and y (orthogonal) axes. Order the points according to this direction, and let $y_1, \dots, y_n$ be their corresponding y-coordinates (distinct). By Erdős-Szekeres, there is a monotone subsequence of the desired length. But this cannot contain an equilateral triangle.

8. (Diestel 2.20.) Let G be a graph, and let α be the size of its largest independent set. Prove that the vertices of G can be covered by $\leq \alpha$ vertex-disjoint subgraphs, each either a cycle, a K_2, or a K_1.

   Solution: Take a longest path, and say that it ends at v. Make a cycle by taking the edge from v back to its earliest neighbor along the path. Delete this cycle from the graph. Importantly, v is not adjacent to any vertex in the rest of the graph! So, in the remainder, the independence number is at most $\alpha - 1$, and we may apply induction. Note that K_2 arises when the longest path has only one edge, in which case we can't close a cycle, and K_1 arises when the longest path is a single vertex.

9. (Gallai, Hasse, Roy, Vitaver.) Let D be a directed graph, and let χ be the chromatic number of its underlying undirected graph. Show that D has a directed path of at least χ vertices.

   Solution: Take a maximal acyclic subgraph, and use it to define level sets, by coloring each vertex by the length of the longest directed path that ends at that vertex.