II. Inequalities

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1 Warm-Up

(Po98) Prove that for all ordered triples (a, b, c) of prime numbers:

$$a^2b + a^2 + ac^2 + 115a + b^2c + b^2 + c^2 + 27c + 176 < 6ab + 22ac + 14bc + 5b.$$

Solution: Complete the square

2 The p-Norm

Here's the easiest way to think of this: let's define the "p-norm" of a sequence as follows: take a sequence $\{a_1, a_2, \ldots, a_n\}$ and call it a. Then write $\|a\|_p$ to denote $\sqrt[p]{|a_1|^p + \cdots + |a_n|^p}$. (By the way, p doesn't have to be an integer; the p-th root is then defined as the $1/p$ power.)

This is kind of intuitive; if $p = 2$, then if we have n-dimensional space and a is a coordinate vector, $\|a\|_2$ is the length of it. And different p's give different notions of length; for instance, $p = 1$ corresponds to the Manhattan (taxicab) distance. For completeness, definte $\|a\|_\infty$ to be the maximum of the $|a_k|$. Amazingly enough, this notion of p-norm appears in many areas of mathematics. You'll see it again and again as you learn more math.

Also, we should define "addition" and "multiplication" on sequences. That is, given $a = \{a_1, a_2, \ldots, a_n\}$ and $b = \{b_1, b_2, \ldots, b_n\}$, define $a + b$ to be $\{a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\}$ and ab to be $\{a_1b_1, a_2b_2, \ldots, a_nb_n\}$.

3 Cute Inequalities (inspired by Kiran97)

AM-GM-HM For positive sequences a:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$$

with equality when all of the a_k are equal.

Cauchy-Schwarz $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$. Note that this is a special case of Hölder. Can you see why?

Minkowski Given the same sequences as above, and $p \geq 1$, $\|a + b\|_p \leq \|a\|_p + \|b\|_p$.

Hölder Given the same sequences as above, and $p, q \geq 1$ such that $1/p + 1/q = 1$,

$$\|ab\|_1 \leq \|a\|_p \cdot \|b\|_q$$

Note: we can take $p = 1$ and $q = \infty$; their reciprocals "add up to 1."

Chebyshev If a and b are increasing positive sequences, then:

$$\frac{a_1b_1+\cdots+a_nb_n}{n} \geq \frac{a_1+\cdots+a_n}{n}\frac{b_1+\cdots+b_n}{n}$$

If one sequence is increasing but the other is decreasing, then the inequality flips.

Rearrangement If we have the series $a_1b_1+\cdots+a_nb_n$, it is maximized when the a's and b's are both sorted in the same direction, and minimized when they are sorted in opposite directions.

Bernoulli If $x > -1$ and $r \geq 1$, then $(1+x)^r \geq 1+rx$.

1. (Titu97) Prove that for all nonzero $a, b, c \in \mathbb{R}$:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

Solution: Cauchy-Schwartz (as a verb) the LHS with a permutation of itself: $(b/c)^2+(c/a)^2+(b/a)^2$.

2. (Titu97) Prove that for $a_k \geq 0$:

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq (1+\sqrt[n]{a_1\cdots a_n})^n$$

Solution: Generalized Hölder. Prove that by induction; main point is to prove the identity $\|a\|_p = \|a\|_1^{1/p}$.

3. (Titu97) Let $x_k \in [1,2]$, $k \in \{1,\ldots,n\}$. Prove:

$$\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n \frac{1}{x_k}\right)^2 \leq n^3$$

Solution: Hölder with $p=3$, $q=3/2$.

4. (IMO95) Let a, b, and c be positive real n umbers such that $abc=1$. Prove:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

5. (Kiran97) Let a, b, c be positive. Prove:

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

with equality iff $a=b=c=1$.

6. (IMO 2001 Shortlist) Prove that for all positive reals a, b, c:

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Solution: Prove that each term exceeds

$$\frac{a^{(4/3)}}{a^{(4/3)}+b^{(4/3)}+c^{(4/3)}}$$

Cross multiply and square. Then factor the following difference of squares

$$(a^{(4/3)}+b^{(4/3)}+c^{(4/3)})^2-(a^{(4/3)})^2$$

and apply AM-GM on the product. We get $8a^{2/3}bc$.

7. (Titu97) Prove that for positive $x_1, x_2, \ldots, x_n$:

$$\frac{x_1^3}{x_1^2 + x_1 x_2 + x_2^2} + \frac{x_2^3}{x_2^2 + x_2 x_3 + x_3^2} + \cdots + \frac{x_n^3}{x_n^2 + x_n x_1 + x_1^2} \geq \frac{1}{3}(x_1 + \cdots + x_n)$$

4 Convexity and Smoothing

Jensen A *convex* function is a function $f(t)$ for which the second derivative is nonnegative. This is equivalent to having the property that for any a, b in the domain, $f((a+b)/2) \leq (f(a) + f(b))/2$. Then given weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive numbers $a_1, a_2, \ldots, a_n$:

$$f\left(\frac{\lambda_1 a_1 + \cdots + \lambda_n a_n}{\lambda_1 + \cdots + \lambda_n}\right) \leq \frac{\lambda_1 f(a_1) + \cdots + \lambda_n f(a_n)}{\lambda_1 + \cdots + \lambda_n}$$

Smoothing Given an inequality, show that it becomes less true when you "squish" the values of the variables together. Then if it's still true after you're done squishing, hey, it must have been true in the first place!

1. (Zvezda98) Prove for all nonnegative number a, b, c:

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

2. (IMO84) For $x, y, z > 0$ and $x + y + z = 1$, prove that $xy + yz + xz - 2xyz \leq 7/27$.

   Solution: Smooth with the following expression: $x(y+z) + yz(1-2x)$. Now, if $x \leq 1/2$, then we can push y and z together. The mushing algorithm is as follows: first, if there is one of them that is greater than $1/2$, pick any other one and mush the other two until all are within $1/2$. Next we will be allowed to mush with any variable taking the place of x. Pick the middle term to be x; then by contradiction, the other two terms must be on opposite sides of $1/3$. Hence we can mush to get one of them to be $1/3$. Finally, use the $1/3$ for x and mush the other two into $1/3$. Plugging in, we get $7/27$.

3. (MMO63) For $a, b, c > 0$, prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

   Solution: Same idea as previous, except first you multiply it out and normalize $a + b + c = 1$. Then you get:

$$(a+b)(a^2 + b^2 + c(a+b)) + c(c+a)(c+b) \geq \frac{3}{2}(a+b)(b+c)(c+a)$$

   Show that the difference is always at least 0, and if you mush together a and b, it gets better as long as $(3/2)(a+b) - c \geq 0$, which happens as long as $c \leq 3/5$. Hence the same algorithm as previous works.

4. (88 Friendship Competition) For $a, b, c > 0$:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$$

5. (USAMO98) Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \ldots + \tan(a_n - \pi/4) \geq n - 1$$

   Prove that $\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$.

   Solution: Let $t_k = \tan(x_k - \pi/4)$. Then $\tan x_k = (1 + t_k)/(1 - t_k)$, and we want this product to be at least n^{n+1}. Next the given inequality is equivalent to $1 + t_k \geq \sum_{j \neq k}(1 - t_j)$, and by AM-GM, it is at least $n\sqrt[n]{\prod_{j \neq k}(1 - t_j)}$. Finally, take the product over all possible LHS and the result falls out.

5 Brute Force (stolen from Kiran98)

Weighted Power Mean Given weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive numbers $a_1, a_2, \ldots, a_n$, and powers p and q such that $p \leq q$:

$$\left(\frac{\lambda_1 a_1^p + \cdots + \lambda_n a_n^p}{\lambda_1 + \cdots + \lambda_n}\right)^{1/p} \leq \left(\frac{\lambda_1 a_1^q + \cdots + \lambda_n a_n^q}{\lambda_1 + \cdots + \lambda_n}\right)^{1/q}$$

with equality when all of the a_k are equal.

Schur's Inequality For x, y, z positive and r real:

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \geq 0$$

with equality when $x = y = z$.

Now in all of these problems, all variables should be assumed positive.

1. $4(a^3 + b^3) \geq (a+b)^3$

   Solution: Expand; to get the $ab(a^2+b^2) \leq a^3+b^3$, take it as a product of two guys and use Weighted Power Mean.

2. $9(a^3 + b^3 + c^3) \geq (a+b+c)^3$

   Solution: Expand and get:

   $$8\sum_{\text{sym}} a^3 \geq 3\sum_{\text{sym}} a^2b + 6abc$$

   (count terms; it works)

   Next by AM-GM, get rid of 6abc; cancels 2 of the LHS. Divide through by 3 and write out the rest (6 terms per side, split cyclically) then use rearrangement.

3. If $abc = 1$ then

   $$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1$$

4. (MOP98) Prove that for $x, y, z > 0$,

   $$\frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \leq \frac{9}{4(x+y+z)}$$

5. If $abc = 1$ then

   $$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

6. If $abc = 1$ then

   $$\frac{c}{a+b+1} + \frac{a}{b+c+1} + \frac{b}{c+a+1} \geq 1$$

7. If $abc = 1$ then

   $$\frac{1}{a+ab} + \frac{1}{b+bc} + \frac{1}{c+ca} \geq \frac{3}{2}$$

8. Prove:

   $$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$$