Brutal Force II

Po-Shen Loh — MOP 2002 — 11 July 2002

brutal (adj.)

1. Extremely ruthless or cruel.
2. Crude or unfeeling in manner or speech.
3. Harsh; unrelenting.
4. Disagreeably precise or penetrating.

force (n.)

1. The capacity to do work or cause physical change; energy, strength, or active power.
2. Power made operative against resistance; exertion.
3. A vector quantity that tends to produce an acceleration of a body in the direction of its application.
4. A unit of a nation's military personnel, especially one deployed into combat.

Brutal force was originally an all-encompassing term that covered all methods of solution involving deformation of the diagram. In recent years, however, it has been specialized to coincide with the notion of a geometrical form of induction. In this lecture, we will consider the latter meaning of the term.

1 Concept

Geometry problems are usually obviously true in their symmetric cases. For example, the following problem is clearly true if A is the midpoint of the outer arc BC:

> **(Rookie Contest 1998, 1999, 2002, Po's Star Theorem)**. Given two congruent circles, ω_1 and ω_2. Let them intersect at B and C. Select a point A on ω_1. Let AB and AC intersect ω_2 at A_1 and A_2. Let X be the midpoint of BC. Let A_1X and A_2X intersect ω_1 at P_1 and P_2. Prove that $AP_1 = AP_2$.
> **Solution:** True for symmetric case; perturb A by θ. Then A_1 and A_2 move by θ (vertical angles), and P_1 and P_2 also move by θ (symmetry through X). Therefore done.

Unfortunately, this only covers one of the infinitely many possible relative configurations, so we must somehow perform an "inductive step" to generalize the symmetric case to all cases. To do this, we perturb our symmetric case to a general case by some parameter and check what happens to the rest of the diagram. For example, move A away from the symmetric location by arc measure θ; the remainder of the solution is left as an exercise to the reader.

2 Problems

1. (Russia 1998.14). A Circle S centered at O meets another circle S' at A and B. Let C be a point on the arc of S contained in S'. Let E, D be the second intersections of S' with AC, BC, respectively. Show that $DE \perp OC$.

   Solution: Clearly true for symmetric case; for perturbation, angles flow around as usual.

2. (UK 1996.3). Let ABC be an acute triangle and O its circumcenter. Let S denote the circle through A, B, O. The lines CA and CB meet S again at P and Q, respectively. Prove that the lines CO and PQ are perpendicular.

3. (Russia 1998.10). In acute triangle ABC, the circle S passes through the circumcenter O and vertices B, C. Let OK be a diameter of S, and let D, E be the second intersections of S with AB, AC, respectively. Show that $ADKE$ is a parallelogram.

   Solution: In symmetric case, A, O, K collinear. Therefore, $\angle OKD = \angle OKE = \angle OCE = \angle OCA$ since $OA = OC$. But perturb A by θ; then D and E also move by θ and we are done.

4. (Zvezda, Complex Numbers 2001.15). In triangle ABC prove that the angle bisector of $\angle A$, the midsegment parallel to AC, and the line joining the tangent points of the incircle with sides BC and CA are concurrent.

   Solution: Let the circle hit AC at F and BC at E. Let A_1 be the foot of the angle bisector on BC, and let B_1 be the reflection of B across AA_1. Now Brianchon on $ABEA_1B_1F$, to get that the intersection of the angle bisector and the line connecting the tangents is halfway between B and B_1, which means that it is on the midline.

   This was inspired by brutal force: suppose we have the isosceles case in which B_0 and C_0 form a line perpendicular to the angle bisector from A. Now construct the line through $BB_1 \cap AA_1$ parallel to AC, and construct the line through $B_0C_0 \cap AA_1$. The distance between these two lines should be exactly half the distance that B moved from B_0 measured in the direction perpendicular to AC. But construct the line through B parallel to AC; this line cuts off a triangle B_0BC_1 similar to B_0BC_0. By similar triangles, we find that the vertical elevation of B and that of $BB_1 \cap EF$ must be the same. Hence we consider BB_1, and discover the elegant solution via Brianchon's Theorem.

5. (St. Petersburg 1997.19). The circles S_1, S_2 intersect at A and B. Let Q be a point on S_1. The lines QA and QB meet S_2 at C and D, respectively, while the tangents to S_1 at A and B meet at P. Assume that Q lies outside S_2, and that C and D lie outside S_1. Prove that the line QP goes through the midpoint of CD.

   Solution: Clearly true for symmetric case; perturb Q by θ and observe that the midpoint M orbits a circle ω centered at O_2 (center of S_2) by θ. Furthermore, by limiting argument as Q approaches A or B, we see that the tangents AP and BP are also tangent to ω. Now we have S_1 and ω with common tangents AP and BP; therefore, by homothety we are done.

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