A C^1 TETRAHEDRAL FINITE ELEMENT WITHOUT EDGE DEGREES OF FREEDOM

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Abstract. A composite C^1 tetrahedral finite element is developed which does not have any edge degrees of freedom. This eliminates the need to associate a basis for the planes perpendicular to each edge; such a basis can not depend continuously upon the edge orientation. The finite element space is piecewise polynomial over the four tetrahedra formed by adding the circumcenter, and their traces on each face belong to the (two dimensional) Bell subspace.

Key words. C^1 Tetrahedron, C^1 Finite Element, Reduced Finite Element.

1. Introduction.

Most C^1 tetrahedral finite elements proposed to date have components of the gradient perpendicular to an edge as degrees of freedom. In particular, for each edge of the mesh it is necessary to specify a basis for the perpendicular plane. Since there does not exist a continuous tangent vector field on the sphere, it follows that this basis can not be determined as a continuous function of the edge orientation. Since many elements will share a single edge, it is necessary to first fix a basis for the plane perpendicular to each edge, and to then pass this (global) data to the elements. This contrasts with the majority of finite elements where the transformation of basis functions on a parent element $\hat{K}$ to a finite element K is local in the sense that it is completely determined by a canonical diffeomorphism $\chi : \hat{K} \to K$. This lack of locality requires element specific book–keeping which breaks the natural modularity inherent in traditional finite element codes.

Below we present a composite C^1 tetrahedral element where each tetrahedron is subdivided into four tetrahedra with the following 45 degrees of freedom:

1. $u(v^{(i)})$, $\nabla u(v^{(i)})$ and $D^2u(v^{(i)})$ at the four vertices $v^{(i)}$.
2. $\nabla u(c_f).n_f$ at the centroids c_f of the four triangular faces f with normal n_f.
3. $u(c_K)$ at the tetrahedron centroid c_K.

This element may be viewed as a tetrahedral analog of the Bell triangle in the sense that the basis functions for both of them are $\mathcal{P}_5$ (polynomials of degree 5) and their normal derivatives on a face are constrained to be in $\mathcal{P}_3$. Moreover, they are both "reduced" elements in the sense that the finite element spaces are formed by restricting the functions of a $\mathcal{P}_5$ element to satisfy the constraint on the normal derivatives on each face (the Bell element is the reduced Argyris element).

Historically the development of C^1 finite elements was motivated by the need to solve the plate and shell equations which are naturally posed in two dimensions. This resulted in a variety of two dimensional C^1 elements [3] and the corresponding theory is covered in most finite element texts [2, 3, 7]. However, there is a dearth of three dimensional C^1 elements in the finite element literature, so problems naturally posed in $H^2(\Omega)$, such as the Cahn Hillard or Monge Ampere equations, or the novel formulation of the Stokes equations in [5], are often treated with mixed

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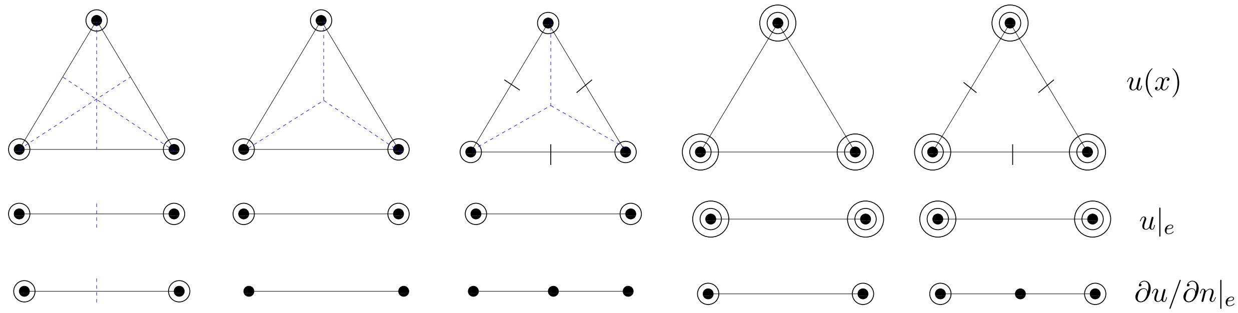

FIG. 1.1. *Composite $\mathcal{P}_2$, reduced HCT, HCT, Bell, and Argyris, C^1 triangles*

methods or non–conforming elements. On rectangular domains an alternative is to use the Bogner Fox Schmit square or cube, i.e. tensor products of one dimensional Hermite polynomials.

Fortunately development of C^1 functions on arbitrary triangulations in both two and three dimensions is an active area of study in the spline literature; moreover, the comprehensive text by Lai and Schumaker [4] makes this body of knowledge very accessible to a wide audience. The lowest order C^1 tetrahedron with basis $\mathcal{P}_k$ on general triangulations has $k = 9$ and involves degrees of freedom with four derivatives (c.f. the Argyris element in two dimensions). These elements are difficult to implement [6], so composite elements with lower degree polynomials are used in three dimensions; that is, the basis functions on each tetrahedron K of a triangulation are piecewise polynomial over a subdivision of K (c.f. the two dimensional Hsieh Clough Tocher (HCT) element). The tradeoffs between practicability, complexity, and accuracy with this approach discussed in [4, pp 171] include:

- Macroelements with fewer subdivisions are preferred.
- Degrees of freedom with fewer (lower) derivatives are preferable.
- The (global) dimension of the finite element space.
- Higher order elements are more accurate but may involve more complicated basis functions.

As an example, the reduced Hsieh Clough Tocher triangle would be preferable to the composite $\mathcal{P}_2$ triangle; they have the same degrees of freedom, but the composite $\mathcal{P}_2$ element subdivides the triangle into twice as many partitions. These criteria also favor the Bell over the HCT triangle; to leading order in the mesh size they have the same number of degrees of freedom, but the Bell element is more accurate and does not require partition of the triangle.

1.1. C^1 Tetrahedra. The C^1 tetrahedra documented in [4] are most easily characterized by the trace of their basis functions on a triangular face. Composite $\mathcal{P}_5$, $\mathcal{P}_3$ and $\mathcal{P}_2$ tetrahedra with traces equal to the Argyris, Hsieh Clough Tocher (HCT), and composite $\mathcal{P}_2$, subspaces (see Figure 1.1) are developed. These elements subdivide tetrahedron into four, 12, and 24 tetrahedron respectively by adding the circumcenters of the tetrahedron, tetrahedron and faces, and tetrahedron faces and edges, respectively, see Figure 1.2. Continuity of the normal derivatives across faces for the composite $\mathcal{P}_5$ and $\mathcal{P}_3$ elements is assured by the inclusion of normal derivative degrees of freedom on the edges and faces. In particular, the derivatives in the plane perpendicular to each edge are specified which gives rise to the issues discussed in the introduction. While the composite $\mathcal{P}_2$ tetrahedron does not encounter this issue, the normal derivatives on each face of

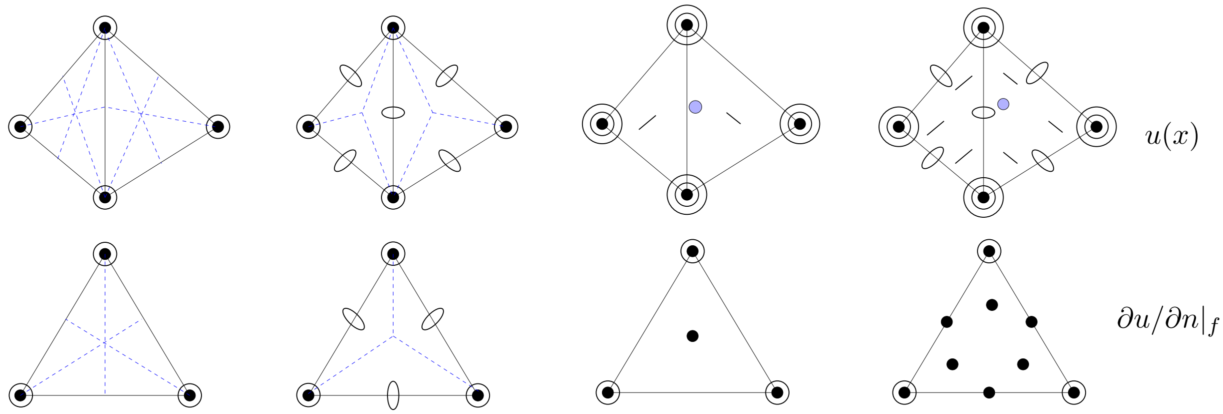

FIG. 1.2. *Composite $\mathcal{P}_2$, $\mathcal{P}_3$, reduced $\mathcal{P}_5$, and $\mathcal{P}_5$, C^1 tetrahedra.*

this and the composite $\mathcal{P}_3$ tetrahedron are piecewise polynomials involving all of the variables on the face (function values and gradients). The large number partitions of both the tetrahedra and the faces for these elements negates the simplicity typically inherent with low order elements.

Absent from [4] is a discussion of reduced elements; for example, the reduced HCT or Bell triangle. The basis functions for these elements are characterized as the subspace of the HCT or Argyris space having normal derivatives in $\mathcal{P}_1$ and $\mathcal{P}_3$ respectively. Motivated by the desire to dispense with edge degrees of freedom, and the observation that the Bell element does not involve these, this paper develops the reduced composite $\mathcal{P}_5$ tetrahedron discussed in the introduction. Note that the corresponding reduced composite $\mathcal{P}_3$ tetrahedron obtained by requiring the normal derivatives along the edges to be linear does not have $\mathcal{P}_1$ normal derivatives on the faces. The normal derivatives on each face are continuous, piecewise $\mathcal{P}_2$, linear on the face edges, and C^1 at the face centroid. Reductions of the composite $\mathcal{P}_2$ tetrahedron are considered in [1].

In the next section the space of reduced $\mathcal{P}_5$ functions is introduced and error estimates of the natural interpolant are established. The subsequent section gives a practical construction of a basis for the reduced element, and a numerical example is presented in the final section. We finish this section with a brief explanation of the notation. Notation of the form $\{v^{(i)}\}_{i=0}^3$ will be used to index quantities like the vertices of a tetrahedron, or the unit vectors $\{e^{(i)}\}_{i=1}^3$, where the parentheses are used to avoid confusion with components which are indexed with subscripts, $e_j^{(i)} = \delta_{ij}$. The standard reference (parent) tetrahedron is denoted by $\hat{K}$, and any quantity associated with this tetrahedron will contain an over-hat; for example, the vertex set is $\{\hat{v}^{(i)}\}_{i=0}^3$.

2. Reduced Composite $\mathcal{P}_5$ Element. The following notation will be used to characterize the composite $\mathcal{P}_5$ tetrahedron and the corresponding reduced element.

NOTATION 2.1. *Let $K \subset \mathbb{R}^3$ denote a tetrahedron.*

- *The vertices of K are denoted by $\{v^{(i)}\}_{i=0}^3$ and the centroid by c_K.*
- *K_r denotes the subdivision of K into four tetrahedra with vertex set $\{v^{(i)}\}_{i=0}^3 \cup \{c_K\}$.*
- *If $f \subset K$ is a triangular face, then its centroid is denoted by c_f, and the three points with barycentric coordinates $(3/5, 1/5, 1/5)$, $(1/5, 3/5, 1/5)$, $(1/5, 1/5, 3/5)$, are denoted by $\{x_f^{(i)}\}_{i=0}^2$. The normal will be denoted by n_f.*
- *If $e \subset K$ is an edge, its center is denoted by c_e.*

- *The* degrees of freedom *for the* composite $\mathcal{P}_5$ tetrahedron*,* $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$*, are*
  1. $\{D^\alpha u(v^{(i)}) \mid 0 \le |\alpha| \le 2,\ 0 \le i \le 3\}$*, (40 dof)*
  2. $\{(I - \mathbf{e} \otimes \mathbf{e})\nabla u(c_e) \mid e \subset \partial K\}$*, (12 dof) where* $\mathbf{e}$ *is the unit vector parallel to* e*.*
  3. $\{\nabla u.n_f(x_f^{(i)}) \mid f \subset \partial K,\ 0 \le i \le 2\}$*, (12 dof)*
  4. $\{u(c_K)\}$*, (1 dof).*

  Here $\mathcal{P}_5(K_r)$ *denotes the functions that are picewise polynomial of degree* 5 *on the four tetrahedra in* K_r*, and* $\mathcal{P}_5(K_r) \cap C^4(c_K)$ *is the subspace with four continuous derivatives at the centroid* c_K *of* K*.*
- *The subspace for the reduced element is*

$$U_K = \{u_h \in \mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K) \mid \nabla u_h.n_f \in \mathcal{P}_3(f) \textit{ for each face } f \subset \partial K\},$$

  and the degrees of freedom for the reduced element *are*
  1. $\{D^\alpha u(v^{(i)}) \mid 0 \le |\alpha| \le 2,\ 0 \le i \le 3\}$*, (40 dof)*
  2. $\{\nabla u.n_f(c_f) \mid f \subset \partial K\}$*, (4 dof)*
  3. $\{u(c_K)\}$*, (1 dof).*

The following properties of $\mathcal{P}_5(K_r) \cap C^1(K)$ are useful.

- C^1 constraints on piecewise polynomial functions gives rise to additional continuity at the vertices; for example, the HCT element is C^2 at the center. For the composite $\mathcal{P}_5$ tetrahedron

$$\mathcal{P}_5(K_r) \cap C^1(K) = \mathcal{P}_5(K_r) \cap C^1(K) \cap C^2(v^{(i)}) \cap C^3(c_K), \quad 0 \le i \le 3.$$

  This shows that the degrees of freedom are well defined.
- If C^4 continuity a the centroid is omitted three additional degrees of freedom, $\{\nabla(c_K)\}$, corresponding to the gradient at the center are required for unisolvence. In particular, $dim(\mathcal{P}_5(K_r) \cap C^1(K)) = 68$.
- Construction of basis functions for $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$ is simplified by the property that C^1 continuity across the interior faces of K_r follows from C^4 continuity at the center and the continuity implied by the degrees of freedom. A similar property holds for the HCT triangle; C^2 continuity at the center and C^1 continuity at the vertices implies C^1 continuity across the internal edges.

2.1. Unisolvence. In [4] it was shown that the 65 degrees of freedom are unisolvent on $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$. Since the normal derivatives of $u_h \in \mathcal{P}_5(K_r) \cap C^1(K)$ on each edge and face are in $\mathcal{P}_4$ it is convenient to characterize $\mathcal{P}_3$ as a subspace of $\mathcal{P}_4$ by writing the edge and face degrees of freedom in terms of the reduced degrees of freedom.

If $e = (v^{(i)}, v^{(j)})$ is a line segment, the relation

$$\mathcal{P}_3(e) = \left\{ p \in \mathcal{P}_4(e) \mid p(c_e) = (1/2)\left(p(v^{(i)}) + p(v^{(j)})\right) + (|e|/8)\left(p'(v^{(i)}) - p'(v^{(j)})\right) \right\}, \tag{2.1}$$

can be used to characterize the Bell basis functions as a subspace of the Argyris basis functions. Specifically, if t is a triangle, then $u_h \in \mathcal{P}_5(t)$ is in the Bell subspace if the normal derivatives on each edge satisfy the above relation. The reduced basis functions can then be formed by subtracting a multiple of the edge basis functions from the vertex basis functions so that this relation holds. The following lemma provides the additional identities required to guarantee the normal derivative on the face of a tetrahedron is cubic.

LEMMA 2.2. *Let t be a triangle with vertices $\{v^{(i)}\}_{i=0}^2$ and let $\{x^{(i)}\}_{i=0}^2 \subset t$ be the points with barycentric coordinates $(3/5, 1/5, 1/5)$, $(1/5, 3/5, 1/5)$, and $(1/5, 1/5, 3/5)$.*

- *Let $\{p_i\}_{i=0}^2 \subset \{\mathcal{P}_4(t) \mid p|_{\partial t} = 0\}$ satisfy $p_i(x^{(j)}) = \delta_{ij}$ and $p \in \{\mathcal{P}_3(t) \mid p|_{\partial t} = 0\}$ satisfy $p(c_t) = 1$ where $c_t \in t$ is the centroid. Then*

$$p(x) = (81/125)\left(p_0(x) + p_1(x) + p_2(x)\right).$$

- *If $p \in \{\mathcal{P}_4(t) \mid p|_e \in \mathcal{P}_3(e),\ e \subset \partial t\}$ then $p \in \{p \in \mathcal{P}_3(t) \mid p(c_t) = 0\}$ if and only if*

$$p(x^{(i)}) = (12/25)p(v^{(i)}) - (8/125)\left(p(v^{(j)}) + p(v^{(k)})\right) + (6/125)(dp^i_{ij} - dp^i_{ki}) - (2/125)(dp^j_{jk} - dp^k_{jk}).$$

where (i, j, k) is an even permutation of $(0, 1, 2)$, and $dp^m_{pq} \equiv \nabla p(v^{(m)}).(v^{(p)} - v^{(q)})$.

Explicit calculation shows this to be true for the reference triangle $\hat{t} = \{\hat{x} \mid \hat{x}_\alpha \geq 0,\ \hat{x}_1 + \hat{x}_2 \leq 1\}$. The lemma then follows since any triangle t is the image of $\hat{t}$ under the affine map $\chi : \hat{K} \to K$ and dp^m_{pq} are affine degrees of freedom, i.e.

$$dp^m_{pq} = \nabla p(v^{(m)}).(v^{(p)} - v^{(q)}) = F^{-T}\hat{\nabla}\hat{p}(\hat{v}^{(m)}).F(\hat{v}^{(p)} - \hat{v}^{(q)}) = d\hat{p}^m_{pq},$$

where $F \in \mathbb{R}^2$ is the Jacobian of χ and $v^{(i)} = \chi(\hat{v}^{(i)})$.

A proof of unisolvence for the degrees of freedom for the reduced element now follows from elementary linear algebra.

LEMMA 2.3. *Let $K \subset \mathbb{R}^3$ be a tetrahedron and*

$$U_K = \{u_h \in \mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K) \mid \nabla u_h.n_f \in \mathcal{P}_3(f) \text{ for each face } f \subset \partial K\}.$$

Then the following degrees of freedom are unisolvent for U_K.

1. *(40 vertex dof) $\{D^\alpha u(v^{(i)}) \mid 0 \leq |\alpha| \leq 2,\ 0 \leq i \leq 3\}$,*
2. *(4 face dof) $\{\nabla u.n_f(c_f) \mid f \subset \partial K\}$,*
3. *(1 cell dof) $\{u(c_K)\}$.*

Proof. If $f \subset \partial K$ is a face, $dim(\mathcal{P}_4(f)) - dim(\mathcal{P}_3(f)) = 5$, and since K has four faces it follows that

$$dim(U_K) \geq dim(\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)) - 4 \times 5 = 45.$$

Next, if $u_h \in U_K$ and the reduced degrees of freedom for u_h vanish then clearly the vertex degrees of freedom for $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$ vanish; moreover,

1. Equation (2.1) shows that the edge degrees of freedom for $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$ also vanish, so $u_h|_e = 0$ for each edge $e \subset \partial K$.
2. Lemma 2.2 shows that the face degrees of freedom for $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$ also vanish.

It follows that $u_h = 0$ and $u_h \in U_K$ is uniquely determined from the reduced degrees of freedom, so $dim(U_K) \leq 45$ and the lemma follows. ∎

$$
\begin{array}{ccccc}
W^{m,p}(K) & \xrightarrow{\circ\chi} & W^{m,p}(\hat{K}) & & \\
I_K \downarrow & & \hat{I}_K \downarrow & & \\
U_K & \xrightarrow{\circ\chi} & \hat{U}_K & \subset \mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})
\end{array}
$$

FIG. 2.1. *Interpolation operators on the finite and parent elements.*

2.2. Error Estimate. Unisolvence of the reduced degrees of freedom guarantee that the canonical interpolation operator $I_K : C^2(K) \to U_K$ is well defined; in this section estimates for the error $u - I_K u$ are established.

The classical parent element construction will be used to develop the error estimates with a minor modification to accommodate the fact that the face degrees of freedom are not invariant under affine maps.

NOTATION 2.4.

1. *The affine map $\chi : \hat{K} \to K$ of the parent element to a tetrahedron K is written as $\chi(\hat{x}) = v^{(0)} + F\hat{x}$ where the Jacobian $F \in \mathbb{R}^{3\times 3}$ has columns $v^{(i)} - v^{(0)}$.*
2. *The mapping $\hat{}\, : C^2(K) \to C^2(\hat{K})$ induced by composition with χ is denoted with a hat, $\hat{u} \equiv u \circ \chi$, and*
$$\hat{U}_K = \{\hat{u}_h \in \mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}}) \mid \hat{u} \circ \chi^{-1} \in U_K\}.$$
3. *$I_K : C^2(K) \to U_K$ is the interpolation operator that preserves the reduced degrees of freedom. Below the following representation is assumed:*
   (a) *I_K preserves affine equivalence of the vertex degrees of freedom; at each vertex $v^{(i)}$*
   $$\begin{aligned} I_K(u) &= u, \\ \nabla I_K(u).(v^{(j)} - v^{(i)}) &= \nabla u.(v^{(j)} - v^{(i)}), \\ (v^{(k)} - v^{(i)})^\top D^2 I_K(u)(v^{(j)} - v^{(i)}) &= (v^{(k)} - v^{(i)})^\top D^2(u)(v^{(j)} - v^{(i)}). \end{aligned}$$
   (b) *I_K preserves scale invariance of the face degrees of freedom; at the centroid $c_{\hat{f}}$ of each face $\hat{f} \subset \partial\hat{K}$,*
   $$|f|^{1/2} \nabla I_K(u).n_f = |f|^{1/2} \nabla u.n_f.$$
   (c) *$I_K(u) = u$ at the centroid c_K of K*
4. *$\hat{I}_K = \chi \circ I_K \circ \chi^{-1} : C^2(\hat{K}) \to \hat{U}_K$ denotes the induced interpolation operator on $\hat{K}$ (see Figure 2.1).*
5. *Quantities with a hat are associated with the parent element ($\hat{f}$ is a face, $n_{\hat{f}}$ it's normal, etc.), and the subscript K indicates a dependence upon K.*

The identities
$$\nabla\phi = F^{-\top}\hat{\nabla}\hat{\phi}, \qquad D^2(\phi) = F^{-\top}\hat{D}^2(\hat{\phi})F^{-1}, \tag{2.2}$$
and
$$n_f = F^{-\top} n_{\hat{f}} / |F^{-\top} n_{\hat{f}}|, \qquad |f| = |\hat{f}|\,|\det(F)\, F^{-\top}\hat{n}_{\hat{f}}|, \qquad \text{where } f = \chi(\hat{f}) \text{ is a face of } K,$$

show that $\hat{I}_K$ can be characterized as

1. $\hat{I}_K$ preserves $\{\hat{D}^\alpha \hat{u}(\hat{v}^{(i)}) \mid 0 \le |\alpha| \le 2,\ 0 \le i \le 3\}$; at each vertex $\hat{v}^{(i)}$ of $\hat{K}$

$$\hat{I}_K(\hat{u}) = \hat{u},$$
$$\hat{\nabla}\hat{I}_K(u).(\hat{v}^{(j)} - \hat{v}^{(i)}) = \hat{\nabla}\hat{u}.(\hat{v}^{(j)} - \hat{v}^{(i)}),$$
$$(\hat{v}^{(j)} - \hat{v}^{(k)})^\top \hat{D}^2 \hat{I}_K(\hat{u})(\hat{v}^{(j)} - \hat{v}^{(i)}) = (\hat{v}^{(k)} - \hat{v}^{(i)})^\top \hat{D}^2(\hat{u})(\hat{v}^{(j)} - \hat{v}^{(i)}).$$

2. At the centroid $c_{\hat{f}}$ of each face $\hat{f} \subset \partial\hat{K}$,

$$\left(\frac{|\hat{f}|\,|\det(F)|}{|F^{-\top} n_{\hat{f}}|}\right)^{1/2} \hat{\nabla}\hat{I}_K(\hat{u}).(F^{-1}F^{-\top})n_{\hat{f}} = \left(\frac{|\hat{f}|\,|\det(F)|}{|F^{-\top} n_{\hat{f}}|}\right)^{1/2} \hat{\nabla}\hat{u}.(F^{-1}F^{-\top})n_{\hat{f}}. \quad (2.3)$$

3. $\hat{I}_K(\hat{u}) = \hat{u}$ at the centroid $c_{\hat{K}}$ of $\hat{K}$

When the degrees of freedom for a finite element are affine invariant the induced mapping $\hat{I}_K = \chi \circ I_K \circ \chi^{-1}$ on the parent element is independent of K. This property is exploited in the classical proof of finite element error estimates [3]; however, a minor modification of the classical argument shows that the same estimates hold when $\hat{I}_K$ is invariant under translation and scaling of K , and depends continuously upon the Jacobian.

Theorem 2.5. *Let $K \subset \mathbb{R}^3$ be a tetrahedron and*

$$U_K = \{u_h \in \mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K) \mid \nabla u_h.n_f \in \mathcal{P}_3(f), f \subset \partial K \text{ a face}\}.$$

Let $I_K : C^2(K) \to U_K$ be the operator that interpolates the following degrees of freedom,

1. *(40 vertex dof) $\{D^\alpha(u)(v^{(i)}) \mid 0 \le |\alpha| \le 2,\ 0 \le i \le 3\}$,*
2. *(4 face center dof) $\{\nabla u(c_f).n_f \mid f \subset \partial K\}$,*
3. *(1 cell center dof) $\{u(c_K)\}$.*

If $2 + 3/p < m \le 5$ then there is a constant depending continuously upon the aspect ratio of K such that

$$|u - I_K u|_{W^{\ell,p}(K)} \le C h_K^{m-\ell} |u|_{W^{m,p}(K)}, \qquad u \in W^{m,p}(K), \qquad 0 \le \ell \le m,$$

where h_K is the diameter of K. Here $W^{m,p}(K)$ is the Sobolev space of functions with m derivatives p integrable, and the aspect ratio of K is the ratio of the diameter to the radius of the largest inscribed sphere in K.

Proof. The lower bound $2 + 3/p < m$ guarantees that $W^{m,p}(\hat{K}) \subset C^2(\hat{K})$, so I_K and $\hat{I}_K$ are well defined,

Since $\mathcal{P}_4(K) \subset U_K$, and the degrees of freedom of the reduced element are unisolvent, it follows that $I_K(p) = p$ for each $p \in \mathcal{P}_4(K)$, and since $\chi : \hat{K} \to K$ is affine it follows that $\hat{I}_K(\hat{p}) = \chi \circ I_K \circ \chi^{-1}(\hat{p}) = \hat{p}$ for each $\hat{p} \in \mathcal{P}_4(\hat{K})$. Classical interpolation theory [3] then shows there exists a constant $\hat{C}$ depending only upon $\hat{K}$ (and the parameters m and p) such that

$$|u - I_K u|_{W^{\ell,p}(K)} \le \hat{C}\left(1 + \|\hat{I}_K\|_{\mathcal{L}(W^{m,p}(\hat{K}), W^{\ell,p}(\hat{K}))}\right)(h_K^m/\rho_K^\ell)|u|_{W^{m,p}(K)}, \qquad 0 \le \ell \le m,$$

where ρ_K is the radius of the largest inscribed sphere in K.

The dependence of $\hat{I}_K$ upon K exhibited in equation (2.3) shows that $\hat{I}_K$ is invariant under translation and dilation of K; $\hat{I}_{y+\lambda K} = \hat{I}_K$ when $y \in \mathbb{R}^3$ and $\lambda \in (0, \infty)$. The space of tetrahedra modulo translations and dilations can be characterized by the relative positions of the vertices, $\{v^{(i)} - v^{(0)} \mid 1 \le i \le 3\} \subset [-1,1]^3$, of the tetrahedra with unit diameter. Moreover the tetrahedra with bounded aspect ratio, $h_K/\rho_K \le \sigma$, correspond to a closed and bounded set $V_\sigma \subset [-1,1]^3$, and their Jacobians F and F^{-1} are continuous on V_σ. It follows that $\|\hat{I}_K\|_{\mathcal{L}(C^2(\hat{K}), W^{\ell,p}(\hat{K}))}$ can be expressed as a continuous function on the compact set V_σ so is bounded by a constant depending only upon $\sigma = h_K/\rho_K$. ◻

3. Constructing Reduced Basis Functions. Let $\hat{K}$ denote the standard reference (parent) tetrahedron and $\chi : \hat{K} \to K$ denote an affine map to a typical tetrahedron K. In this section basis functions $\{\phi_i\}_{i=1}^{45}$ for the reduced element on K will be constructed as linear combinations of the basis functions of $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ composed with χ^{-1}. This construction shows that the only basis functions that need to be coded are those for $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ on the reference element. These functions can conveniently be generated using a computer algebra package. Figure 3.1 shows an algorithm to compute the reduced bases functions using the formula derived in this section.

The following notation and terminology will be used below.

Notation 3.1.

1. *Basis functions for $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ will be denoted by $\{\hat{\psi}_i\}_{i=1}^{65}$, and basis functions which map to the reduced element under χ^{-1} are denoted by $\{\hat{\phi}_i\}_{i=1}^{45}$.*
2. *Vertex basis functions corresponding to the degrees of freedom $\hat{u}$, $\hat{D}_\alpha(\hat{u})$ and $\hat{D}^2_{\alpha\beta}(\hat{u})$ be referred to as* function *or* gradient *or* Hessian *basis functions respectively.*
3. *If $\hat{e} = \hat{f}_p \cap \hat{f}_q$ is the edge of $\hat{K}$ common to faces $\hat{f}_p$ and $\hat{f}_q$, the discussion below considers the edge degrees of freedom to be $\hat{\nabla}\hat{u}(c_{\hat{e}}).\hat{n}_p$ and $\hat{\nabla}\hat{u}(c_{\hat{e}}).\hat{n}_q$, where $\hat{n}_p$ and $\hat{n}_q$ are the face normals.*

3.1. Centroid Basis Function. If $\hat{\psi}$ is the basis function corresponding to $\hat{u}(c_{\hat{K}})$, then $\hat{\psi}$ and $\hat{\nabla}\hat{\psi}$ both vanish on $\partial\hat{K}$. In this situation $\phi = \hat{\psi} \circ \chi^{-1}$ is the basis function on K corresponding to the degree of freedom $u(c_K)$.

3.2. Face Basis Functions. If $\hat{f} \subset \partial\hat{K}$ is a face and $\{\hat{\psi}^{(i)}\}_{i=0}^{2}$ are the basis functions corresponding to the degrees of freedom $\hat{\nabla}\hat{u}(\hat{x}_{\hat{f}}^{(i)}).n_{\hat{f}}$ on $\hat{f}$, then $\hat{\psi}^{(i)}|_{\partial\hat{K}} = 0$ and $\hat{\nabla}\hat{\psi}^{(i)}$ vanishes at the vertices and mid–points of the edges of $\hat{K}$. In particular, $\hat{\nabla}\hat{\psi}^{(i)}(\hat{x}_{\hat{f}}^{(i)}) = \hat{n}_{\hat{f}}$ and if $\psi^{(i)} = \hat{\psi}^{(i)} \circ \chi^{-1}$ then $\nabla\psi^{(i)}(x_f^{(j)}) = |F^{-\top} n_{\hat{f}}| n_f \delta_{ij}$. The first identity in Lemma 2.2 then shows that the corresponding face basis function for the reduced element is $\phi_f = \hat{\phi}_f \circ \chi^{-1}$ where

$$\hat{\phi}_f = (1/|F^{-\top} n_{\hat{f}}|)\,(81/125)\left(\hat{\psi}^{(0)} + \hat{\psi}^{(1)} + \hat{\psi}^{(2)}\right). \tag{3.1}$$

The final step is to determine a global normal to f and scale $\hat{\phi}_f$ by ± 1 accordingly. If the mesh generator returns a consistent orientation of each tetrahedron this is done combinatorially using the ordering of the vertices on the face.

3.3. Vertex Basis Functions. The reduction of each vertex basis function proceeds in two stages.

TRANSFORM($\{v^{(i)}\}_{i=0}^{3}$, F, $\{\hat{\psi}_i\}_{i=1}^{65}$, $\{\phi_i\}_{i=1}^{45}$)
Input: $\{v^{(i)}\}_{i=0}^{3}$ are the global vertex indices of K, F is the Jacobian, and $\{\hat{\psi}_i\}_{i=1}^{65}$ are the basis functions for $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$
Output: $\{\phi_i\}_{i=1}^{45}$ are the reduced basis functions on K
(1) Assign the centroid basis function, typically $\hat{\phi}_{45} = \hat{\psi}_{65}$.
(2) **foreach** face $\hat{f} \subset \hat{K}$
(3) Use equation (3.1) and the vertex indices to compute the four face basis functions consistently signed.
(4) Initialize the vertex basis functions $\hat{\phi}_i = \hat{\psi}_i$, and initialize the gradients $\nabla\hat{\phi}_i(\hat{p}) = \nabla\hat{\psi}_i(\hat{p})$ at face points.
(5) **foreach** edge $\hat{e} \subset \hat{K}$
(6) Let $\hat{\psi}_p$ and $\hat{\psi}_q$ be the edge basis functions associated with $\hat{e}$.
(7) **foreach** vertex $\hat{v}^{(i)}$ of $\hat{e}$
(8) Form the linear combination

$$\hat{\phi}_i \leftarrow \hat{\phi}_i + \theta_p\hat{\psi}_p + \theta_q\hat{\psi}_q,$$

of each of the 10 vertex basis functions at $\hat{v}^{(i)}$ with the edge basis functions using the formula in equations (3.2) and (3.3) to compute the coefficients θ_p and θ_q.
(9) Update the values of $\hat{\nabla}\hat{\phi}_i(\hat{p})$ at the faces points on the faces containing $\hat{e}$.
(10) **foreach** face $\hat{f} \subset \hat{K}$
(11) Let $\{\hat{\psi}^{(m)}\}_{m=0}^{2}$ be the face basis functions associated with $\hat{f}$.
(12) **foreach** vertex $\hat{v}^{(i)}$ of $\hat{f}$
(13) Form the linear combination

$$\hat{\phi}_i \leftarrow \hat{\phi}_i + \theta_i\hat{\psi}^{(i)} + \theta_j\hat{\psi}^{(j)} + \theta_k\hat{\psi}^{(k)}$$

of each of the 10 vertex basis functions at $\hat{v}^{(i)}$ with the face basis functions where the coefficients $\{\theta_i, \theta_j, \theta_k\}$ are computed using the formula in equations (3.4) and (3.5).
(14) Transform $\{\hat{\phi}_i\}_{i=1}^{45}$ to a canonical basis $\{\phi_i\}_{i=1}^{45}$ on K using equations (3.6) and (3.7). Equation (2.2) is used to evaluate derivatives of the basis functions.

FIG. 3.1. *Algorithm to transform the basis of $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ to the reduced basis on K.*

1. First a linear combination is formed with the two basis functions on each edge incident to the vertex with coefficients chosen to guarantee that the normal derivatives on each edge of K are cubic.
   The gradients of the vertex basis functions at the edge centers are used in this step (these gradients are parallel to the edge). Arrays containing this data can conveniently be generated by the computer algebra code used to generate the basis functions.

2. A linear combination with the three basis functions on each face incident to the vertex is then formed with coefficients chosen to guarantee that the normal derivative on each face of K is cubic.
   The gradients of the vertex and edge basis functions at the face points are used in this step (these gradients are in the plane of the face).

The order is important; gradients of the face basis functions vanish on the edges, so addition of a face basis function does not not change the normal derivative of a function on edges of K. However, the gradients of the edge basis functions do not vanish at the face points, so reduction of the edge basis functions needs to be completed before reduction of the face basis functions.

Edge Basis Functions: If $\hat{\psi}_i$ is a vertex basis function for $\mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ let $\hat{e} = (\hat{v}^{(i)}, \hat{v}^{(j)}) = \hat{f}_p \cap \hat{f}_q$ be an incident edge. Let $\hat{\psi}_p$ and $\hat{\psi}_q$ denote the two edge basis functions with $\hat{\nabla}\hat{\psi}_p(c_{\hat{e}}).\hat{n}_p = 1$ and $\hat{\nabla}\hat{\psi}_p(c_{\hat{e}}).\hat{n}_q = 0$ where $\hat{n}_p$ and $\hat{n}_q$ are the normals to $\hat{f}_p$ and $\hat{f}_q$, and similarly $\hat{\nabla}\hat{\psi}_q(c_{\hat{e}}).\hat{n}_p = 0$ and $\hat{\nabla}\hat{\psi}_q(c_{\hat{e}}).\hat{n}_q = 1$. Write

$$\phi_i = (\hat{\psi}_i + \theta_p \hat{\psi}_p + \theta_q \hat{\psi}_q) \circ \chi^{-1},$$

and $e \equiv \chi(\hat{e})$, $f_p = \chi(\hat{f}_p)$, $f_q = \chi(\hat{f}_q)$.

1. If $\hat{\psi}_i$ is a *function* or *gradient* basis function then $\nabla\phi_i(v^{(j)}) = 0$, and $D^2(\phi_i)$ vanishes at the vertices. Equation (2.1) then shows that $\nabla\phi_i.n_p|_e$ and $\nabla\phi_i.n_q|_e$ will be cubic if

$$\nabla\phi_i(c_e).n_p = (1/2)\nabla\phi_i(v^{(i)}).n_p \quad \text{and} \quad \nabla\phi_i(c_e).n_q = (1/2)\nabla\phi_i(v^{(i)}).n_q.$$

   Pulling these formula back to the parent using equation (2.2) gives the pair of linear equations for θ_p and θ_q,

$$\begin{aligned} \hat{\nabla}(\hat{\psi}_i + \theta_p\hat{\psi}_p + \theta_q\hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{n}_p &= (1/2)\hat{\nabla}\hat{\psi}_i(v^{(i)}).(F^{-1}F^{-\top})\hat{n}_p \\ \hat{\nabla}(\hat{\psi}_i + \theta_p\hat{\psi}_p + \theta_q\hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{n}_q &= (1/2)\hat{\nabla}\hat{\psi}_i(v^{(i)}).(F^{-1}F^{-\top})\hat{n}_q. \end{aligned} \tag{3.2}$$

   Note that $\hat{\nabla}\hat{\psi}_i(v^{(i)})$ vanishes if $\hat{\psi}_i$ is the basis function corresponding to $\hat{u}(\hat{v}^{(i)})$.
2. If $\hat{\psi}_i$ is a *Hessian* basis function then $\nabla\phi_i$ vanishes at the vertices and $D^2(\phi_i)(v^{(j)}) = 0$. Equation (2.1) then shows that $\nabla\phi_i.n_p|_e$ and $\nabla\phi_i.n_q|_e$ will be cubic if

$$\nabla\phi_i(c_e).n_p = (1/8)D^2(\phi_i)(v^{(i)})e^{ij}.n_p \quad \text{and} \quad \nabla\phi_i(c_e).n_q = (1/8)D^2(\phi_i)(v^{(i)})e^{ij}.n_q;$$

   here $e^{ij} = \hat{v}^{(j)} - \hat{v}^{(i)}$ is the edge vector. Pulling these formula back to the parent using equation (2.2) gives the pair of linear equations for θ_p and θ_q,

$$\begin{aligned} \hat{\nabla}(\hat{\psi}_i + \theta_p\hat{\psi}_p + \theta_q\hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{n}_p &= (1/8)\hat{D}^2(\hat{\psi}_i)(v^{(i)})\hat{e}^{ij}.(F^{-1}F^{-\top})\hat{n}_p \\ \hat{\nabla}(\hat{\psi}_i + \theta_p\hat{\psi}_p + \theta_q\hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{n}_q &= (1/8)\hat{D}^2(\hat{\psi}_i)(v^{(i)})\hat{e}^{ij}.(F^{-1}F^{-\top})\hat{n}_q. \end{aligned} \tag{3.3}$$

Face Basis Functions: Let $\tilde{\psi}_i \in \mathcal{P}_5(\hat{K}_r) \cap C^1(\hat{K}) \cap C^4(c_{\hat{K}})$ be an "edge reduced" vertex basis function at $\hat{v}^{(i)}$; that is, the normal derivatives of $\tilde{\psi}_i \circ \chi^{-1}$ along any edge of K are cubic. Let $\hat{f}$ be a face incident face incident to $\hat{v}^{(i)}$ and denote this faces basis functions by $\{\hat{\psi}^{(i)}, \hat{\psi}^{(j)}, \hat{\psi}^{(k)}\}$ where $\hat{\psi}^{(i)}$ is the basis function with unit normal derivative at $\hat{x}^{(i)\hat{f}} \equiv (3/5)\hat{v}^{(i)} + (1/5)\hat{v}^{(j)} + (1/5)\hat{v}^{(k)}$. Write

$$\phi_i = (\tilde{\psi}_i + \theta_i\hat{\psi}^{(i)} + \theta_j\hat{\psi}^{(j)} + \theta_k\hat{\psi}^{(k)}) \circ \chi^{-1},$$

and $x_f^{(m)} = \chi(\hat{x}_{\hat{f}}^{(m)})$, $f = \chi(\hat{f})$, and recall that

$$\hat{\nabla}\hat{\psi}^{(m)}(\hat{x}_{\hat{f}}^{(n)}) = \delta_{mn} n_{\hat{f}} \qquad \text{and} \qquad \nabla \psi^{(m)}(x_f^{(n)}) = (1/|F^{-\top} n_{\hat{f}}|)\delta_{mn} n_f,$$

where $\psi^{(m)} = \hat{\psi}^{(m)} \circ \chi^{-1}$.

1. If $\tilde{\psi}_i$ is a *function* or *gradient* basis function then $D^2(\phi_i)$ vanishes at the vertices, so the second statement in Lemma 2.2 shows $\nabla \phi_i . n_f$ will be cubic on f if

$$\nabla \phi_i(x_f^{(m)}).n_f = c_m \nabla \phi_i(v^{(i)}).n_f,$$

where $c_i = 12/25$ and $c_j = c_k = -8/125$. Pulling these formula back to the parent using equation (2.2) shows

$$\theta_m = \left(c_m \nabla \tilde{\psi}_i(\hat{v}^{(i)}) - \nabla \tilde{\psi}_i(\hat{x}_{\hat{f}}^{(m)})\right).(F^{-1}F^{-\top})n_{\hat{f}}/|F^{-\top} n_{\hat{f}}|^2. \tag{3.4}$$

2. If $\hat{\psi}_i$ is a *Hessian* basis function then $\nabla \phi_i$ vanishes at the vertices and $D^2(\phi_i)(v^{(m)}) = 0$ for $m \neq i$. The second statement in Lemma 2.2 then shows that $\nabla \phi_i . n_f$ will be cubic on f if

$$\nabla \phi_i(x^{(m)}).n_f = \begin{cases} (6/125)D^2(\phi_i)(x_f^{(i)})e^{kj}.n_f & m = i \\ +(2/125)D^2(\phi_i)(x^{(i)f})e^{ki}.n_f & m = j \\ -(2/125)D^2(\phi_i)(x^{(i)f})e^{ij}.n_f & m = k \end{cases}$$

where $e^{mn} = v^{(n)} - v^{(m)}$ are the edge vectors of f. Using equation (2.2) to pull these relations back to the parent yield formula for θ_m, $m \in \{i, j, k\}$,

$$\theta_m = \left(c_m \hat{D}^2(\tilde{\psi}_i)(\hat{v}^{(i)})\hat{e}^{(m)} - \nabla \tilde{\psi}_i(\hat{x}^{(m)})\right).(F^{-1}F^{-\top})n_{\hat{f}}/|F^{-\top} n_{\hat{f}}|^2. \tag{3.5}$$

where

$$(c_i, c_j, c_k) = (6/25, 2/125, -2/125) \quad \text{and} \quad (\hat{e}^{(i)}, \hat{e}^{(j)}, \hat{e}^{(k)}) = (\hat{v}^{(j)} - \hat{v}^{(k)}, \hat{v}^{(i)} - \hat{v}^{(k)}, \hat{v}^{(j)} - \hat{v}^{(i)}).$$

Continuous Dependence of Coefficients: The coefficients θ_p, θ_q and θ_m computed from equations (3.2)–(3.5) are invariant under dilation $F \mapsto \lambda F$. We verify that the dependence upon F is continuous over the set of non–singular matrices. Continuity of the coefficients θ_m computed using equations (3.4)–(3.5) is immediate. To verify continuous dependence of the solution (θ_p, θ_q) of equations (3.2) and (3.3) it is convenient to introduce the unit vectors

$$\hat{m}_p = \hat{\nabla}\hat{\phi}_p(c_{\hat{e}})/|\hat{\nabla}\hat{\phi}_p(c_{\hat{e}})| \quad \text{and} \quad \hat{m}_q = \hat{\nabla}\hat{\phi}_q(c_{\hat{e}})/|\hat{\nabla}\hat{\phi}_q(c_{\hat{e}})|.$$

This pair of vectors form a basis for the plane perpendicular to $\hat{e}$, so writing $\hat{n}_p$ and $\hat{n}_q$ in terms of this basis allows the equations (3.2) to be written as

$$\begin{aligned} \hat{\nabla}(\hat{\psi}_i + \theta_p \hat{\psi}_p + \theta_q \hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{m}_p &= (1/2)\hat{\nabla}\hat{\psi}_i(v^{(i)}).(F^{-1}F^{-\top})\hat{m}_p \\ \hat{\nabla}(\hat{\psi}_i + \theta_p \hat{\psi}_p + \theta_q \hat{\psi}_q)(c_{\hat{e}}).(F^{-1}F^{-\top})\hat{m}_q &= (1/2)\hat{\nabla}\hat{\psi}_i(v^{(i)}).(F^{-1}F^{-\top})\hat{m}_q, \end{aligned}$$

and similarly for equations (3.3). The determinant of this pair of equations for (θ_p, θ_q) is

$$|\hat{\nabla}\hat{\psi}_p|\,|\hat{\nabla}\hat{\psi}_q|\left(|F^{-\top}\hat{m}_p|^2|F^{-\top}\hat{m}_q|^2 - (F^{-\top}\hat{m}_p.F^{-\top}\hat{m}_q)^2\right).$$

The Cauchy Schwarz inequality shows that this determinant is non–negative and vanishes if and only if $F^{-\top}\hat{m}_p$ is parallel to $F^{-\top}\hat{m}_q$ which can only happen if F is singular. It follows that θ_p and θ_q are continuous functions of F on the set of non–singular matrices.

h	Reduced composite $\mathcal{P}_5$ Tetrahedron $\|u-u_h\|_{L^2(\Omega)}$	$\lvert u-u_h\rvert_{H^1(\Omega)}$	$\lvert u-u_h\rvert_{H^2(\Omega)}$	BFS Cube $\|u-u_h\|_{L^2(\Omega)}$	$\lvert u-u_h\rvert_{H^1(\Omega)}$	$\lvert u-u_h\rvert_{H^2(\Omega)}$
1/2	5.024363e-04	4.678325e-03	8.074063e-02	6.675721e-04	6.440100e-03	9.355929e-02
1/4	1.853231e-05	3.691489e-04	1.461122e-02	7.005153e-05	1.267368e-03	3.376314e-02
1/8	5.254082e-07	2.523825e-05	2.126684e-03	4.514568e-06	1.650011e-04	8.611408e-03
1/16	1.476834e-08	1.627254e-06	2.815166e-04	2.830667e-07	2.077966e-05	2.158125e-03
Rate	5.1529	3.9551	2.9173	3.9954	2.9892	1.9965
Norms	3.162278	2.452105	4.125031	3.162278	2.452105	4.125031

FIG. 4.1. *Error $u - u_h$ for the numerical example.*

3.4. Transformation to the Canonical Basis. The final step is to transform the gradient and Hessian basis functions on the parent element to give canonical basis on K.

1. If $\{\hat{\phi}_{i\alpha}\}_{\alpha=1}^3$ are *gradient* basis functions at vertex $\hat{v}^{(i)}$ satisfying $\hat{\nabla}\hat{\phi}_{i\alpha}(\hat{v}^{(i)}) = \hat{e}^{(\alpha)}$ where $\hat{e}^\alpha_\beta = \delta_{\alpha\beta}$, let

$$\phi_{ij} = \left(\sum_{\alpha=1}^3 F_{j\alpha}\hat{\phi}_{i\alpha}\right) \circ \chi^{-1}. \tag{3.6}$$

   Then $\nabla\phi_{ij}(v^{(i)}) = e^{(j)}$ where $e^{(j)}_k = \delta_{jk}$.

2. Let $\{\hat{\phi}_{i\alpha\beta} \mid 1 \le \alpha \le \beta \le 3\}$ be *Hessian* basis functions at vertex $\hat{v}^{(i)}$ satisfying

$$\begin{aligned} \hat{D}^2(\hat{\phi}_{i\alpha\alpha})(\hat{v}^{(i)}) &= \hat{e}^{(\alpha)} \otimes \hat{e}^{(\alpha)}, && 0 \le \alpha < d \\ \hat{D}^2(\hat{\phi}_{i\alpha\beta})(\hat{v}^{(i)}) &= e^{(\alpha)} \otimes \hat{e}^{(\beta)} + \hat{e}^{(\beta)} \otimes \hat{e}^{(\alpha)}, && 0 \le \alpha < \beta < d. \end{aligned}$$

   Setting

$$\phi_{ijk} = (1/2)\left(\sum_{\alpha\le\beta}(F_{j\alpha}F_{k\beta} + F_{k\alpha}F_{j\beta})\hat{\phi}_{i\alpha\beta}\right) \circ \chi^{-1}, \qquad 0 \le i, j < d. \tag{3.7}$$

   gives basis functions satisfying

$$D^2(\phi_{ijk})(v^{(i)}) = (1/2)(e^{(j)} \otimes e^{(k)} + e^{(k)} \otimes e^{(j)}), \qquad 0 \le i, j < d.$$

4. Numerical Example. We consider the fourth order equation

$$D^2 : \mathbb{C}(D^2u) = f, \quad \text{on } \Omega = (-1,1)^3, \quad \text{where} \quad \mathbb{C}(D) \equiv 2\mu D + \lambda tr(D)I,$$

with essential boundary conditions $u|_{\Gamma_0} = u_0$, $\partial u/\partial n|_{\Gamma_1} = g_1$, and (with div_Γ denoting surface divergence) natural boundary conditions

$$\left(div\,\mathbb{C}(D^2u).n + div_\Gamma\,\mathbb{C}(D^2u)n\right)|_{\Gamma_0'} = g_3, \qquad n^\top\mathbb{C}(D^2u)n|_{\Gamma_1'} = g_2.$$

Here the partitions of the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_0'$ and $\partial\Omega = \Gamma_1 \cup \Gamma_1'$ are taken to be

$$\Gamma_0 = \{x = \pm 1, \text{ or } z = \pm 1\}, \quad \text{and} \quad \Gamma_1 = \{x = -1, \text{ or } y = -1, \text{ or } z = \pm 1\}.$$

When λ and μ are constant the equation reduces to the biharmonic equation $(2\mu + \lambda)\Delta^2 u = f$. The natural weak statement seeks

$$u \in U(u_0, g_1) \equiv \{H^2(\Omega) \mid u|_{\Gamma_0} = u_0,\ \partial u/\partial n|_{\Gamma_1} = g_1\},$$

such that

$$\int_\Omega \mathbb{C}(D^2 u) : D^2 v = \int_\Omega f v + \int_{\Gamma_0'} (-g_{30} v + g_{31}.\nabla_\Gamma v) + \int_{\Gamma_1'} g_2 \frac{\partial v}{\partial n}, \qquad v \in U \equiv U(0,0),$$

where $g_3 \in H^{-3/2}(\Gamma_0')$ has been decomposed into a sum of $g_{30} \sim div\, \mathbb{C}(D^2 u).n \in H^{-3/2}(\Gamma_0')$ and $g_{31} \sim (I - n \otimes n)\mathbb{C}(D^2 u) n \in H^{-1/2}(\Gamma_0')^3$ acting on the surface gradient, $\nabla_\Gamma v$.

When the degrees of freedom contain high order derivatives the specification of boundary values of u and $\partial u/\partial n$ becomes non-trivial, so a penalization is used,

$$\begin{aligned}
&\int_\Omega \mathbb{C}(D^2 u) : D^2 v + (1/\epsilon)\left(\int_{\Gamma_0} uv + \int_{\Gamma_1} \frac{\partial u}{\partial n}\frac{\partial v}{\partial v}\right) \\
&= \int_\Omega f v + \int_{\Gamma_0'} (-g_{30} v + g_{31}.\nabla_\Gamma v) + \int_{\Gamma_1'} g_2 \frac{\partial v}{\partial n} + (1/\epsilon)\left(\int_{\Gamma_0} u_0 v + \int_{\Gamma_1} g_1 \frac{\partial v}{\partial n}\right).
\end{aligned}$$

For the example we set $\mu = 1$, $\lambda = 1/4$ and $\epsilon = 10^{-50}$, and manufactured the right hand side and boundary data for the solution to be the regularized Greens function $u(x) = \sqrt{|x|^2 + 1/4}$. Meshes were formed by dividing the cubes of a uniform subdivision of $[-1,1]^3$ into six tetrahedra. Errors for the reduced $\mathcal{P}_5(K_r) \cap C^1(K) \cap C^4(c_K)$ tetrahedral element and the (cubic) Bogner Fox Schmit (BFS) cube are tabulated in Figure 4.1. The asymptotic rates of $5 - \ell$ in the $H^\ell(\Omega)$ semi–norm for the element developed in this paper are achieved, as are the rates of $4 - \ell$ for the BFS element.

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